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Decompositions of Response Times: an Almost General Theory*

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Response time (RT) whose distribution depends on two external factors can sometimes be presented as an algebraic combination of two random variables, component times, each of which is selectively influenced by one of these factors. The algebraic operation connecting these component times (such as addition or "minimum of the two") is referred to as the decomposition rule. We consider a broad subclass of associative and commutative decomposition rules, and for any operation from this subclass we construct a decomposition test, a relationship between observable RTs that must hold if these RTs are decomposable by means of this operation. The decomposition tests are constructed under the assumption that RT components are either stochastically independent or perfectly positively stochastically interdependent (in which case they are increasing functions of a common random variable). The decomposition tests generalize the summation test proposed by Ashby & Townsend (1980) and Roberts & Sternberg (1992) for additive decompositions into stochastically independent components. Under the assumption of perfect positive stochastic interdependence, a successful decomposition test is not only necessary but also sufficient for the RT decomposability by means of the corresponding operation. Under the assumption of stochastic independence, it is possible that a decomposition test is successful but RTs cannot be decomposed by any operation. Under both assumptions, however, a successful decomposition test recovers the true decomposition rule essentially uniquely. For a given decomposition rule, the component times themselves cannot be determined uniquely, and the stochastic relationship between them generally has to be assumed rather than recovered from the decomposition tests.

1. INTRODUCTION

This paper deals with one of the classical problems of response time analysis, dating back to Sternberg's (1969) pioneering work: the problem is that of decomposing response time (RT) into component durations *selectively influenced* by different external variables (factors). A prototypical example is the familiar *additive* decomposition of RT into *stochastically independent* selectively influenced components. Let **T** denote an observable RT, a random variable (simple RT, choice RT for a given response or aggregated over several responses). Let the distribution of **T** depend on two factors, α and β, all other external variables being fixed or counterbalanced across trials: $\mathbf{T}=\mathbf{T}(\alpha, \beta)$. We say that the RT in question is additively decomposable into stochastically independent and selectively influenced component times $\mathbf{A}=\mathbf{A}(\alpha)$ and $\mathbf{B}=\mathbf{B}(\beta)$, if the following three assumptions hold.

(i) *Selective influence assumption*: the (distributions of the) random variables $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ only depend on α and β, respectively (so that **B** does not depend on α, or **A** on β).

(ii) *Stochastic independence assumption*: for any given values of α and β, the random variables $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ are stochastically independent.

(iii) *Additive decomposition rule assumption*: the observable RT, $\mathbf{T}(\alpha, \beta)$, is distributed as $\mathbf{A}(\alpha)+\mathbf{B}(\beta)$.

The conjunction of these three assumptions can be referred to as the (principal) *architecture* of the RT in question. A concise representation of this architecture is provided by the formula

$$\mathbf{T}(\alpha, \beta) \overset{d}{=} \mathbf{A}(\alpha)+\mathbf{B}(\beta), \qquad \mathbf{A}(\alpha) \perp \mathbf{B}(\beta),$$

where the symbol $\overset{d}{=}$ should be read "is distributed as," the symbol $\perp$ indicates stochastic independence, and the random variables **T**, **A**, and **B** are assumed to depend only on those factors that are explicitly shown as their arguments. (The reason for using the equidistribution symbol $\overset{d}{=}$ rather

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than a numerical equality = will be explained later; it should be noted now that the two must not be confused.)

Due to the obvious interpretation of the additive decomposition rule as reflecting a physical concatenation of processes in time, this architecture is commonly referred to as "serial" (with stochastically independent components). Replacing in the formulation above the operation + with the operations max ("maximum of") or min ("minimum of"), one gets two other familiar architectures, commonly referred to as "parallel" ones (with stochastically independent components):

$$\mathbf{T}(\alpha, \beta) \overset{\mathrm{d}}{=} \max[\mathbf{A}(\alpha), \mathbf{B}(\beta)], \qquad \mathbf{A}(\alpha) \perp \mathbf{B}(\beta),$$

$$\mathbf{T}(\alpha, \beta) \overset{\mathrm{d}}{=} \min[\mathbf{A}(\alpha), \mathbf{B}(\beta)], \qquad \mathbf{A}(\alpha) \perp \mathbf{B}(\beta).$$

Most of the previous studies of RT decompositions have been confined to these three architectures.

In this paper the notion of a RT decomposition into selectively influenced components is extended along two dimensions. First, we incorporate, within a unified theoretical framework, architectures that widely differ in their *decomposition rule*, the mathematical operation connecting hypothetical component times $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$. Among logically possible decomposition rules we focus specifically on a broad subclass of commutative and associative operations, termed *simple operations*. This subclass includes the traditional operations of arithmetic addition, minimum, and maximum as special cases, but it also includes a variety of theoretically interesting decomposition rules that have not been previously considered in the context of RT analysis, such as "Minkowski-norm" decompositions,

$$\mathbf{T}(\alpha, \beta) \overset{\mathrm{d}}{=} [\mathbf{A}(\alpha)^p + \mathbf{B}(\beta)^p]^{1/p}, \qquad p > 1,$$

and multiplicative decompositions,

$$\mathbf{T}(\alpha, \beta) \overset{\mathrm{d}}{=} k\mathbf{A}(\alpha)\,\mathbf{B}(\beta), \qquad k > 0.$$

We propose an interpretation of RT components that naturally leads to decomposition rules other than the familiar "serial" and "parallel" connections. In this interpretation, component times $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ reflect certain "properties," rather than "parts" of a hypothetical process evoked by a signal and developing until it reaches a preset criterion level (Dzhafarov, 1993).

Second, our analysis is not confined to stochastically independent component times exclusively. Stochastic independence of component times by no means follows from their being selectively influenced by different factors. The essence of our approach to this issue, derived from Dzhafarov (1992), is that $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ are selectively influenced (by α and β, respectively) if, and only if, the proposition

$$\{\mathbf{A}(\alpha), \mathbf{B}(\beta)\} \overset{\mathrm{d}}{=} \{A(\alpha, \mathbf{X}), B(\beta, \mathbf{Y})\}$$

holds for their joint distribution, where A and B are some functions, whereas $(\mathbf{X}, \mathbf{Y})$ is a pair of random variables ("internal sources of variability") whose joint distribution does not depend on the factors α and β. Stochastic independence, $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, is obtained as a special case, by assuming that the internal sources of variability are themselves stochastically independent, $\mathbf{X} \perp \mathbf{Y}$.

The analytic tools used in this paper, however, are not powerful enough to allow us to operate with arbitrary joint distributions of the internal sources of variability, $(\mathbf{X}, \mathbf{Y})$. Instead we contrast stochastic independence with only the simplest case of stochastic interdependence, termed *perfect positive stochastic interdependence*. This is obtained by assuming that the two component durations have one common source of variability, $\mathbf{X} = \mathbf{Y}$, whereas both functions A and B increase with the value of $\mathbf{X}$. In this case (generalizing the "single-variate RT decomposition model," Dzhafarov, 1992), the component times $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ are increasing functions of each other, for any given values of α and β: we denote this relationship by $\mathbf{A}(\alpha) \parallel \mathbf{B}(\beta)$.

Thus, the focus of this paper is on the RT architectures representable as

$$\mathbf{T}(\alpha, \beta) \overset{\mathrm{d}}{=} \mathbf{A}(\alpha) \blacklozenge \mathbf{B}(\beta), \qquad \mathbf{A}(\alpha) \perp \mathbf{B}(\beta),$$

or

$$\mathbf{T}(\alpha, \beta) \overset{\mathrm{d}}{=} \mathbf{A}(\alpha) \blacklozenge \mathbf{B}(\beta), \qquad \mathbf{A}(\alpha) \parallel \mathbf{B}(\beta),$$

where $\blacklozenge$ is an operation chosen from the subclass of associative and commutative operations mentioned earlier.

The principal point of interest is, of course, whether and how such a RT architecture can be empirically recovered, based on a relationship between observable RT distributions only. Ashby & Townsend (1980) proposed a simple empirical procedure that, if not successful, proves that the RT cannot be *additively* decomposed into *stochastically independent* components. Later this procedure was subjected to a thorough experimental analysis by Roberts & Sternberg (1992), as the "summation test." We show in this paper that analogous tests can be constructed for all simple operations (in fact, all associative and commutative operations) connecting $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$, under both stochastic independence, $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, and perfect positive stochastic interdependence, $\mathbf{A}(\alpha) \parallel \mathbf{B}(\beta)$. We term the variety of these tests the *decomposition tests*, and this paper is dedicated to studying their most basic properties.

Logically, a decomposition test constructed for a specific simple operation (say, $+$) and for one of the two forms of stochastic relationship ($\perp$ or $\|$) is a *necessary condition* for the existence of the corresponding architecture (in this example, a "serial" architecture with the same stochastic relationship). The aim of this study is to investigate whether and under which conditions such a test is also *sufficient* for the existence of the corresponding decomposition, as well as whether and under which conditions the decomposition rule is identified by the test *uniquely*. This investigation is conducted with no a priori constraints imposed on observable RT distributions (which makes the results potentially applicable to decompositions of random variables in empirical domains other than RT).

We show that under the assumption of stochastic independence, $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, a decomposition test constructed for a specific simple operation, if successful, is not generally *sufficient for the existence of the corresponding architecture.* If, for example, observable RTs $\mathbf{T}(\alpha, \beta)$, for several appropriately chosen values of α and β, satisfy the original Ashby–Townsend–Roberts–Sternberg summation test, these RTs nevertheless may be indecomposable into sums of selectively influenced stochastically independent component times. We show, however, that no two tests designed for two different simple operations can be successful simultaneously (with rather mild constraints imposed on the relationship between these operations). This means that, under the assumption of stochastic independence, a successful decomposition test identifies the decomposition rule in the RT architecture uniquely, if this architecture exists at all (which the test generally does not guarantee).

The situation is much simpler under the assumption of perfect positive stochastic interdependence, $\mathbf{A}(\alpha) \| \mathbf{B}(\beta)$. Here, a successful decomposition test constructed for a specific simple operation is always both necessary and sufficient for the existence of the corresponding architecture. At the same time, for all practical purposes (the precise meaning of which will be explained later), the decomposition rule here is identified by the test as uniquely as in the independence case, under the same mild constraints imposed on the algebraic relationship between "competing" operations.

A unique recovery of the decomposition rule under the assumption of either stochastic independence or perfect positive stochastic interdependence does not mean that the component times themselves, $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$, can be reconstructed uniquely (the opposite is de facto true). Nor does it mean that the decomposition tests allow one to empirically identify, rather than to assume, the stochastic relationship between component times, even for as limited a choice as that between independence and perfect positive interdependence: two tests designed for one and the same simple operation, one with $\mathbf{A}(\alpha) \| \mathbf{B}(\beta)$, another with $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, may very well be successful simultaneously (i.e., on the same observable RTs).

2. STRUCTURE OF THE PAPER AND TECHNICALITIES

In Section 3 we discuss the original Ashby–Townsend–Roberts–Sternberg summation test as a necessary condition for an additive decomposition of RT into stochastically independent components. In Section 4 we introduce the concept of a simple operation and construct decomposition tests, analogous to the summation test, as necessary conditions for decomposability involving any of such operations (including, notably, the operations of maximum and minimum that have been previously characterized only by the pattern of failure of the original summation test, with no tests specifically designed for "parallel" architectures). In Section 5 we study the uniqueness-of-identification problem for simple operations under the assumption of stochastic independence. We show that a simple operation is identified uniquely among all "algebraically distinct" competing simple operations. Note that the uniqueness of identification does not imply that the identified architecture exists: it only rules out other, competing architectures. In Section 6 we point out that a wide variety of stochastic relationships between component times are consistent with their being selectively influenced by different factors, and we introduce the second (after stochastic independence) focal relationship: perfect positive stochastic interdependence. We show in the same section that decomposition tests involving simple operations can also be constructed for perfectly positively stochastically interdependent component times, that these tests are necessary conditions for the existence of the corresponding architectures, and that the decomposition rules are identified by these tests uniquely (or "essentially" uniquely). In Section 7 we turn to the problem of the existence of an architecture if the corresponding decomposition test is successful. We show that the existence is guaranteed under the assumption of perfect positive stochastic interdependence, but that under the assumption of stochastic independence it is guaranteed only in the case of "min-parallel" and "max-parallel" architectures (and even then, only if we allow for "incomplete" random variables as component times). For all other simple operations (including the traditional "serial" one, with stochastically independent components) a decomposition test may very well be successful while no corresponding decomposition of RTs is possible. We also show that the two forms of stochastic relationship considered in this paper, stochastic independence and perfect positive stochastic interdependence, are not uniquely recovered by the decomposition tests: generally, they have to be assumed rather than found out. In Section 8 we show how the concept of a RT decomposition into selectively influenced component times can be interpreted to make all simple compositions a priori equally "realistic." The concluding section briefly summarizes the main results established in this paper.

The proofs of all rigorously established results are presented in the Appendices, which, also contain necessary auxiliary lemmas and rigorous definitions of key concepts. In the main text these results are summarized in the form of numbered Statements. Generally, we allow the discussion in the main text to be mathematically looser than in the corresponding Appendices, especially in such (important) technical details as domains of functions and spectra of random variables. Although all random variables representing RTs can be safely assumed to be positive and have continuous density functions, no such restrictions are imposed in this paper (not even continuity of distribution functions), as all our results turn out to hold for arbitrary random variables.

Random variables are always denoted by boldface letters. Besides the common abbreviations "RT" and "RTs", we use the following abbreviations for frequently used terms: "r.v.('s)" for "random variable(s)"; "s.-independence" and "s.-independent" for "stochastic independence" and "stochastically independent"; "p.p.s.-interdependence" and "p.p.s.-interdependent" for "perfect positive stochastic interdependence" and "perfectly positively stochastically interdependent."

3. SUMMATION TEST

We begin with the empirical test proposed by Ashby & Townsend (1980) and termed the "summation test" by Roberts & Sternberg (1992) for additive decompositions of RT into s.-independent selectively influenced component times:

$$\mathbf{T}(\alpha, \beta) \overset{d}{=} \mathbf{A}(\alpha) + \mathbf{B}(\beta), \qquad \mathbf{A}(\alpha) \perp \mathbf{B}(\beta). \tag{1}$$

Confining one's attention to a 2×2 crossed factorial design, where factors α and β attain two values each, $(\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$, and denoting

$$\mathbf{T}_{ij} = \mathbf{T}(\alpha_i, \beta_j), \quad \mathbf{A}_i = \mathbf{A}(\alpha_i), \quad \mathbf{B}_j = \mathbf{B}(\beta_j)$$
$$(i = 1, 2, j = 1, 2),$$

one can rewrite (1) as a system of four distributional equations:

$$\begin{aligned} \mathbf{T}_{11} &\overset{d}{=} \mathbf{A}_1 + \mathbf{B}_1 \quad (\mathbf{A}_1 \perp \mathbf{B}_1) \\ \mathbf{T}_{12} &\overset{d}{=} \mathbf{A}_1 + \mathbf{B}_2 \quad (\mathbf{A}_1 \perp \mathbf{B}_2) \\ \mathbf{T}_{21} &\overset{d}{=} \mathbf{A}_2 + \mathbf{B}_1 \quad (\mathbf{A}_2 \perp \mathbf{B}_1) \\ \mathbf{T}_{22} &\overset{d}{=} \mathbf{A}_2 + \mathbf{B}_2 \quad (\mathbf{A}_2 \perp \mathbf{B}_2). \end{aligned} \tag{2}$$

The reason for using the equidistribution symbol $\overset{d}{=}$ in (1) and (2), rather than the numerical equality $=$, is that the component times occurring in different equations (i.e., corresponding to different treatments) are *unrelated* r.v.'s: only their individual distributions, and not their joint distributions, are defined. In particular, the component times denoted by one and the same symbol (say, $\mathbf{A}_1$) in different equations are unrelated r.v.'s with identical distributions. If, for instance, the decompositions of $\mathbf{T}_{11}$ and $\mathbf{T}_{12}$ were presented as

$$\mathbf{T}_{11} = \mathbf{A}_1 + \mathbf{B}_1 \quad (\mathbf{A}_1 \perp \mathbf{B}_1)$$
$$\mathbf{T}_{12} = \mathbf{A}_1 + \mathbf{B}_2 \quad (\mathbf{A}_1 \perp \mathbf{B}_2)$$

this would have wrongly implied a definite stochastic relationship between $\mathbf{T}_{11}$ and $\mathbf{T}_{12}$, by the virtue of their dependence on a common r.v., $\mathbf{A}_1$. One should be cautious not to transfer algebraic properties of numerical equations (such as the equivalence of $a + b = c$ and $a = c - b$) to distributional equations.

To construct an empirical test for decomposition (2), one has to subject the observable RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ to numerical transformations resulting in new r.v.'s (such as $\mathbf{T}_{12} + \mathbf{T}_{21}$). The distribution of such a r.v., however, is not defined unless one specifies the joint distribution of the argument r.v.'s (in this example, $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$). It must be clear from our comments on the meaning of $\overset{d}{=}$ that stochastic relationships among $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ are not constrained in any way by their hypothetical decompositions: the joint distribution of, say, $(\mathbf{T}_{12}, \mathbf{T}_{21})$ can be chosen arbitrarily, insofar as its marginals coincide with the individual distributions of $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$. In constructing the Ashby–Townsend–Roberts–Sternberg test one chooses to treat $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$ as pairs of s.-independent r.v.'s, making thereby such r.v.'s as $\mathbf{T}_{11} + \mathbf{T}_{22}$ and $\mathbf{T}_{12} + \mathbf{T}_{21}$ will defined. It is quite easy to see now that if decomposition (2) holds, then

$$\mathbf{T}_{11} + \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} + \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}). \tag{3}$$

Indeed, if one chooses to treat the component times $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$ in (2) as mutually s.-independent r.v.'s, then

$$(\mathbf{A}_1 + \mathbf{B}_1) + (\mathbf{A}_2 + \mathbf{B}_2) \overset{d}{=} \mathbf{T}_{11} + \mathbf{T}_{22} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22})$$
$$(\mathbf{A}_1 + \mathbf{B}_2) + (\mathbf{A}_2 + \mathbf{B}_1) \overset{d}{=} \mathbf{T}_{12} + \mathbf{T}_{21} \quad (\mathbf{T}_{12} \perp \mathbf{T}_{21}),$$

whereas

$$(\mathbf{A}_1 + \mathbf{B}_1) + (\mathbf{A}_2 + \mathbf{B}_2) \overset{d}{=} (\mathbf{A}_1 + \mathbf{B}_2) + (\mathbf{A}_2 + \mathbf{B}_1)$$

for any fixed joint distribution of $(\mathbf{A}_1, \mathbf{B}_1, \mathbf{A}_2, \mathbf{B}_2)$, because arithmetic addition is associative and commutative. (We present this simple derivation in detail, as it lends itself to an immediate generalization, discussed below.)

The empirical value of proposition (3) is in that it relates only observable RTs to each other, completely circumventing the hypothetical component times. As a result, (3) provides an empirical test for the decomposition assumptions represented by (2), in the sense that if (3) is empirically rejected, then RT cannot be decomposable according to (2). It does not follow from this, of course, that if (3) is empirically corroborated, then RTs are necessarily decomposable according to (2) (later we will show this to be de facto false). Nor does it follow that if (3) is empirically corroborated, then RT cannot be decomposed in some other, non-additive, way (this will be shown to be de facto true, under certain constraints).

Clearly, empirical rejection or corroboration of (3) can only be statistical, based on finite samples of RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$: the distributions of these r.v.'s are only observable, not actually observed. Strictly speaking, therefore, the truth or falsity of proposition (3) is always an inference, rather than an empirical fact, and our use of the word "empirical" in this paper should be understood as an abridged version of "providing a basis for constructing empirical procedures." Unless specifically mentioned, however, in this paper we predicate all our considerations on the assumption that the truth or falsity of propositions involving observable RTs has been inferred correctly. It should be mentioned here that such an inference does not necessarily presuppose a successful reconstruction of true RT distribution or characteristic functions (as would be the case in the convolution-based procedure suggested by Ashby & Townsend, 1980): Roberts & Sternberg's (1992) procedure (the "summation test" proper) circumvents such a reconstruction completely. We briefly describe this procedure next, not only for its own sake, but also because it elucidates the logical structure of proposition (3).

Refer to Fig. 1. Let each of the four observable RTs $\mathbf{T}_{ij}$ be represented by a finite sample $\{t_{ij}^{(1)}, t_{ij}^{(2)}, ..., t_{ij}^{(n)}\}$ $(i = 1, 2;$ $j = 1, 2)$, the values being listed (for convenience only) in an ascending order; the equality of all four sample sizes to n is also immaterial and assumed for convenience only. Then the set of the n^2 equally weighed values $\{t_{11}^{(k)} + t_{22}^{(l)}\}$, $k, l \in \{1, 2, ..., n\}$, is a sample from $\mathbf{T}_{11} + \mathbf{T}_{22}$, and the set of the n^2 equally weighed values $\{t_{12}^{(k)} + t_{21}^{(l)}\}$ is a sample from $\mathbf{T}_{12} + \mathbf{T}_{21}$. Note that the assignment of equal weights to all possible pairs $(t_{11}^{(k)}, t_{22}^{(l)})$ and all possible pairs $(t_{12}^{(k)}, t_{21}^{(l)})$ is the operational meaning of $\mathbf{T}_{11} \perp \mathbf{T}_{22}$ and $\mathbf{T}_{12} \perp \mathbf{T}_{21}$, respectively (the term "weight" is synonymous here to "probability mass"). Other stochastic relationships would have resulted in weighing different pairs differently, up to an exclusion of some pairs from consideration (i.e., assigning to them zero weights). Now, if proposition (3) is true, then the empirical distribution functions constructed from the two samples are mutually s.-independent estimates of one and the same distribution, which can be corroborated or rejected by an appropriate non-parametric procedure (note, however, that within either of the two samples, the n^2 values, viewed as r.v.'s, are stochastically interdependent, making the use of conventional procedures, such as Smirnov–Kolmogorov's, dubious).

$$T_{11} \stackrel{d}{=} A_1 + B_1 \quad (A_1 \perp B_1)$$

$$\begin{array}{c|c||cccc|}
 & & t_{11}^{1} & t_{11}^{2} & \cdots & t_{11}^{n} \\
\hline
\multirow{4}{*}{$T_{22} \stackrel{d}{=} A_2 + B_2 \ (A_2 \perp B_2)$} & t_{22}^{1} & t_{11}^{1} + t_{22}^{1} & t_{11}^{2} + t_{22}^{1} & \cdots & t_{11}^{n} + t_{22}^{1} \\
 & t_{22}^{2} & t_{11}^{1} + t_{22}^{2} & t_{11}^{2} + t_{22}^{2} & \cdots & t_{11}^{n} + t_{22}^{2} \\
 & \vdots & \vdots & T_{11} + T_{22} \ (T_{11} \perp T_{22}) & & \vdots \\
 & t_{22}^{n} & t_{11}^{1} + t_{22}^{n} & t_{11}^{2} + t_{22}^{n} & \cdots & t_{11}^{n} + t_{22}^{n} \\
\hline
\end{array}$$

$$T_{12} \stackrel{d}{=} A_1 + B_2 \quad (A_1 \perp B_2)$$

$$\begin{array}{c|c||cccc|}
 & & t_{12}^{1} & t_{12}^{2} & \cdots & t_{12}^{n} \\
\hline
\multirow{4}{*}{$T_{21} \stackrel{d}{=} A_2 + B_1 \ (A_2 \perp B_1)$} & t_{21}^{1} & t_{12}^{1} + t_{21}^{1} & t_{12}^{2} + t_{21}^{1} & \cdots & t_{12}^{n} + t_{21}^{1} \\
 & t_{21}^{2} & t_{12}^{1} + t_{21}^{2} & t_{12}^{2} + t_{21}^{2} & \cdots & t_{12}^{n} + t_{21}^{2} \\
 & \vdots & \vdots & T_{12} + T_{21} \ (T_{12} \perp T_{21}) & & \vdots \\
 & t_{21}^{n} & t_{12}^{1} + t_{21}^{n} & t_{12}^{2} + t_{21}^{n} & \cdots & t_{12}^{n} + t_{21}^{n} \\
\hline
\end{array}$$

FIGURE 1

4. DECOMPOSITION TESTS INVOLVING SIMPLE OPERATIONS

The following generalization suggests itself immediately.

STATEMENT 1. *Let* $\blacklozenge$ *be any associative and commutative operation, and let RTs* $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ *in a* 2×2 *crossed factorial design be decomposable as*

$$\begin{aligned}
\mathbf{T}_{11} &\stackrel{d}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_1 \quad (\mathbf{A}_1 \perp \mathbf{B}_1) \\
\mathbf{T}_{12} &\stackrel{d}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_2 \quad (\mathbf{A}_1 \perp \mathbf{B}_2) \\
\mathbf{T}_{21} &\stackrel{d}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_1 \quad (\mathbf{A}_2 \perp \mathbf{B}_1) \\
\mathbf{T}_{22} &\stackrel{d}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_2 \quad (\mathbf{A}_2 \perp \mathbf{B}_2).
\end{aligned} \tag{4}$$

Then

$$\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} \blacklozenge \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}). \tag{5}$$

In other words, proposition (5) *is a necessary condition for the existence of a decomposition described by* (4).

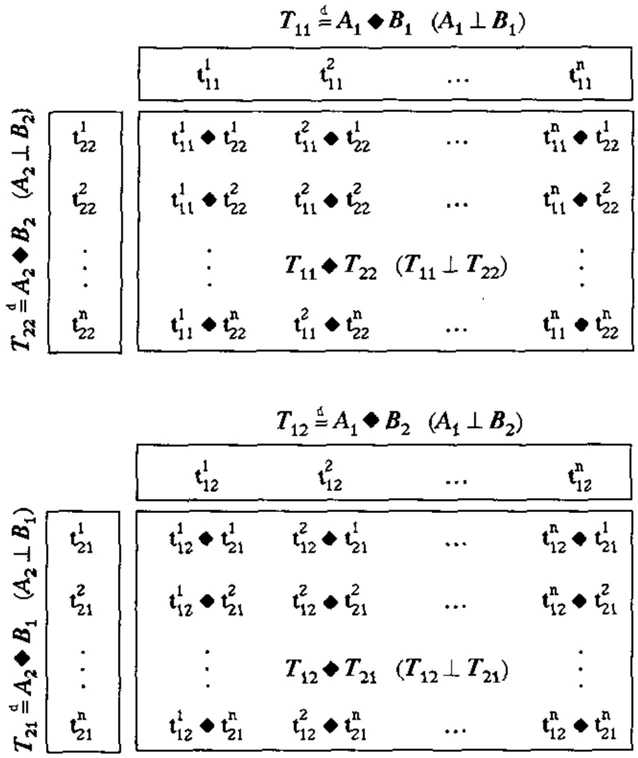

FIGURE 2

No formal proof of this statement is necessary here, because it has already been given: in deriving (3) from (2) the only property of arithmetic addition used has been its associativity and commutativity. Proposition (5) will be referred to as the *decomposition test* for the architecture described by (4). The operational meaning of Statement 1 is illustrated in Fig. 2. As in the original summation test, we form all possible pairs $(t_{11}^{(k)}, t_{22}^{(l)})$ from samples representing $\mathbf{T}_{11}$ and $\mathbf{T}_{22}$ and all possible pairs $(t_{12}^{(k)}, t_{21}^{(l)})$ from samples representing $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$, and assign to all these pairs equal probability masses. Instead of arithmetically summing the paired elements, however, we subject them to the operation that is assumed to create observable RTs from their hypothetical component times, $\blacklozenge$. Finally we compare the empirical distributions of $(t_{11}^{(k)} \blacklozenge t_{22}^{(l)})$-values and $(t_{12}^{(k)} \blacklozenge t_{21}^{(l)})$-values to corroborate or falsify the hypothesis that they are sampled from one and the same population.

Since the operations minimum and maximum are also both associative and commutative,

$$
\begin{aligned}
\min\{a, b\} &= \min\{b, a\},\\
\max\{a, b\} &= \max\{b, a\},\\
\min\{a, \min\{b, c\}\} &= \min\{\min\{a, b\}, c\},\\
\max\{a, \max\{b, c\}\} &= \max\{\max\{a, b\}, c\},
\end{aligned}
\tag{6}
$$

we have, as a corollary, a straightforward transfer of the summation test from "serial" to "parallel" architectures:

$$
\left.\begin{aligned}
\mathbf{T}_{11} &\overset{d}{=} \max\{\mathbf{A}_1, \mathbf{B}_1\} \quad (\mathbf{A}_1 \perp \mathbf{B}_1)\\
\mathbf{T}_{12} &\overset{d}{=} \max\{\mathbf{A}_1, \mathbf{B}_2\} \quad (\mathbf{A}_1 \perp \mathbf{B}_2)\\
\mathbf{T}_{21} &\overset{d}{=} \max\{\mathbf{A}_2, \mathbf{B}_1\} \quad (\mathbf{A}_2 \perp \mathbf{B}_1)\\
\mathbf{T}_{22} &\overset{d}{=} \max\{\mathbf{A}_2, \mathbf{B}_2\} \quad (\mathbf{A}_2 \perp \mathbf{B}_2)
\end{aligned}\right\}
\Rightarrow \max\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \overset{d}{=} \max\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22},\ \mathbf{T}_{12} \perp \mathbf{T}_{21}),
\tag{7}
$$

$$
\left.\begin{aligned}
\mathbf{T}_{11} &\overset{d}{=} \min\{\mathbf{A}_1, \mathbf{B}_1\} \quad (\mathbf{A}_1 \perp \mathbf{B}_1)\\
\mathbf{T}_{12} &\overset{d}{=} \min\{\mathbf{A}_1, \mathbf{B}_2\} \quad (\mathbf{A}_1 \perp \mathbf{B}_2)\\
\mathbf{T}_{21} &\overset{d}{=} \min\{\mathbf{A}_2, \mathbf{B}_1\} \quad (\mathbf{A}_2 \perp \mathbf{B}_1)\\
\mathbf{T}_{22} &\overset{d}{=} \min\{\mathbf{A}_2, \mathbf{B}_2\} \quad (\mathbf{A}_2 \perp \mathbf{B}_2)
\end{aligned}\right\}
\Rightarrow \min\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \overset{d}{=} \min\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}).
\tag{8}
$$

In the modified Roberts–Sternberg procedure for these operations, one takes the larger of the two values in every pair in the case of maximum and the smaller of the two in the case of minimum. It is remarkable that although the summation test was proposed by Ashby & Townsend as early as in 1980, the precise analogues of this test for "parallel" architectures have not been previously noted. Instead, these architectures have been characterized by the pattern of failure of the original summation test (Nozawa, 1992; Schweickert, 1978; Townsend & Ashby, 1983; Townsend & Nozawa, 1988; for surveys see Luce, 1986; Massaro & Cowan, 1993; Schweickert, 1992; Townsend, 1990).

At this point one might pose an important question: Are different architectures, say, "parallel" and "serial," objectively different (i.e., empirically distinguishable), or are they merely different languages for describing the dependence of observable distributions on external factors, languages one is free to arbitrarily choose among? It has been shown by Dzhafarov (1993) that any family of RT distributions can be modeled in terms of a certain number of processes developing in time and the corresponding number of criteria (critical levels), such that a response is observed when one of the processes reaches its criterion first. The criteria can be thought of as r.v.'s, and their joint distribution can be chosen arbitrarily. In particular, the criteria can always be chosen to be mutually s.-independent, in which case the observable RT is distributed as the minimum of several s.-independent r.v.'s, a result independently established by Marley & Colonius (1992) and Townsend (1976). Superficially, this corresponds to the "min-parallel" architec-

ture described by (8), *seemingly suggesting that all other* architectures should be reducible to this one. Such a conclusion is not true, however, because it overlooks the key assumption that RT components are selectively influenced. The assumption of selectivity has not been made in Dzhafarov (1993), where different processes correspond to different responses, not factors. The question, therefore, should be reformulated as follows: Is it always true, or can it ever happen, that one of the tests holds for some $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ (say, the summation test described by proposition 5), and another test also holds for the same $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ (say, the max-test or min-test in propositions 7 and 8)? This question (the answer to which is that this cannot happen, under some constraints) is a special version of the uniqueness-of-identification problem, discussed later.

Another question arises in relation to the fact that from a mathematical point of view there seems to be no reason for confining one's attention to the three architectures just considered. Are these architectures more "realistic" than those involving other decomposition rules, say, multiplication? A multiplicative architecture was, in fact, proposed, albeit in a different context, by Roberts (1987). According to his model, actions such as bar presses are controlled by pulses generated at a source and transmitted through regulators to the response system. For any given stimulus each regulator transmits a fixed fraction of the pulses it receives to the next regulator, but this fraction may vary from one experimental condition to another. Suppose now that a factor α (a characteristic of the stimulus) selectively influences a regulator A, while another factor, β, selectively influences a regulator B: the combined effect of the two factors on the rate of responding then will be the product of their individual effects. It is easy now to recast this model in terms of randomly varying component times and to obtain a multiplicative architecture, as well as the corresponding empirical test,

$$\left.\begin{array}{ll} \mathbf{T}_{11} \overset{\mathrm{d}}{=} k\mathbf{A}_1\mathbf{B}_1 & (\mathbf{A}_1 \perp \mathbf{B}_1) \\ \mathbf{T}_{12} \overset{\mathrm{d}}{=} k\mathbf{A}_1\mathbf{B}_2 & (\mathbf{A}_1 \perp \mathbf{B}_2) \\ \mathbf{T}_{21} \overset{\mathrm{d}}{=} k\mathbf{A}_2\mathbf{B}_1 & (\mathbf{A}_2 \perp \mathbf{B}_1) \\ \mathbf{T}_{22} \overset{\mathrm{d}}{=} k\mathbf{A}_2\mathbf{B}_2 & (\mathbf{A}_2 \perp \mathbf{B}_2) \end{array}\right\} \Rightarrow \begin{array}{l} \mathbf{T}_{11}\mathbf{T}_{22} \overset{\mathrm{d}}{=} \mathbf{T}_{12}\mathbf{T}_{21} \\ (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}), \end{array} \tag{9}$$

where $k > 0$ is a conversion coefficient included to match the dimensionality of time products and time.

One may still be unconvinced of the "possibility" of such an architecture, and ultimately such a question can only be resolved empirically, by reliably demonstrating (or failing to demonstrate in a long run) the existence of RTs that conform with the right-hand side of proposition (9). In the concluding section of this paper we discuss a possible interpretation, within the framework of the "McGill modelling language" (Dzhafarov, 1993), of the notion of a RT component, according to which no mathematically valid decomposition rule can be excluded from consideration a priori. The essence of this interpretation is that the component times $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$, even in the familiar "serial" and "parallel" schemes, are thought of as time-dimensioned characteristics of a single process, rather than durations of separate processes. The component $\mathbf{A}(\alpha)$ may be said to be the "RT that would be observed if the factor β did not intervene (exist)," the component $\mathbf{B}(\beta)$ may be interpreted analogously, and the factual RT turns out to be computable from these two "would-be" RTs by means of an algebraic operation. The reader who knows of compelling arguments for not considering any operations beyond arithmetic addition, minimum, and maximum may still find this paper of interest, as the applicability of our results to these three decomposition rules is by no means diminished by their applicability to other decomposition rules.

Moreover, the arithmetic addition, minimum, and maximum are in fact the prototypical operations from which all other operations considered in this paper can be obtained by a unified mathematical procedure, derived from Aczél's (1966) investigation of the associativity equation (see also Krantz, Luce, Suppes, & Tversky, 1971, pp. 99–102). In the present context, the class of all possible associative and commutative operations is too broad to lead to useful results beyond the necessity property established in Statement 1: stronger constraints must be imposed to obtain stronger results. These constraints are described in Definition A1, and their mathematical structure is elucidated in Lemma A1 (of Appendix A). Here, in order to make the motivation for these constraints clear, we choose a more constructive (and less formal) way of presentation.

The operation $a + b$ is not only associative and commutative, it is also increasing in both arguments and continuous in both arguments, and it maps positive reals (negative reals, all reals) onto positive reals (respectively, negative reals, all reals). Consider an arbitrary real-valued, strictly monotonic (increasing or decreasing), and continuous function g whose domain is some open interval of reals I. Then the expression $g^{-1}[g(a) + g(b)]$ defines a new binary operation that can be denoted by $a \oplus b$. This operation is closed on I and maps onto I, and is associative, commutative, increasing in both arguments, and continuous in both arguments. In other words, $a \oplus b$ shares all principal properties of $a + b$, except, perhaps, for its domain. It is natural to call all such operations *addition-like*. The remarkable mathematical fact is that an operation cannot have the properties just mentioned without being representable as $g^{-1}[g(a) + g(b)]$. The function g creating an addition-like operation $\oplus$ from arithmetic addition is denoted by $g_{\oplus}$; two different functions $g_{\oplus}$ create two different operations, except when they are scaling transforms of each other. As an example, the multiplication $\times$ is an addition-like operation, with I being the half-line of positive reals and $g_{\times}$ being logarithm:

$$a \times b \equiv \exp[\log(a) + \log(b)].$$

As a result, the multiplicative architecture described by (9) can be reduced to a "serial" architecture by log-transforming all r.v.'s involved. Another interesting addition-like operation, from the point of view of its potential applicability to experimental data, is obtained by making $g_{\oplus}$ a power function (the domain I being again the half-line of positive reals)

$$a \oplus b \equiv [a^p + b^p]^{1/p},$$

where $|p| \geqslant 1$. The RT architecture and decomposition test corresponding to this operation (under the assumption of s.-independence) are

$$\left.\begin{array}{ll} \mathbf{T}_{11} \overset{\mathrm{d}}{=} [\mathbf{A}_1^p + \mathbf{B}_1^p]^{1/p} & (\mathbf{A}_1 \perp \mathbf{B}_1) \\ \mathbf{T}_{12} \overset{\mathrm{d}}{=} [\mathbf{A}_1^p + \mathbf{B}_2^p]^{1/p} & (\mathbf{A}_1 \perp \mathbf{B}_2) \\ \mathbf{T}_{21} \overset{\mathrm{d}}{=} [\mathbf{A}_2^p + \mathbf{B}_1^p]^{1/p} & (\mathbf{A}_2 \perp \mathbf{B}_1) \\ \mathbf{T}_{22} \overset{\mathrm{d}}{=} [\mathbf{A}_2^p + \mathbf{B}_2^p]^{1/p} & (\mathbf{A}_2 \perp \mathbf{B}_2) \end{array}\right\}$$
$$\Rightarrow \mathbf{T}_{11}^p + \mathbf{T}_{22}^p \overset{\mathrm{d}}{=} \mathbf{T}_{12}^p + \mathbf{T}_{21}^p$$
$$(\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}). \tag{10}$$

The empirical value of this scheme for positive values of p stems from the fact that as p increases, the corresponding architecture gradually changes from being purely additive ("serial," $p = 1$) to being purely supremal ("max-parallel," $p = \infty$), exhibiting intermediate properties in between. This is a consequence of the well-known property of the Minkowski norm:

$$[a^p + b^p]^{1/p} \to \max\{a, b\} \qquad \text{as} \quad p \to \infty.$$

As a result, the "Minkowski-norm" operations ($p > 1$) may provide a reasonable approximation, if not an adequate description, in the situations when the factors α and β appear to have only partially additive effects. For negative values of p (i.e., for $p \leqslant -1$), we have

$$[a^p + b^p]^{1/p} \to \min\{a, b\} \qquad p \to -\infty,$$

from which it follows that an architecture corresponding to a negative value of p might be utilized when the combined effect of the factors α and β appears to be between the "min-parallel" ($p = -\infty$) and "harmonic" ($p = -1$) combinations.

It must be clear now that a broad class of potentially useful decomposition rules can be obtained from arithmetic addition as a prototype, by means of a unified mathematical algorithm,

$$a \oplus b \equiv g_{\oplus}^{-1}[g_{\oplus}(a) + g_{\oplus}(b)],$$

and that the operations $\min\{a, b\}$ and $\max\{a, b\}$ can be appended to this class of the addition-like operations as its limit cases. Since they are not the only limit cases, however, it would be more elucidating, as well as more veridical to the factual motivation, to consider $\min\{a, b\}$ and $\max\{a, b\}$ as two independent prototypical cases, on a par with arithmetic addition. It is easy to see then that, unlike the addition, maximum and minimum, when one applies to them the same mathematical algorithm, do not generate "maximum-like" and "minimum-like" operations beyond themselves. Indeed, for any strictly monotonic continuous function g,

$$g^{-1}[\max\{g(a), g(b)\}] = \max\{a, b\},$$
$$g^{-1}[\min\{g(a), g(b)\}] = \min\{a, b\},$$

if g is increasing, and

$$g^{-1}[\max\{g(a), g(b)\}] = \min\{a, b\},$$
$$g^{-1}[\min\{g(a), g(b)\}] = \max\{a, b\},$$

if g is decreasing. As a result, having applied this algorithm to the three prototypical operations, +, min, and max, we end up with the class of "simple operations" (for the lack of a better term), consisting of all addition-like operations $a \oplus b$, and the operations $\min\{a, b\}$ and $\max\{a, b\}$. Within the class of simple operations, Statement 1 becomes equivalent to the conjunction of three propositions: (7) for maximum, (8) for minimum, and the proposition below for all addition-like operations $\oplus$:

$$\left.\begin{array}{ll} g_{\oplus}(\mathbf{T}_{11}) \overset{\mathrm{d}}{=} g_{\oplus}(\mathbf{A}_1) + g_{\oplus}(\mathbf{B}_1) & (\mathbf{A}_1 \perp \mathbf{B}_1) \\ g_{\oplus}(\mathbf{T}_{12}) \overset{\mathrm{d}}{=} g_{\oplus}(\mathbf{A}_1) + g_{\oplus}(\mathbf{B}_2) & (\mathbf{A}_1 \perp \mathbf{B}_2) \\ g_{\oplus}(\mathbf{T}_{21}) \overset{\mathrm{d}}{=} g_{\oplus}(\mathbf{A}_2) + g_{\oplus}(\mathbf{B}_1) & (\mathbf{A}_2 \perp \mathbf{B}_1) \\ g_{\oplus}(\mathbf{T}_{22}) \overset{\mathrm{d}}{=} g_{\oplus}(\mathbf{A}_2) + g_{\oplus}(\mathbf{B}_2) & (\mathbf{A}_2 \perp \mathbf{B}_2) \end{array}\right\}$$
$$\Rightarrow g_{\oplus}(\mathbf{T}_{11}) + g_{\oplus}(\mathbf{T}_{22}) \overset{\mathrm{d}}{=} g_{\oplus}(\mathbf{T}_{12}) + g_{\oplus}(\mathbf{T}_{21})$$
$$(\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}) \tag{11}$$

5. UNIQUENESS OF SIMPLE OPERATIONS UNDER STOCHASTIC INDEPENDENCE

We can now rigorously formulate and answer the question posed in the previous section: Under the assumption of s.-independence, is it possible that two decomposition tests involving two different simple operations are successful for one and the same quadruple of observable RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$? The answer turns out to be affirmative if no additional constraints are imposed, but it changes to negative if one excludes from consideration one trivial case and imposes a rather natural condition on the algebraic relationship between two competing simple operations. The trivial case just mentioned is the one when

$$\mathbf{T}_{11} \stackrel{d}{=} \mathbf{T}_{12}, \quad \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{21} \quad \text{or} \quad \mathbf{T}_{11} \stackrel{d}{=} \mathbf{T}_{21}, \quad \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12}.$$

Obviously, then

$$\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} \blacklozenge \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21})$$

for any commutative operation $\blacklozenge$, and in particular, for any simple operation. It is also obvious, however, that this case should be excluded from consideration because this is the case when at least one of the factors α and β is *ineffective*. Indeed, if $\mathbf{T}_{11} \stackrel{d}{=} \mathbf{T}_{12}$, $\mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{21}$, then changes in the second factor have no effect at any level of the first factor, and we have the opposite situation if $\mathbf{T}_{11} \stackrel{d}{=} \mathbf{T}_{21}$, $\mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12}$. To be of any interest, the analysis should be restricted to the cases when both factors are effective, that is, when $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$, ignoring the order within the pairs, are different pairs of r.v.'s. We will refer to such quadruples of RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ as having *effective index factors*.

The following example shows that even if the index factors are effective, one can still think of pairs of decomposition tests that may be successful simultaneously. Let one of the tests be the original summation test, comparing the distribution of $\mathbf{T}_{11} + \mathbf{T}_{22}$ $(\mathbf{T}_{11} \perp \mathbf{T}_{22})$ with that of $\mathbf{T}_{12} + \mathbf{T}_{21}$ $(\mathbf{T}_{12} \perp \mathbf{T}_{21})$. Let the competing decomposition test be designed for the addition-like operation $\oplus$ defined (by the algorithm discussed in the previous section) by the function $g_{\oplus}(x) \equiv 2x + \sin x$ (this function is strictly increasing and continuous). In other words, this test compares the distribution of $[2\mathbf{T}_{11} + \sin \mathbf{T}_{11}] + [2\mathbf{T}_{22} + \sin \mathbf{T}_{22}]$ $(\mathbf{T}_{11} \perp \mathbf{T}_{22})$ with that of $[2\mathbf{T}_{12} + \sin \mathbf{T}_{12}] + [2\mathbf{T}_{21} + \sin \mathbf{T}_{21}]$ $(\mathbf{T}_{12} \perp \mathbf{T}_{21})$. Choose now an arbitrary r.v. X, such that lext $\mathbf{X} \geqslant 2\pi$ (lext $\mathbf{X}$ is the infimum of the spectrum of $\mathbf{X}$) and put

$$\mathbf{T}_{11} \stackrel{d}{=} \mathbf{X} - 2\pi; \quad \mathbf{T}_{22} \stackrel{d}{=} \mathbf{X} + 2\pi; \quad \mathbf{T}_{21} \stackrel{d}{=} \mathbf{T}_{12} \stackrel{d}{=} \mathbf{X}.$$

Obviously, $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$ are different (unordered) pairs of r.v.'s, and it is easy to verify that for these $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ both our decomposition tests hold:

$$\mathbf{T}_{11} + \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} + \mathbf{T}_{21}$$
$$[2\mathbf{T}_{11} + \sin \mathbf{T}_{11}] + [2\mathbf{T}_{22} + \sin \mathbf{T}_{22}]$$
$$\stackrel{d}{=} [2\mathbf{T}_{12} + \sin \mathbf{T}_{12}] + [2\mathbf{T}_{21} + \sin \mathbf{T}_{21}]$$
$$(\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}).$$

This disappointing result has a clear algebraic cause: the two simple operations in question, $+$ and $\oplus$, are such that the system of numerical equations

$$x + y = a$$
$$x \oplus y = b$$

can have more than one (unordered) pair of reals (x, y) satisfying this system. This is not the case in most other cases, involving more interesting and less artificially constructed simple operations. For example, with appropriately chosen domains and coefficients, each of the systems

$$\begin{cases} x + y = a \\ \max(x, y) = b, \end{cases}$$

$$\begin{cases} x + y = a \\ \min(x, y) = b, \end{cases}$$

$$\begin{cases} x + y = a \\ xy = b, \end{cases}$$

$$\begin{cases} [x^p + y^p]^{1/p} = a \\ [x^q + y^q]^{1/q} = b, \end{cases}$$

has at most one (unordered) solution (x, y), for any (a, b). Let us call two operations exhibiting this property *algebraically distinct decomposition rules*, the motivation for the term being as follows. Consider two such operations, for example, arithmetic addition and arithmetic multiplication on positive reals, and let (c_{11}, c_{22}) and (c_{12}, c_{21}) be different (unordered) pairs of positive reals. Then if $c_{11} + c_{22} = c_{12} + c_{21}$, it is impossible that $c_{11} c_{22} = c_{12} c_{21}$, and vice versa.

Necessary and sufficient conditions for simple operations to be algebraically distinct decomposition rules are established in Lemmas A2 and A3. We summarize and illustrate them here informally, as they play a key role throughout this paper. Fig. 3 illustrates the situation when two algebraically distinct decomposition rules are both addition-like (denoted $\oplus$ and $\otimes$). According to Lemma A2, this happens if, and only if, the defining functions $g_{\oplus}$ and $g_{\otimes}$ are strictly convex–concave with respect to each other (see Definition A2). The geometric meaning of this property in Fig. 3 is that if one chooses one of these operations, say $\oplus$, and $g_{\oplus}$-transforms the axes, so that the contour $x \oplus y = a$ is represented by a straight line (for any a), then the contour $x \otimes y = b$ (for any b) is either convex (Fig. 3a) or concave (Fig. 3b) with respect to this straight line. As a result the two contours cannot intersect at more than two points, and because of their symmetry with respect to the main bisector, these two intersection points are identical when viewed as unordered pairs. Figure 4 illustrates the facts established in Lemma A3: any addition-like operation $\oplus$ and maximum are algebraically distinct decomposition rules (Fig. 4a); any

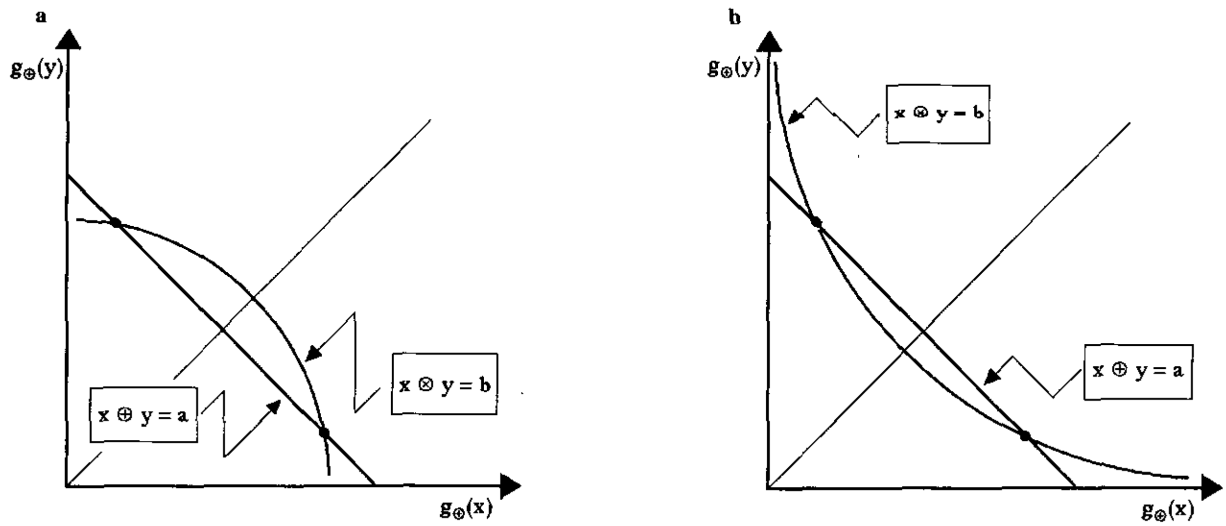

FIGURE 3

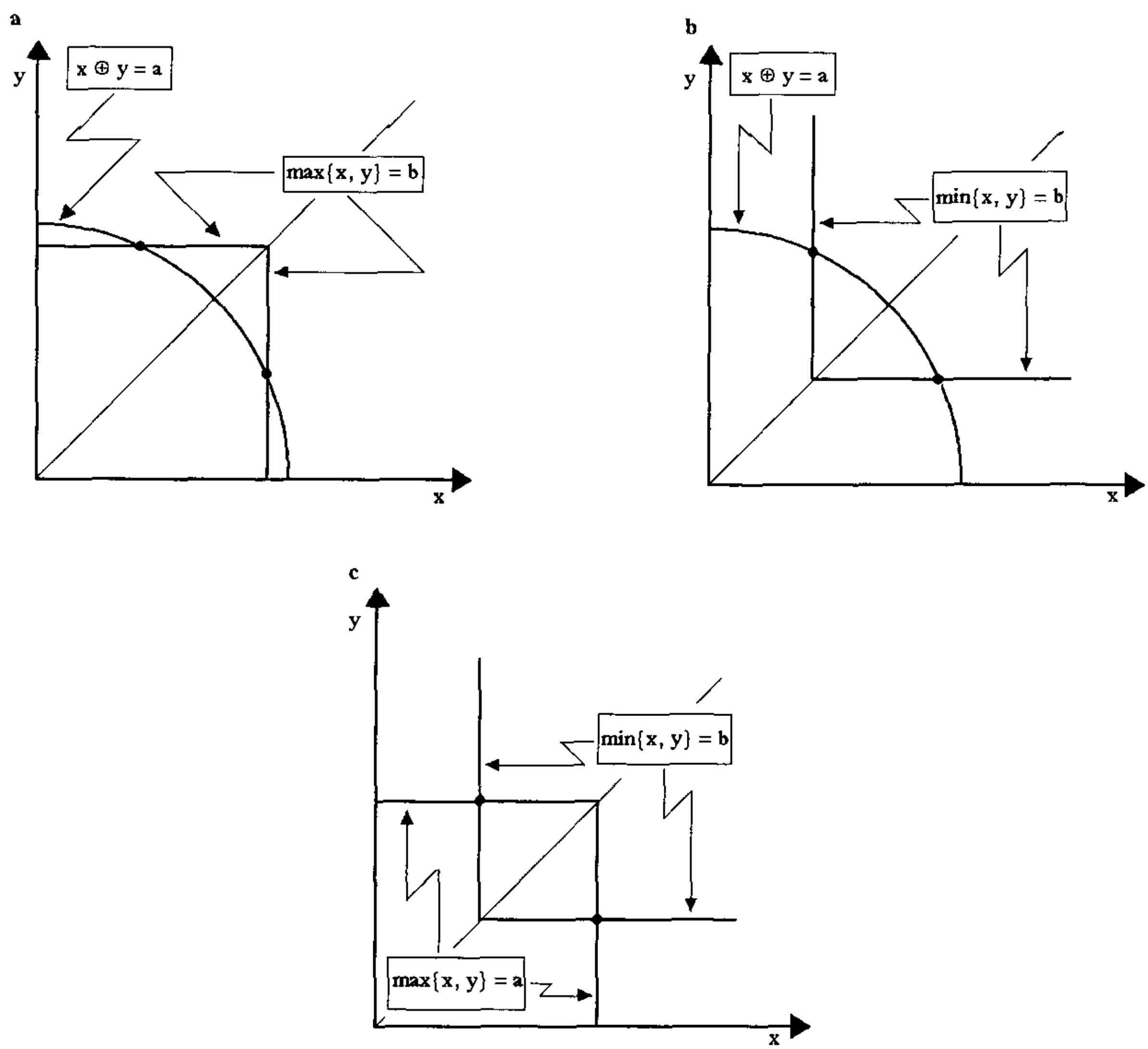

FIGURE 4

addition-like operation $\oplus$ and minimum are algebraically distinct decomposition rules (Fig. 4b); minimum and maximum are algebraically distinct decomposition rules (Fig. 4c).

The question posed at the beginning of this section can now be reformulated as follows: Under the s.-independence assumption, can two decomposition tests involving two algebraically distinct decomposition rules hold simultaneously for one and the same quadruple of observable RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$, provided that both index factors are effective, that is, $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$, are different as unordered pairs? Put in a shorter form, does a successful decomposition test identify the decomposition rule uniquely among algebraically distinct decomposition rules? The reader may be tempted to say that the answer is trivial, for the following reason: using again arithmetic addition and multiplication as examples, since the system of numerical equations

$$x+y=a$$
$$xy=b$$

has no more than one (unordered) solution, it must also be true (one might think) that the system of distributional equations

$$\begin{aligned} \mathbf{X}+\mathbf{Y} &\overset{d}{=} \mathbf{A} \\ \mathbf{XY} &\overset{d}{=} \mathbf{B} \end{aligned} \qquad (\mathbf{X} \perp \mathbf{Y})$$

has no more than one (unordered) pair of r.v.'s satisfying it; ergo the two decomposition tests

$$\begin{aligned} \mathbf{T}_{11}+\mathbf{T}_{22} &\overset{d}{=} \mathbf{T}_{12}+\mathbf{T}_{21} \\ \mathbf{T}_{11}\mathbf{T}_{22} &\overset{d}{=} \mathbf{T}_{12}\mathbf{T}_{21} \end{aligned} \qquad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21})$$

cannot hold simultaneously, unless $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$ are identical. Although the resulting statement, as we will see, happens to be correct, the reasoning by which it was obtained is expressly false. We will see, for example, that this reasoning would have led to a wrong conclusion if the two operations were minimum and maximum, rather than addition and multiplication. The best example known to us, however, follows from Rényi's (1950) theory of linear equations in r.v.'s: the numerical system

$$x+y=a$$
$$x-y=b$$

always has a unique solution, but the distributional system

$$\begin{aligned} \mathbf{X}+\mathbf{Y} &\overset{d}{=} \mathbf{A} \\ \mathbf{X}-\mathbf{Y} &\overset{d}{=} \mathbf{B} \end{aligned} \qquad (\mathbf{X} \perp \mathbf{Y})$$

can nevertheless be satisfied by infinitely many $(\mathbf{X}, \mathbf{Y})$-pairs (e.g., if both $\mathbf{A}$ and $\mathbf{B}$ are unit-normally distributed, then any two normal distributions whose variances add to 1 and whose means are zero would constitute a solution). The following two statements, therefore, based on Theorems B1, B2, and B3, are nontrivial.

STATEMENT 2. *Under the assumption of s.-independence, if a decomposition test involving an addition-like operation is successful for observable RTs* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *(with effective index factors) then no other decomposition test that involves an algebraically distinct addition-like decomposition rule, or maximum, or minimum can be successful on the same RTs.*

STATEMENT 3. *Under the assumption of s.-independence, if a decomposition test involving minimum or maximum is successful for observable RTs* $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ *(with effective index factors), then no decomposition test involving an addition-like decomposition rule may be successful on the same RTs.*

One can summarize these two Statements by saying that for any $\mathbf{T}_{11} \perp \mathbf{T}_{22}$, $\mathbf{T}_{12} \perp \mathbf{T}_{21}$ with effective index factors, none of the systems of distributional equations

$$\begin{cases} \mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \\ \mathbf{T}_{11} \otimes \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \otimes \mathbf{T}_{21}, \end{cases}$$
$$\begin{cases} \mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \\ \min(\mathbf{T}_{11}, \mathbf{T}_{22}) \overset{d}{=} \min(\mathbf{T}_{12}, \mathbf{T}_{21}), \end{cases} \tag{12}$$
$$\begin{cases} \mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \\ \max(\mathbf{T}_{11}, \mathbf{T}_{22}) \overset{d}{=} \max(\mathbf{T}_{12}, \mathbf{T}_{21}), \end{cases}$$

can hold, where $\oplus$ and $\otimes$ are algebraically distinct addition-like decomposition rules. Each of the two equations within a system excludes the other.

The only remaining case is that in which the competing operations are maximum and minimum. It turns out that here, in spite of the fact that these two operations are algebraically distinct, it is possible that for some $\mathbf{T}_{11} \perp \mathbf{T}_{22}$, $\mathbf{T}_{12} \perp \mathbf{T}_{21}$

$$\begin{aligned} \min(\mathbf{T}_{11}, \mathbf{T}_{22}) &\overset{d}{=} \min(\mathbf{T}_{12}, \mathbf{T}_{21}) \\ \max(\mathbf{T}_{11}, \mathbf{T}_{22}) &\overset{d}{=} \max(\mathbf{T}_{12}, \mathbf{T}_{21}). \end{aligned} \tag{13}$$

even when the index factors are effective, that is, when $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$ are not identical as unordered pairs. It is easy to see, however, that for this to happen, the relationship between $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$ must be of a rather artificial nature, both readily identifiable and unlikely to be observed.

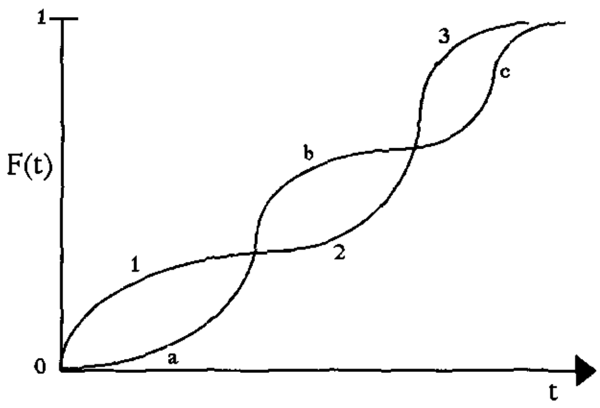

FIGURE 5

Refer to Fig. 5 that shows two repeatedly intersecting distribution functions, one following the path 1–2–3, the other the path a–b–c. Let them be the distribution functions $F_{11}(t)$ and $F_{22}(t)$ for RTs $\mathbf{T}_{11}$ and $\mathbf{T}_{22}$, respectively. Consider now the following procedure, called a *cross-over rearrangement* (Definition B2): choose any two successive crossing points, say, the end-points of the line segments 2 and b, and interchange the two segments in between, in this case 2 and b. We obtain thereby two other functions, one following the path 1–b–3, the other the path a–2–c. This step can be repeated as many times as there are pairs of successive crossing points (possibly counting as such ∞ and $-\infty$). Thus in Fig. 5 we can form eight different pairs of curves that are cross-over rearrangements of the functions $F_{11}(t)$ and $F_{22}(t)$ (counting themselves). It is easy to verify that any such a pair of curves can be viewed as a pair of distribution functions, and can be, therefore, taken to be the pair of distribution functions $F_{12}(t)$, $F_{21}(t)$ for some RTs $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$. By Theorem B4, the decomposition tests involving minimum and maximum can succeed simultaneously, on one and the same quadruple of RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$, if, and only if, the distribution functions $F_{12}(t)$, $F_{21}(t)$ are cross-over rearrangements of the distribution functions $F_{11}(t)$, $F_{22}(t)$. Intuitively, this situation seems highly unlikely, and it seems safe to compartmentalize it and exclude it from consideration. Observe that cross-over rearrangements of distribution functions $F_{12}(t)$, $F_{21}(t)$ will typically be less smooth than the functions themselves: for example, in Fig. 5 any cross-over rearrangement of the curves 1–2–3 and a–b–c (except for the exchange of the entire curves) will have at least one "cusp," indicating a discontinuity in the density function. One could, therefore, exclude such rearrangements by imposing a formal requirement that $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ must all have continuous density functions. Now we can formulate the final proposition of this section, based on Theorem B4.

STATEMENT 4. *Let the observable RTs* $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ *be such that (the distribution functions for)* $\mathbf{T}_{12}$, $\mathbf{T}_{21}$ *are not cross-over rearrangement of (the distribution functions for)* $\mathbf{T}_{11}$, $\mathbf{T}_{22}$. *Then, under the assumption of s.-independence, the decomposition tests involving minimum and maximum cannot be simultaneously successful on* $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$.

Observe that one does not have to mention here the effectiveness of the index factors, because the identity of $(\mathbf{T}_{11}, \mathbf{T}_{22})$ and $(\mathbf{T}_{12}, \mathbf{T}_{21})$ is a special case of the distribution functions for $\mathbf{T}_{12}$, $\mathbf{T}_{21}$ being cross-over rearrangements of those for $\mathbf{T}_{11}$, $\mathbf{T}_{22}$.

It should be emphasized that the uniqueness-of-identification results obtained here and below (Statements 2, 3, 4, and 6) tell us nothing about statistical power of competing decomposition tests. Obviously, algebraically distinct decomposition rules may very well be so similar that they can be decided between only on unrealistically large samples (think, for example, of two "Minkowski-norm" operations with exponents 12 and 13). At the same time, a computer simulation work using a wide variety of Weibull-distributed RT components (Cortese & Dzhafarov, 1995) shows that at the level of only a few thousand RTs per treatment one can achieve a virtually perfect discriminability among the three traditional decomposition rules: plus, maximum, and minimum.

6. PERFECT POSITIVE STOCHASTIC INTERDEPENDENCE

That the (hypothetical) selectively influenced component times $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ in a decomposition of an observable RT are s.-independent, $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, has been assumed by most researchers primarily because, in the absence of empirical information for or against, this is thought to be the simplest choice, both conceptually and technically. Dzhafarov (1992) has analyzed an alternative possibility, termed the "single-variate RT decomposition model," that, if one is to be guided by simplicity considerations, is at least a viable alternative to the s.-independence hypothesis. Moreover, the investigation presented by Dzhafarov (1992) leaves little doubt that this alternative is, in fact, more simple conceptually and more manageable technically. In the context of decomposition tests, as we will see, this conclusion is even more called for. For the moment we are not concerned with the issue of empirical testability of the single-variate RT decomposition model against the s.-independence model, both being treated as assumptions that predicate decomposition tests.

As mentioned in the Introduction, the theoretical framework for the variety of possible stochastic relationships between component times $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ is formed by the definition of selective influence,

$$\{\mathbf{A}(\alpha), \mathbf{B}(\beta)\} \stackrel{d}{=} \{A(\alpha, \mathbf{X}), B(\beta, \mathbf{Y})\},$$

where A and B are some functions, whereas $(\mathbf{X}, \mathbf{Y})$ is a pair of r.v.'s (termed "internal sources of variability") whose joint distribution does not depend on the factors α and β. Possible interpretations for internal sources of variability in terms of randomly preset criteria or stochastic components of a response formation process are discussed by Dzhafarov (1993); one such interpretation will also be addressed in the concluding section of this paper. For our present purposes it is sufficient and convenient to assume that both $\mathbf{X}$ and $\mathbf{Y}$ are *uniformly distributed between* 0 *and* 1, and the functions A and B are *quantile functions* for the r.v.'s $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ (see Definition C1):

$$\mathbf{A}(\alpha) \stackrel{d}{=} Q_A(\mathbf{X}, \alpha), \qquad \mathbf{B}(\beta) \stackrel{d}{=} Q_B(\mathbf{Y}, \beta).$$

In other words, for any $0 \leqslant p \leqslant 1$, $Q_A(p, \alpha)$ is simply the quantile of rank p of the r.v. $\mathbf{A}$ at the factor level α; analogously for $Q_B(\mathbf{Y}, \beta)$. This choice of $\mathbf{X}$, $\mathbf{Y}$ and A, B outlines an important subclass of possible stochastic relationships between $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$: each such a relationship is determined by a fixed *pairing scheme* of quantiles of $\mathbf{A}(\alpha)$ with the quantiles of $\mathbf{B}(\beta)$. In the case of s.-independence, $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, the pairing scheme is all-with-all, and it induces the all-with-all pairing of the observable RTs, as reflected in the generalized Roberts–Sternberg algorithm (Fig. 2).

In the single-variate RT decomposition model we assume that there is only one, common source of variability, $\mathbf{X} = \mathbf{Y}$, because of which the r.v.'s $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ are *perfectly positively stochastically (p.p.s.-)interdependent*, $\mathbf{A}(\alpha) \parallel \mathbf{B}(\beta)$. It is trivial to prove now the following analogue of Statement 1.

STATEMENT 5. *Let* $\blacklozenge$ *be a simple operation (in fact, any associative and commutative operation), and let RTs* $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ *in a* 2×2 *crossed factorial design be decomposable as*

$$\begin{aligned} \mathbf{T}_{11} &\stackrel{d}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_1 \quad (\mathbf{A}_1 \parallel \mathbf{B}_1) \\ \mathbf{T}_{12} &\stackrel{d}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_2 \quad (\mathbf{A}_1 \parallel \mathbf{B}_2) \\ \mathbf{T}_{21} &\stackrel{d}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_1 \quad (\mathbf{A}_2 \parallel \mathbf{B}_1) \\ \mathbf{T}_{22} &\stackrel{d}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_2 \quad (\mathbf{A}_2 \parallel \mathbf{B}_2) \end{aligned} \tag{14}$$

Then,

$$\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} \blacklozenge \mathbf{T}_{21} \quad (\mathbf{T}_{11} \parallel \mathbf{T}_{22}, \mathbf{T}_{12} \parallel \mathbf{T}_{21}). \tag{15}$$

In other words, proposition (15) *is a necessary condition for the existence of a decomposition described by* (14).

Proposition (15) is referred to as the decomposition test (for operation $\blacklozenge$) under the assumption of p.p.s.-interdependence. The operational meaning of (15) is illustrated in Fig. 6. Here the equality of sizes of the paired samples, as well as the ascending order in which the samples are arranged, is critical, not just a matter of convenience, as it was in Fig. 2. The value $t_{11}^{(k)}$ is an estimator of a fixed-rank quantile of $\mathbf{T}_{11}$, and it can only be paired with the estimator $t_{22}^{(k)}$ of the same-rank quantile of $\mathbf{T}_{22}$; analogously for $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$. In other words, among the n^2 pairs $(t_{11}^{(k)}, t_{22}^{(l)})$ and n^2 pairs $(t_{12}^{(k)}, t_{21}^{(l)})$, $k, l \in \{1, 2, ..., n\}$, all are assigned zero weights (i.e., ignored) except for the diagonal pairs $(t_{11}^{(k)}, t_{22}^{(k)})$, $(t_{12}^{(k)}, t_{21}^{(k)})$ that are taken with equal weights. The rest is analogous to the case of s.-independence: one computes n values $(t_{11}^{(k)} \blacklozenge t_{22}^{(k)})$ and n values $(t_{12}^{(k)} \blacklozenge t_{21}^{(k)})$, considering them samples from $\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22}$ $(\mathbf{T}_{11} \parallel \mathbf{T}_{22})$ and $\mathbf{T}_{12} \blacklozenge \mathbf{T}_{21}$ $(\mathbf{T}_{12} \parallel \mathbf{T}_{21})$, respectively. If (15) is true, then the *empirical distribution functions constructed from these two* samples are mutually s.-independent estimates of one and the same distribution, which can be corroborated or rejected by an appropriate non-parametric procedure. (Here, unlike in the case of s.-independence, the n values, viewed as r.v.'s, are stochastically independent within either of the two samples. The pairing of *sample* quantiles of the same rank, however, implies a stronger form of stochastic interdependence than p.p.s.-interdependence—the latter only requires pairing of the same-rank *population* quantiles. As a result, the use of conventional procedures, such as Smirnov–Kolmogorov's, remains dubious in this case, too.)

We pose now the analogue of the question studied in the previous section: Under the p.p.s.-interdependence

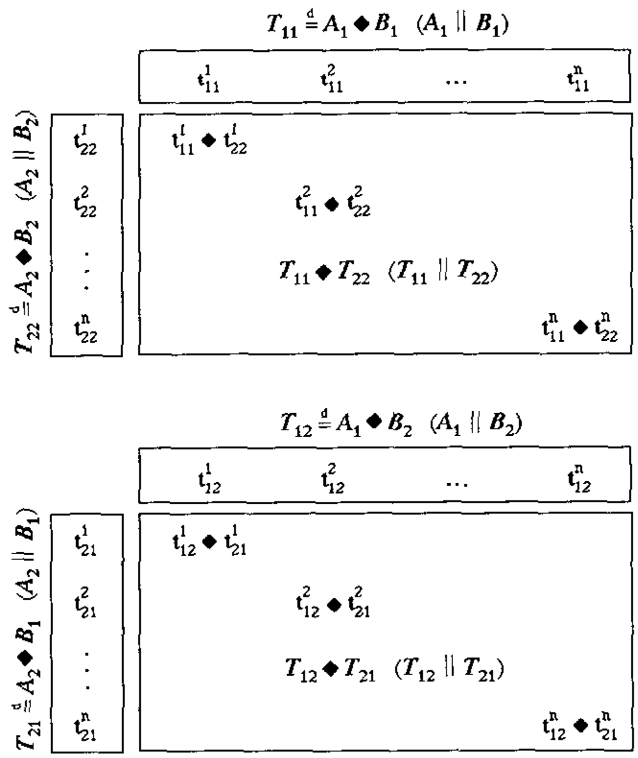

FIGURE 6

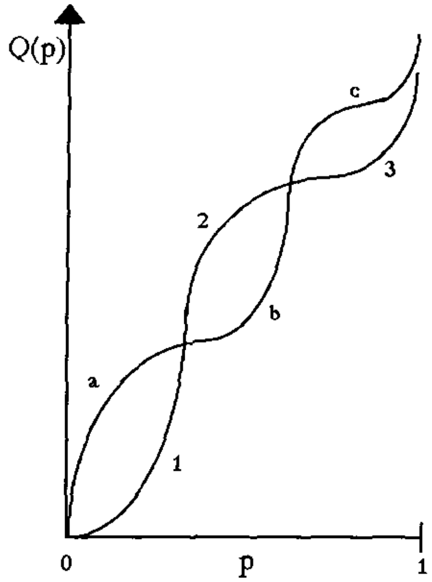

FIGURE 7

assumption, can two decomposition tests involving two algebraically distinct decomposition rules hold simultaneously for one and the same quadruple of observable RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$, provided that both index factors are effective? In other words, does a successful decomposition test identify the decomposition rule as uniquely here as it does under the assumption of s.-independence? The answer requires that we deal with cross-over rearrangements of quantile functions. Figure 7 illustrates the concept. The two curves shown, 1–2–3 and a–b–c, are the same as in Fig. 5, only viewed this time as functions of quantile rank p: let them be the quantile functions $Q_{11}(p)$ and $Q_{22}(p)$ for RTs $\mathbf{T}_{11}$ and $\mathbf{T}_{22}$, respectively. Cross-over rearrangements of these two functions (such as 1–b–c and a–2–3) are obtained as in Fig. 5, by choosing successive crossing points, and interchanging the line segments between them. The resulting curves can also be viewed as quantile functions $Q_{12}(p)$, $Q_{21}(p)$, for some RTs $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$. It is easy to see that the quantile functions $Q_{12}(p)$, $Q_{21}(p)$ for $(\mathbf{T}_{12}, \mathbf{T}_{21})$ are cross-over rearrangements of the quantile functions $Q_{11}(p)$, $Q_{22}(p)$ for $(\mathbf{T}_{11}, \mathbf{T}_{22})$ if, and only if, the same is true about their distribution functions. As a result, we may simply say, by some abuse of language, that the RTs $\mathbf{T}_{12}, \mathbf{T}_{21}$ are cross-over rearrangements of the RTs $\mathbf{T}_{11}, \mathbf{T}_{22}$, without mentioning either quantile or distribution functions. It is also easy to see that all comments made in the previous section about the justifiability of compartmentalizing and excluding from consideration the RTs quadruples $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ exhibiting this relationship apply here with no modifications. The following statement is based on Theorems C1 and C2.

STATEMENT 6. *Let the observable RTs* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *be such that* $\mathbf{T}_{12}, \mathbf{T}_{21}$ *are not cross-over rearrangements of* $\mathbf{T}_{11}, \mathbf{T}_{22}$. *Then, under the assumption of p.p.s.-interdependence, two decomposition tests involving two algebraically distinct decomposition rules cannot be simultaneously successful on* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$.

Put differently, if $\mathbf{T}_{11} \| \mathbf{T}_{22}$ and $\mathbf{T}_{12} \| \mathbf{T}_{21}$ are not cross-over rearrangements of each other (which also means that the index factors are effective), then none of the systems of distributional equations

$$\begin{cases}\mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21}\\ \mathbf{T}_{11} \otimes \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \otimes \mathbf{T}_{21},\end{cases}$$
$$\begin{cases}\mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21}\\ \min(\mathbf{T}_{11}, \mathbf{T}_{22}) \overset{d}{=} \min(\mathbf{T}_{12}, \mathbf{T}_{21}),\end{cases}$$
$$\begin{cases}\mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21}\\ \max(\mathbf{T}_{11}, \mathbf{T}_{22}) \overset{d}{=} \max\mathbf{T}_{12}, \mathbf{T}_{21},\end{cases} \tag{16}$$
$$\begin{cases}\min(\mathbf{T}_{11}, \mathbf{T}_{22}) \overset{d}{=} \min(\mathbf{T}_{12}, \mathbf{T}_{21})\\ \max(\mathbf{T}_{11}, \mathbf{T}_{22}) \overset{d}{=} \max(\mathbf{T}_{12}, \mathbf{T}_{21}),\end{cases}$$

can hold, where $\oplus$ and $\otimes$ are algebraically distinct addition-like decomposition rules. As in the case of s.-independence, each of the two equations within a system excludes the other.

The computer simulation work already mentioned (Cortese & Dzhafarov, 1995; see the last paragraph of Section 5) shows that statistical discriminability of the three traditional decomposition rules (plus, max, min) under the assumption of p.p.s.-interdependence is even better than under the assumption of s.-independence: a virtually perfect discriminability is achieved at a level of only a few hundred RTs per treatment.

Observe that the algebraic distinctiveness of the simple operations involved is as critical here as it is under the assumption of s.-independence. This can be demonstrated by using the same example (see the previous section): the operation $+$ competes against the addition-like operation $\oplus$ defined by the function $g_{\oplus}(x) \equiv 2x + \sin x$. Again, we choose an arbitrary $\mathbf{X}$ such that lext$\mathbf{X} \geqslant 2\pi$, and put

$$\mathbf{T}_{11} \overset{d}{=} \mathbf{X} - 2\pi; \quad \mathbf{T}_{22} \overset{d}{=} \mathbf{X} + 2\pi; \quad \mathbf{T}_{21} \overset{d}{=} \mathbf{T}_{12} \overset{d}{=} \mathbf{X}.$$

This time we put $\mathbf{T}_{11} \| \mathbf{T}_{22}$ and $\mathbf{T}_{12} \| \mathbf{T}_{21}$, and it is easy to verify that the decomposition tests involving $+$ and $\oplus$ hold simultaneously:

$$\mathbf{T}_{11} + \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{11} + \mathbf{T}_{22}$$
$$[2\mathbf{T}_{11} + \sin \mathbf{T}_{11}] + [2\mathbf{T}_{22} + \sin \mathbf{T}_{22}]$$
$$\overset{d}{=} [2\mathbf{T}_{12} + \sin \mathbf{T}_{12}] + [2\mathbf{T}_{21} + \sin \mathbf{T}_{21}]$$
$$(\mathbf{T}_{11} \| \mathbf{T}_{22}, \mathbf{T}_{12} \| \mathbf{T}_{21}).$$

7. SUFFICIENCY OF DECOMPOSITION TESTS FOR DECOMPOSABILITY

So far, we have seen that the two forms of stochastic relationship between hypothetical component times, s.-independence and p.p.s.-interdependence, exhibit an almost perfect parallelism. Under both assumptions the decomposition tests are necessary conditions for the existence of the corresponding architectures (i.e., the architectures involving the same simple operation and the same form of stochastic relationship; see Statements 1 and 5). Under both assumptions, excluding the peculiar case of cross-over rearrangements, only one decomposition rule can be successfully used in a decomposition test, providing it competes against algebraically distinct decomposition rules (see Statements 2, 3, 4, and 6). None of these results however, implies that a successful decomposition test guarantees the existence of the corresponding architecture: we do not know, for example, whether it follows from the success of the original summation test that the observable RTs are additively decomposable into s.-independent components. This problem, the sufficiency of decomposition tests for the existence of corresponding architectures, is the one we consider next. We will see that the mentioned parallelism between the two forms of stochastic relationship breaks down on this problem: the analysis of the p.p.s.-interdependence case turns out to be much simpler. The following statement is based on Theorems D1 and D2.

STATEMENT 7. *Under the assumption of p.p.s.-interdependence, let a decomposition test involving a simple operation* $\blacklozenge$ *(addition-like, minimum, or maximum) be successful when applied to RTs* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$, *that is,*

$$\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \blacklozenge \mathbf{T}_{21} \qquad (\mathbf{T}_{11} \parallel \mathbf{T}_{22}, \mathbf{T}_{12} \parallel \mathbf{T}_{21}).$$

Then there exist r.v.'s $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *such that*

$$\mathbf{T}_{11} \overset{d}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_1 \quad (\mathbf{A}_1 \parallel \mathbf{B}_1)$$
$$\mathbf{T}_{12} \overset{d}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_2 \quad (\mathbf{A}_1 \parallel \mathbf{B}_2)$$
$$\mathbf{T}_{21} \overset{d}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_1 \quad (\mathbf{A}_2 \parallel \mathbf{B}_1)$$
$$\mathbf{T}_{22} \overset{d}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_2 \quad (\mathbf{A}_2 \parallel \mathbf{B}_2).$$

The component times $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *are not generally determined uniquely, but their spectra can always be chosen to lie within the same domain of the operation* $\blacklozenge$ *that contains the spectra of RTs* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$.

Combining this statement with Statement 5, we can say that proposition (15) is both necessary and sufficient for the decomposability of RTs according to (14). The importance of the concluding sentence in Statement 7, concerning the choice of the interval in which the component times may attain their values, is in establishing that the additive component times can always be chosen "to make sense." For example, one would not want to additively decompose RTs that are inherently positive into components that may attain negative values. Since the interval of positive reals is a domain for arithmetic addition (because the addition is defined on this interval and maps onto it), Statement 7 tells us that the component times can always be chosen to lie within this interval. Note that even though the domain of an operation is part of this operation's definition, there can be different domains for operations denoted by one and the same symbol and computed according to one and the same algorithm. For example, addition may have one of the three domains, $(0\ \infty)$, $(-\infty, 0)$, and $(-\infty, \infty)$, whereas any open interval of reals can serve as a domain for maximum or minimum.

One could add to Statement 7 (though this issue is not pursued in Appendix D beyond the most obvious continuity characteristics of the quantile functions involved) that the component times $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ can always be chosen so that their quantile functions (and distribution functions) have the same degree of smoothness as the quantile (respectively, distribution) functions for the decomposed RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$.

As already mentioned, the situation is much more complex if one assumes that the component times are s.-independent. The decomposability is guaranteed here only if the operations involved are minimum or maximum, and even then in a somewhat less straightforward sense. We consider these operations first, preceded by necessary clarifications.

Let $\mathbf{T}$ be a continuous r.v., say, a non-negative one, as would be natural to assume in the case of a RT. Recall that, by conventional definition, the distribution function $F_T(t)$ for $\mathbf{T}$ should converge to zero as t decreases (here, $\to 0$), and it should converge to 1 as t increases ($\to \infty$). If the first of these requirements is not satisfied, that is, $F_T(t) \to p > 0$ as $t \to 0$, we will say that $\mathbf{T}$ is left-incomplete (in probabilistic terminology, has an atom at 0). If the second requirement is not satisfied, that is, $F_T(t) \to q < 1$ as $t \to \infty$, we will say that $\mathbf{T}$ is right-incomplete (has an atom at infinity; in Feller, 1968, such r.v.'s are called "defective"; in Dzhafarov, 1993, right-incomplete r.v.'s are called incomplete). In the literature on the "parallel" versus "serial" RT architectures right-incomplete r.v.'s have sometimes been treated as a physical impossibility (Luce, 1986; Townsend, 1976; Townsend & Ashby, 1983). It has been shown, by Dzhafarov (1993), however, that right-incomplete component times are physically realizable and that they arise naturally in the context of "min-parallel" architectures. In the Grice-representability language, a component time is modelled as the time it takes for a deterministic process (evoked by stimulus) to cross a randomly preset criterion, as shown in Fig. 8. It may happen then, as in Fig. 8a, that

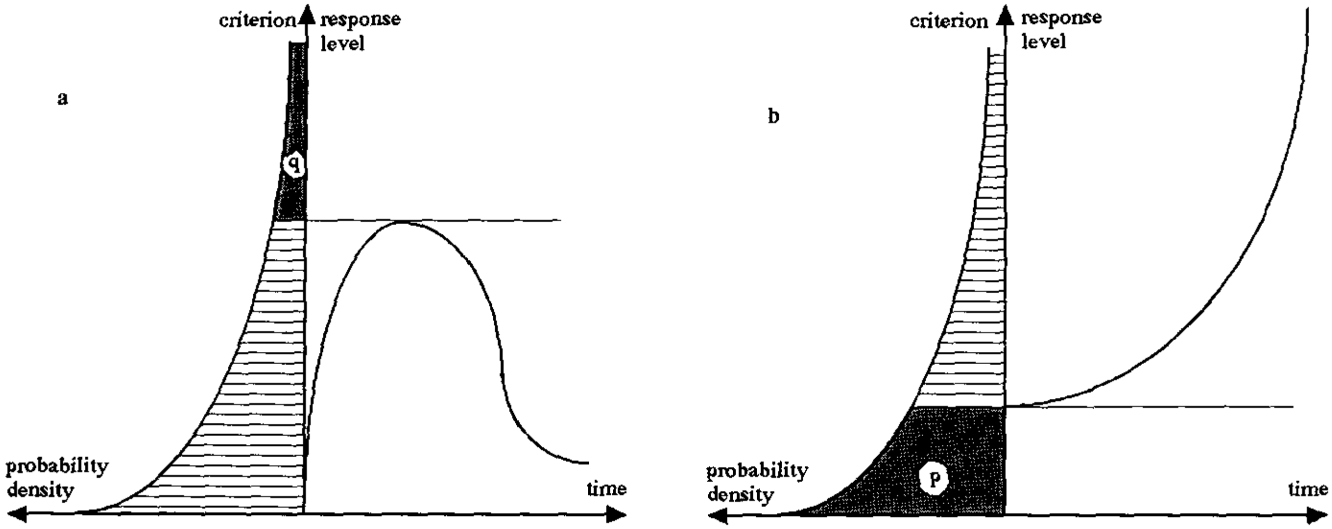

FIGURE 8

with a non-zero probability q the criterion exceeds the maximum level of the process, making the crossing time an incomplete r.v. If such a process is assumed to compete with other processes (the winner being the one that reaches its criterion first), then the observable minimum of the competing crossing times may be right-complete (so that a finite RT is observed in every trial) even if all but one of the component times are right-incomplete. The modelling of the left-incompleteness in the Grice-representability language is equally simple (Fig. 8b): the starting level for a process may be non-zero, in which case the criterion may fall below it with some non-zero probability p. If the overall observable RT is assumed to be the longest of the crossing times for several processes (the "max-parallel" architecture), then this RT will be left-complete even if all but one of the component times are left-incomplete. Both forms of incompleteness, left and right, can be equivalently modelled in the McGill-representability language (Dzhafarov, 1993). Having established this, we can formulate the following statement, derived from Theorems E1 and E2.

STATEMENT 8. *Under the assumption of s.-independence, let a decomposition test involving maximum (or minimum) as its operation be successful when applied to RTs* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$, *that is,*

$$\max\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \overset{d}{=} \max\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}),$$

or

$$\min\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \overset{d}{=} \min\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}).$$

Then there exist r.v.'s $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *such that, respectively,*

$$\mathbf{T}_{11} \overset{d}{=} \max\{\mathbf{A}_1, \mathbf{B}_1\} \quad (\mathbf{A}_1 \perp \mathbf{B}_1)$$
$$\mathbf{T}_{12} \overset{d}{=} \max\{\mathbf{A}_1, \mathbf{B}_2\} \quad (\mathbf{A}_1 \perp \mathbf{B}_2)$$
$$\mathbf{T}_{21} \overset{d}{=} \max\{\mathbf{A}_2, \mathbf{B}_1\} \quad (\mathbf{A}_2 \perp \mathbf{B}_1)$$
$$\mathbf{T}_{22} \overset{d}{=} \max\{\mathbf{A}_2, \mathbf{B}_2\} \quad (\mathbf{A}_2 \perp \mathbf{B}_2)$$

or

$$\mathbf{T}_{11} \overset{d}{=} \min\{\mathbf{A}_1, \mathbf{B}_1\} \quad (\mathbf{A}_1 \perp \mathbf{B}_1)$$
$$\mathbf{T}_{12} \overset{d}{=} \min\{\mathbf{A}_1, \mathbf{B}_2\} \quad (\mathbf{A}_1 \perp \mathbf{B}_2)$$
$$\mathbf{T}_{21} \overset{d}{=} \min\{\mathbf{A}_2, \mathbf{B}_1\} \quad (\mathbf{A}_2 \perp \mathbf{B}_1)$$
$$\mathbf{T}_{22} \overset{d}{=} \min\{\mathbf{A}_2, \mathbf{B}_2\} \quad (\mathbf{A}_2 \perp \mathbf{B}_2).$$

The component times $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *are not generally determined uniquely, but their spectra can always be chosen to lie within the same domain of the operation max (or min) that contains the spectra of RTs* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$. *The component times* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *are all right-complete in the case of maximum, but some of them (not all) may be left-incomplete; in the case of minimum, they are all left-complete, but some of them (not all) may be right-incomplete.*

Combining this statement with propositions (7) and (8), and keeping in mind the possibility of incomplete component times, we can say that, under the assumption of s.-independence, a successful decomposition test involving minimum or maximum is both necessary and sufficient for the existence of the corresponding "parallel" architecture.

The remaining decomposition tests involve addition-like operations. Mathematically, any such decomposition test can be immediately reduced to one with arithmetic addition, because

$$\mathbf{T}_{11} \oplus \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21})$$

is equivalent to

$$g_{\oplus}(\mathbf{T}_{11}) + g_{\oplus}(\mathbf{T}_{22}) \overset{d}{=} g_{\oplus}(\mathbf{T}_{12}) + g_{\oplus}(\mathbf{T}_{21})$$
$$[g_{\oplus}(\mathbf{T}_{11}) \perp g_{\oplus}(\mathbf{T}_{22}), g_{\oplus}(\mathbf{T}_{12}) \perp g_{\oplus}(\mathbf{T}_{21})].$$

Renaming the $g_{\oplus}$-transformed RTs as $\tilde{\mathbf{T}}_{11}$, $\tilde{\mathbf{T}}_{12}$, $\tilde{\mathbf{T}}_{21}$, $\tilde{\mathbf{T}}_{22}$, we observe that these new r.v.'s are either all positive or all negative, or else they can attain all real values (see Lemma A1). The problem can now be formulated as follows: Given that

$$\tilde{\mathbf{T}}_{11} + \tilde{\mathbf{T}}_{22} \overset{d}{=} \tilde{\mathbf{T}}_{12} + \tilde{\mathbf{T}}_{21} \quad (\tilde{\mathbf{T}}_{11} \perp \tilde{\mathbf{T}}_{22}, \tilde{\mathbf{T}}_{12} \perp \tilde{\mathbf{T}}_{21}), \tag{17}$$

can one always find real-valued r.v.'s $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$ such that

$$\begin{aligned} \tilde{\mathbf{T}}_{11} &\overset{d}{=} \tilde{\mathbf{A}}_1 + \tilde{\mathbf{B}}_1 \quad (\tilde{\mathbf{A}}_1 \perp \tilde{\mathbf{B}}_1) \\ \tilde{\mathbf{T}}_{12} &\overset{d}{=} \tilde{\mathbf{A}}_1 + \tilde{\mathbf{B}}_2 \quad (\tilde{\mathbf{A}}_1 \perp \tilde{\mathbf{B}}_2) \\ \tilde{\mathbf{T}}_{21} &\overset{d}{=} \tilde{\mathbf{A}}_2 + \tilde{\mathbf{B}}_1 \quad (\tilde{\mathbf{A}}_2 \perp \tilde{\mathbf{B}}_1) \\ \tilde{\mathbf{T}}_{22} &\overset{d}{=} \tilde{\mathbf{A}}_2 + \tilde{\mathbf{B}}_2 \quad (\tilde{\mathbf{A}}_2 \perp \tilde{\mathbf{B}}_2), \end{aligned} \tag{18}$$

on the condition that they are all positive (negative, arbitrary) if $\tilde{\mathbf{T}}_{11}$, $\tilde{\mathbf{T}}_{12}$, $\tilde{\mathbf{T}}_{21}$, $\tilde{\mathbf{T}}_{22}$ are positive (respectively, negative, arbitrary)? If, and only if, such $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$ can be found, then by $g_{\oplus}^{-1}$-transforming them one gets the decomposition sought:

$$\begin{aligned} \mathbf{T}_{11} &\overset{d}{=} \mathbf{A}_1 \oplus \mathbf{B}_1 \quad (\mathbf{A}_1 \perp \mathbf{B}_1) \\ \mathbf{T}_{12} &\overset{d}{=} \mathbf{A}_1 \oplus \mathbf{B}_2 \quad (\mathbf{A}_1 \perp \mathbf{B}_2) \\ \mathbf{T}_{21} &\overset{d}{=} \mathbf{A}_2 \oplus \mathbf{B}_1 \quad (\mathbf{A}_2 \perp \mathbf{B}_1) \\ \mathbf{T}_{22} &\overset{d}{=} \mathbf{A}_2 \oplus \mathbf{B}_2 \quad (\mathbf{A}_2 \perp \mathbf{B}_2). \end{aligned}$$

Although the mathematical theory of decompositions of r.v.'s into sums of s.-independent r.v.'s is conspicuously void of results that might be judged both general and constructive, a variety of decompositions associated with so-called *irreducible* r.v.'s have been found (Linnik & Ostrovskii, 1972; Lukacs, 1960) that imply that (17) may very well hold without (18). A "classical" example (albeit artificial in the present context, as the r.v.'s involved are all discrete) is obtained by putting

$$\tilde{\mathbf{T}}_{11} = \begin{cases} 0, \text{ prob. } \frac{1}{3} \\ 2, \text{ prob. } \frac{1}{3} \\ 4, \text{ prob. } \frac{1}{3}, \end{cases} \qquad \tilde{\mathbf{T}}_{22} = \begin{cases} 0, \text{ prob. } \frac{1}{2} \\ 1, \text{ prob. } \frac{1}{2}, \end{cases}$$

$$\tilde{\mathbf{T}}_{12} = \begin{cases} 0, \text{ prob. } \frac{1}{3} \\ 1, \text{ prob. } \frac{1}{3} \\ 2, \text{ prob. } \frac{1}{3} \end{cases} \qquad \tilde{\mathbf{T}}_{21} = \begin{cases} 0, \text{ prob. } \frac{1}{2} \\ 3, \text{ prob. } \frac{1}{2}, \end{cases}$$

and verifying that (17) holds. It can be proved (Lukacs, 1960) that all these r.v.'s are irreducible, that is, all their possible decompositions into sums of s.-independent components have the form

$$\tilde{\mathbf{T}}_{ij} \overset{d}{=} (\tilde{\mathbf{T}}_{ij} - c) + c \quad (i = 1, 2; j = 1, 2),$$

where c is a constant (formally, a constant and a r.v. are s.-independent). From this it follows easily that no decomposition (18) is possible for these r.v.'s, whatever the permissible spectra of $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$, and even if some of $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{B}}_1$, $\tilde{\mathbf{B}}_2$ are allowed to be constants. (The essence of the example, using the language defined in Appendix E, is that both "RTs" $\tilde{\mathbf{T}}_{11}$, $\tilde{\mathbf{T}}_{22}$ here are irreducible, but neither is an *independent additive component of either of the* "RTs" $\tilde{\mathbf{T}}_{12}$, $\tilde{\mathbf{T}}_{21}$.) A "classical" example involving continuous r.v.'s is as follows. Let $\tilde{\mathbf{T}}_{11}$, $\tilde{\mathbf{T}}_{12}$, $\tilde{\mathbf{T}}_{21}$, $\tilde{\mathbf{T}}_{22}$ (with unconstrained spectra) have the following characteristic functions (Lukacs, 1960):

$$\begin{aligned} \varphi_{11}(s) &= (1 - s^2/2) \exp(-s^2/4), \\ \varphi_{22}(s) &= \varphi_{11}(s), \\ \varphi_{12}(s) &= (1 - s^2/2)^2 \exp(-3s^2/8), \\ \varphi_{21}(s) &= \exp(-s^2/8). \end{aligned}$$

Again, (17) holds, because $\varphi_{11}(s)\, \varphi_{22}(s) = \varphi_{12}(s)\, \varphi_{21}(s)$, but *no decomposition* (18) *is possible* (because both $\tilde{\mathbf{T}}_{11}$, $\tilde{\mathbf{T}}_{22}$ are irreducible, but neither is an independent additive component of either $\tilde{\mathbf{T}}_{12}$, which is irreducible itself, or $\tilde{\mathbf{T}}_{21}$, which is normally distributed, and hence can have only normally distributed independent additive components). Other examples, or references to the literature containing such examples, can be found in Feller (1968), Linnik & Ostrovskii (1972), Lukacs (1960), and Rusza & Szekely (1988). Irreducible r.v.'s, upon which all these examples are based, are not as rare as one might judge from the difficulty of constructing them. In fact, for any distribution function *there exists an irreducible r.v. with a distribution function* arbitrarily close to it (see Definition E1 and Lemma E1 for more rigorous formulations). We are in the position now to formulate the following statement, based on Theorem E3.

STATEMENT 9. *Under the assumption of s.-independence, a decomposition test involving any addition-like operation* $\oplus$ *may be successful for some r.v.'s* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *that do not have the corresponding architecture. In other words, for any addition-like operation* $\oplus$ *one can find a quadruple of r.v.'s* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *such that*

$$\mathbf{T}_{11} \oplus \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}),$$

but that cannot be decomposed as

$$\mathbf{T}_{11} \stackrel{d}{=} \mathbf{A}_1 \oplus \mathbf{B}_1 \quad (\mathbf{A}_1 \perp \mathbf{B}_1)$$

$$\mathbf{T}_{12} \stackrel{d}{=} \mathbf{A}_1 \oplus \mathbf{B}_2 \quad (\mathbf{A}_1 \perp \mathbf{B}_2)$$

$$\mathbf{T}_{21} \stackrel{d}{=} \mathbf{A}_2 \oplus \mathbf{B}_1 \quad (\mathbf{A}_2 \perp \mathbf{B}_1)$$

$$\mathbf{T}_{22} \stackrel{d}{=} \mathbf{A}_2 \oplus \mathbf{B}_2 \quad (\mathbf{A}_2 \perp \mathbf{B}_2).$$

Comparing this statement with Statement 1, we conclude that, under the assumption of s.-independence, a successful decomposition test involving an addition-like operation is necessary but not sufficient for the existence of the corresponding architecture.

We see that addition-like decompositions into s.-independent components stand in sharp contrast to all other decomposition tests. Having verified that

$$\mathbf{T}_{11} \oplus \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}),$$

one rules out the possibility that the RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ might have an architecture involving another, algebraically distinct, decomposition rule, but one cannot be sure that they have the architecture involving $\oplus$: this has to be proved by actually finding some quadruple of $\oplus$-decomposing component times $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$. This result is certainly disappointing, but it is not clear at the moment just how grave its consequences are for RT analysis. We cannot exclude the possibility that all empirically observable RTs de facto belong to a subclass of r.v.'s for which the decomposability in question is guaranteed by a success of the corresponding decomposition test, at least for some decomposition rules. Just as an example, this would be the case for additive decompositions if all RT distributions were gamma-distributions with one and the same exponential parameter. It would be very useful, therefore, to investigate the validity of Statement 9 for a relatively narrow family of r.v.'s that would include (according to some, so far non-existing, theory) all possible RTs or all possible RTs in a certain experiment. The mentioned (non-existing) theory constraining the family of RT distributions would also be likely to constrain the distributions of the component times, which might considerably simplify mathematical analysis.

Irrespective of the results of such an investigation, however, one practical conclusion that can be drawn from this paper is that s.-independence in the context of RT decompositions should not be invoked for the sake of simplicity, either conceptual (one's intuition should be considerably "educated" to successfully deal with independent additive components of r.v.'s) or technical (the actual computation of independent additive components is both nontrivial and ad hoc). The p.p.s.-interdependence assumption seems at least equally plausible a priori, and it does make matters simple. Dzhafarov (1992) showed that for some experimental situations one can impose restrictions on hypothetical component times that would make the two assumptions, s.-independence and p.p.s.-interdependence, empirically testable against each other (for a detailed analysis, see Dzhafarov & Rouder, 1995).

Unfortunately, these restrictions do not apply to the decomposition tests based on a 2×2 factorial design. A decomposition test involving an addition-like operation can very well hold on one and the same quadruple of RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ under both assumptions, s.-independence and p.p.s.-interdependence. As an example, consider the RTs

$$\mathbf{T}_{11} \stackrel{d}{=} \mathbf{X} - a; \quad \mathbf{T}_{22} \stackrel{d}{=} \mathbf{X} + a; \quad \mathbf{T}_{21} \stackrel{d}{=} \mathbf{T}_{12} \stackrel{d}{=} \mathbf{X},$$

where lext$\mathbf{X} \geqslant$ a. It is easy to verify that for these RTs

$$\mathbf{T}_{11} + \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} + \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21})$$

$$\mathbf{T}_{11} + \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} + \mathbf{T}_{21} \quad (\mathbf{T}_{11} \parallel \mathbf{T}_{22}, \mathbf{T}_{12} \parallel \mathbf{T}_{21})$$

hold simultaneously. It would be useful to investigate what additional constraints have to be imposed on RT distributions (or those of the hypothetical RT components) that would prevent this from happening (making thereby the two forms of stochastic relationship testable against each other).

8. PHYSICAL REALIZABILITY OF NON-TRADITIONAL DECOMPOSITION RULES

Perhaps the most unusual notion introduced in this paper is that of using operations other than arithmetic addition, minimum, and maximum. As stated earlier, decisive arguments in favor of the "possibility" of RT architectures involving such operations can only be empirical. It is legitimate, however, in the absence of empirical information, to ask whether these alternative operations can at all be *physically realizable*, that is, whether they can be shown to reflect some properties of hypothetical processes whose durations are observable RTs. In the case of the common operations (+, min, max) their physical realizability is obvious: one can think of each component time as the duration of a separate subprocess (selectively influenced by a certain factor) with uniquely defined starting–stopping rules. In such a "naturalistic" interpretation, the alternative addition-like decomposition rules do appear artificial, if not non-realizable. We propose, therefore, another interpretation, at this stage as a logical possibility only. We present this interpretation in the McGill modelling language (Dzhafarov, 1993), in which observable RTs are durations of stochastic processes evoked by stimuli and developing until they reach a fixed critical level (criterion).

The simplest special case of a stochastic process is a deterministic function of time with randomly preset initial conditions. Let $C_{\alpha,\beta}(t \mid \mathbf{X}, \mathbf{Y})$ be such a process, where t stands for time, $\mathbf{X}$, $\mathbf{Y}$ are r.v.'s representing initial conditions, and $C_{\alpha,\beta}$ is some increasing function of t that depends on factors α and β as its parameters. The process is assumed to be evoked at moment $t=0$ and increase until it crosses some fixed criterion, whose value can be taken to be unity. The time of the crossing is determined by solving for t the equation

$$C_{\alpha,\beta}(t \mid \mathbf{X}, \mathbf{Y}) = 1,$$

and the solution

$$\mathbf{T} = T(\alpha, \beta, \mathbf{X}, \mathbf{Y})$$

is the observable RT. Suppose that it happens that this solution is representable as

$$T(\alpha, \beta, \mathbf{X}, \mathbf{Y}) = A(\alpha, \mathbf{X}) \blacklozenge B(\beta, \mathbf{Y}), \tag{19}$$

where $\blacklozenge$ is a simple operation, and A, B are some functions. Then, putting

$$\mathbf{T}(\alpha, \beta) \stackrel{d}{=} T(\alpha, \beta, \mathbf{X}, \mathbf{Y}),$$

$$\mathbf{A}(\alpha) \stackrel{d}{=} A(\alpha, \mathbf{X}),$$

$$\mathbf{B}(\beta) \stackrel{d}{=} B(\beta, \mathbf{Y}),$$

one can say that the RT has the architecture

$$\mathbf{T}(\alpha, \beta) \stackrel{d}{=} \mathbf{A}(\alpha) \blacklozenge \mathbf{B}(\beta),$$

where the stochastic relationship between $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ is determined by the joint distribution of $\mathbf{X}$, $\mathbf{Y}$ (the initial conditions that now can be called "internal sources of variability" for the component times). As we see, the component times here are merely time-dimensioned descriptors of a single process, and they are not interpretable as durations of its separate "parts." It remains to demonstrate now that one can indeed construct the process $C_{\alpha,\beta}(t \mid \mathbf{X}, \mathbf{Y})$ so that proposition (19) holds. Consider the process representable as

$$C_{\alpha,\beta}(t \mid \mathbf{X}, \mathbf{Y}) = \frac{1}{\exp(\mathbf{Y})} \times \exp\{B_\beta^{-1}\{g^{-1}[g(t) - g(A_\alpha(\mathbf{X}))]\}\},$$

with the following properties: g is some monotonic continuous function (increasing or decreasing) mapping positive time values onto one of the three intervals $(-\infty, 0)$, $(0, \infty)$, or $(-\infty, \infty)$; A_α and B_β are some increasing functions mapping reals into positive time values and depending on the factors α and β as their respective parameters. (If the expression for the process appears too artificial, observe that by simple renaming it can be simplified as

$$\mathbf{S}_1 \exp\{\Phi_\alpha[g(t) - \Psi_\beta(\mathbf{S}_2)]\};$$

it will save us time, however, to use the more explicit version.) The process level is considered undefined or equal to zero at all moments $t > 0$ at which $g(t) - g(A_\alpha(\mathbf{X}))$ falls outside the domain of g^{-1}. It is easy to prove that at all other moments the process increases. The solution (with respect to t) of the equation

$$\frac{1}{\exp(\mathbf{Y})} \exp\{B_\beta^{-1}\{g^{-1}[g(t) - g(A_\alpha(\mathbf{X}))]\}\} = 1$$

is, writing $A(\alpha, \mathbf{X})$, $B(\beta, \mathbf{Y})$ instead of $A_\alpha(\mathbf{X})$, $B_\beta(\mathbf{Y})$,

$$\begin{aligned} T(\alpha, \beta, \mathbf{X}, \mathbf{Y}) &= g^{-1}\{g[A(\alpha, \mathbf{X})] + g[B(\beta, \mathbf{Y})]\} \\ &= A(\alpha, \mathbf{X}) \oplus B(\beta, \mathbf{Y}). \end{aligned}$$

where $\oplus$ is the addition-like operation determined by $g_\oplus \equiv g$. This agrees with (19) and makes our point for addition-like operations: architectures involving such operations are physically realizable. The case of s.-independence is obtained by assuming that the two initial conditions are mutually s.-independent, $\mathbf{X} \perp \mathbf{Y}$; the case of p.p.s.-interdependence is obtained by assuming that the process is single-variate, $\mathbf{X} = \mathbf{Y}$. The traditional serial architecture with s.- independent components corresponds in this scheme to the process

$$C_{\alpha,\beta}(t \mid \mathbf{X}, \mathbf{Y}) = \frac{1}{\exp(\mathbf{Y})} \exp\{B_\beta^{-1}[t - A_\alpha(\mathbf{X})]\},$$

$$\mathbf{X} \perp \mathbf{Y}.$$

The minimum and maximum operations can be appended as limit cases, through a procedure involving "Minkowski-norm" operations for maximum ($g(t) = t^p$, $p \geqslant 1$) and their negative-exponent counterparts for minimum ($g(t) = t^p$, $p \leqslant -1$). To avoid technical complications, however, a more straightforward approach is to simply define stochastic processes as

$$C_{\alpha,\beta}^{(\max)}(t \mid \mathbf{X}, \mathbf{Y}) = \min\{c_\alpha(t \mid \mathbf{X}), c_\beta(t \mid \mathbf{Y})\},$$

$$C_{\alpha,\beta}^{(\min)}(t \mid \mathbf{X}, \mathbf{Y}) = \max\{c_\alpha(t \mid \mathbf{X}), c_\beta(t \mid \mathbf{Y})\},$$

where c_α and c_β are some increasing functions. Denoting the solutions of the equations $c_\alpha(t \mid \mathbf{X}) = 1$ and $c_\beta(t \mid \mathbf{Y}) = 1$ by $A(\alpha, \mathbf{X})$ and $B(\beta, \mathbf{Y})$, respectively, it is easy to see that the processes $C_{\alpha,\beta}^{(\max)}(t \mid \mathbf{X}, \mathbf{Y})$ and $C_{\alpha,\beta}^{(\min)}(t \mid \mathbf{X}, \mathbf{Y})$ cross a unity level at the moments $\max\{A(\alpha, \mathbf{X}), B(\beta, \mathbf{Y})\}$ and $\min\{A(\alpha, \mathbf{X}), B(\beta, \mathbf{Y})\}$, respectively. The difference between this interpretation and the traditional one, with two parallel processes, is obviously inconsequential, and our only reason for presenting it is to emphasize that here, too, as is the case with addition-like decomposition rules, the component times can be interpreted as time-dimensioned characteristics of a single process, rather than durations of separate processes.

9. SUMMARY OF MAIN RESULTS

To state the main results succinctly, we adopt the following conventions. A decomposition test involving a decomposition rule ◆ is referred to as the (◆)-test. Analogously, we say that RT is (◆)-decomposable, meaning that it can be decomposed into two components connected by the simple operation ◆. It is implied but not mentioned explicitly that we deal with a 2×2 crossed factorial design and that the component times are selectively influenced by different factors. It is also implied without mentioning that the RTs involved do not form a pattern of cross-over rearrangements (in particular, the index factors are effective), and that competing decomposition rules (◆ and ◇) are algebraically distinct. With these conventions in mind, the results are as follows.

(A) *Necessity and uniqueness of identification.* Under either of the two forms of stochastic relationship, p.p.s.-interdependence or s.-independence, and for any two decomposition rules ◆ and ◇,

(A1) a successful (◆)-test is necessary for (◆)-decomposability of RTs;

(A2) (◆)-test and (◇)-test cannot be successful together; that is, no RTs can be both (◆)-decomposable and (◇)-decomposable.

(B) *Sufficiency.*

(B1) Under the assumption of p.p.s.-interdependence a successful (◆)-test is sufficient for (◆)-decomposability of RTs.

(B2) Under the assumption of s.-independence, a successful (◆)-test is sufficient for (◆)-decomposability of RTs if the operation ◆ is min or max (some of the component times in these cases can be incomplete random variables); if the operation ◆ is addition-like, then a (◆)-test may be successful even if the RTs are not (◆)-decomposable (in which case they are not decomposable by means of any other operation either).

(C) *Recovery of component times.* The component times of (◆)-decomposable RTs generally cannot be recovered uniquely, but their spectra can always be chosen to lie within the same domain of the operation ◆ that contains the spectra of the RTs themselves.

(D) *Recovery of stochastic relationship.* The decomposition tests do not recover the stochastic relationship: a (◆)-test may be successful under both p.p.s.-interdependence and s.-independence.

(E) The results of this paper are not predicated on any constraints imposed on RT distributions. The results therefore are equally applicable to random variables in other empirical domains.

APPENDIX A: SIMPLE OPERATIONS

Definition A1. An operation $\oplus$ is called an addition-like operation on some open interval $I \subseteq \mathrm{Re}$ if $\oplus$ maps $I \times I \to I$ onto, is associative, continuous, and strictly increasing in both arguments. An operation on I is called simple if it is $\min\{a, b\}$, $\max\{a, b\}$ or $a \oplus b$, where $\oplus$ is addition-like ($a, b \in I$).

Lemma A1. (Aczél). *An operation $\oplus$ is addition-like on some interval $I \subseteq \mathrm{Re}$ iff there is a strictly monotonic continuous function $g_\oplus : I \to \mathrm{Re}$, such that $a \oplus b = g_\oplus^{-1}[g_\oplus(a) + g_\oplus(b)]$; $g_\oplus$ is determined uniquely up to a multiplication $cg_\oplus$. Addition-like operations are commutative, and $g_\oplus(I)$ is one of the three intervals $(-\infty, 0)$, $(0, \infty)$, $(-\infty, \infty)$.*

Proof. The main statement, that $a \oplus b \equiv g_\oplus^{-1}[g_\oplus(a) + g_\oplus(b)]$, and the uniqueness statement are proved in Aczél (1966, pp. 256–267), with the commutativity following as an obvious corollary. That $g_\oplus(I)$ is $(-\infty, 0)$, $(0, \infty)$, or $(-\infty, \infty)$, is proved as follows. If $\sup g_\oplus(I) + \sup g_\oplus(I) > \sup g_\oplus(I)$, then, for sufficiently large $x, y \in g_\oplus(I)$, $x + y > \sup g_\oplus(I)$. Hence $g_\oplus^{-1}(x) \oplus g_\oplus^{-1}(y) \notin I$ while $g_\oplus^{-1}(x), g_\oplus^{-1}(y) \in I$, which contradicts the fact that $\oplus$ is closed. If $\sup g_\oplus(I) + \sup g_\oplus(I) < \sup g_\oplus(I)$, then for a sufficiently large $z \in g_\oplus(I)$, $x + y < z$ for all $x, y \in g_\oplus(I)$. Hence $g_\oplus^{-1}(z)$ cannot be presented as $g_\oplus^{-1}(x) \oplus g_\oplus^{-1}(y)$, which contradicts the fact that $\oplus$ maps onto. It follows that $\sup g_\oplus(I) + \sup g_\oplus(I) = \sup g_\oplus(I)$, and hence $\sup g_\oplus(I) = -\infty$, 0, or $+\infty$. Analogously one proves that $\inf g_\oplus(I) + \inf g_\oplus(I) = \inf g_\oplus(I)$, that is, $\inf g_\oplus(I) = -\infty$, 0, or $+\infty$.

This completes the proof, because the only three non-degenerate intervals that can be formed from these values are $(-\infty, 0)$, $(0, \infty)$, or $(-\infty, \infty)$.

DEFINITION A2. Given two addition-like operations $\oplus$ and $\otimes$ on some interval $I \subseteq \mathrm{Re}$, $g_\otimes$ is called strictly convex (strictly concave) on I with respect to $g_\oplus$, if the function g defined by $g_\otimes \equiv g g_\oplus$ is strictly convex (strictly concave) on $g_\oplus(I)$. The function g is well defined and continuous because both $g_\otimes$ and $g_\oplus$ are strictly monotonic and continuous. Obviously, $g_\otimes$ is strictly convex (strictly concave) with respect to $g_\oplus$ on some interval iff $g_\oplus$ is strictly concave (strictly convex) with respect to $g_\otimes$ on the same interval.

The proof of the following lemma, except for minor modifications, was provided to us by Donald Burkholder (personal communication, June 1994).

LEMMA A2. (Burkholder). *Given two addition-like operations $\oplus$ and $\otimes$ on some interval $I \subseteq \mathrm{Re}$, the following two statements are equivalent:*

(i) the system of equations

$$\begin{cases} x \oplus y = a \\ x \otimes y = b, \end{cases} \quad (x, y) \in I \times I \subseteq \mathrm{Re} \times \mathrm{Re}$$

is satisfied by no more than one unordered pair (x, y) for any ordered pair (a, b);

(ii) *$g_\oplus$ is strictly convex or strictly concave on I with respect to $g_\oplus$.*

Proof. Let $g_\otimes \equiv g g_\oplus$, and let $\tilde{c}$ denote $g_\oplus(c)$. This allows us to rewrite the system as

$$\begin{cases} \tilde{x} + \tilde{y} = \tilde{a} \\ g(\tilde{x}) + g(\tilde{y}) = g(\tilde{b}), \end{cases} \quad (\tilde{x}, \tilde{y}) \in g_\oplus(I) \times g_\oplus(I).$$

Any two different pairs $(\tilde{x}_1, \tilde{y}_1)$, $(\tilde{x}_2, \tilde{y}_2)$, such that $\tilde{x}_1 + \tilde{y}_1 = \tilde{x}_2 + \tilde{y}_2$ can be presented (except for trivial permutations of indices) as

$$\begin{aligned} \tilde{x}_1 &= \tilde{x} \\ \tilde{x}_2 &= \tilde{x} + \Delta_1 \tilde{x} \\ \tilde{y}_2 &= \tilde{x} + \Delta_2 \tilde{x} \\ \tilde{y}_1 &= \tilde{x} + \Delta_1 \tilde{x} + \Delta_2 \tilde{x}, \end{aligned}$$

where $\tilde{x}$, $\Delta_1 \tilde{x}$, and $\Delta_2 \tilde{x}$ are chosen so that

$$\begin{aligned} \inf g_\oplus(I) &\leqslant \tilde{x} < \tilde{x} + \Delta_1 \tilde{x} \leqslant \tilde{x} + \Delta_2 \tilde{x} \\ &< \tilde{x} + \Delta_1 \tilde{x} + \Delta_2 \tilde{x} \leqslant \sup g_\oplus(I). \end{aligned}$$

Proposition (i) then can be viewed as stating that for all such $\tilde{x}$, $\Delta_1 \tilde{x}$, and $\Delta_2 \tilde{x}$, the function

$$\begin{aligned} F(\tilde{x}, \Delta_1 \tilde{x}, \Delta_2 \tilde{x}) &\equiv [g(\tilde{x} + \Delta_1 \tilde{x} + \Delta_2 \tilde{x}) + g(\tilde{x})] \\ &\quad - [g(\tilde{x} + \Delta_1 \tilde{x}) + g(\tilde{x} + \Delta_2 \tilde{x})] \\ &\equiv [g(\tilde{x} + \Delta_1 \tilde{x} + \Delta_2 \tilde{x}) - g(\tilde{x} + \Delta_2 \tilde{x})] \\ &\quad - [g(\tilde{x} + \Delta_1 \tilde{x}) - g(\tilde{x})] \end{aligned}$$

does not vanish. At the same time, g is strictly convex (concave) on $g_\oplus(I)$ iff $F(\tilde{x}, \Delta_1 \tilde{x}, \Delta_2 \tilde{x})$ is strictly positive (negative) for all admissible values of $\tilde{x}$, $\Delta_1 \tilde{x}$, and $\Delta_1 \tilde{x}$. This proves that (ii) implies (i). Assume now that (i) holds, but, for some triads $(\tilde{x}_1, \Delta_1 \tilde{x}_1, \Delta_2 \tilde{x}_1)$, and $(\tilde{x}_2, \Delta_1 \tilde{x}_2, \Delta_2 \tilde{x}_2)$, $F(\tilde{x}_1, \Delta_1 \tilde{x}_1, \Delta_2 \tilde{x}_1) > 0$ and $F(\tilde{x}_2, \Delta_1 \tilde{x}_2, \Delta_2 \tilde{x}_2) < 0$. Let $f(\alpha)$, $0 \leqslant \alpha \leqslant 1$, be defined by

$$\begin{aligned} f(\alpha) \equiv F(&\alpha \tilde{x}_1 + (1-\alpha)\, \tilde{x}_2, \alpha \Delta_1 \tilde{x}_1 \\ &+ (1-\alpha) \Delta_1 \tilde{x}_2, \alpha \Delta_2 \tilde{x}_1 + (1-\alpha)\, \Delta_2 \tilde{x}_2). \end{aligned}$$

Observe that $f(\alpha)$ is continuous, and

$$\begin{aligned} \inf g_\oplus(I) &\leqslant \alpha \tilde{x}_1 + (1-\alpha)\, \tilde{x}_2 \\ &< [\alpha \tilde{x}_1 + (1-\alpha)\, \tilde{x}_2] + [\alpha \Delta_1 \tilde{x}_1 + (1-\alpha)\, \Delta_1 \tilde{x}_2] \\ &\leqslant [\alpha \tilde{x}_1 + (1-\alpha)\, \tilde{x}_2] + [\alpha \Delta_2 \tilde{x}_1 + (1-\alpha)\, \Delta_2 \tilde{x}_2] \\ &< [\alpha \tilde{x}_1 + (1-\alpha)\, \tilde{x}_2] + [\alpha \Delta_1 \tilde{x}_1 + (1-\alpha)\, \Delta_1 \tilde{x}_2] \\ &\quad + [\alpha \Delta_2 \tilde{x}_1 + (1-\alpha)\, \Delta_2 \tilde{x}_2] \\ &\leqslant \sup g_\oplus(I). \end{aligned}$$

Since, by assumption, $f(0) > 0$ and $f(1) < 0$, there must be a point α between 0 and 1 at which $f(\alpha) = 0$. This is impossible, however, because it follows from (i) that $F(\tilde{x}, \Delta_1 \tilde{x}, \Delta_2 \tilde{x})$ never vanishes. This contradiction proves that (i) implies (ii) and completes the proof.

LEMMA A3. *Given an addition-like operation $\oplus$ on some interval $I \subseteq \mathrm{Re}$, each of the systems of equations*

$$\begin{cases} x \oplus y = a \\ \max(x, y) = b, \end{cases} \quad \begin{cases} x \oplus y = a \\ \min(x, y) = b, \end{cases} \quad \begin{cases} \max(x, y) = a \\ \min(x, y) = b, \end{cases}$$

$$(x, y) \in I \times I \subseteq \mathrm{Re} \times \mathrm{Re}$$

is satisfied by no more than one unordered pair (x, y) for any ordered pair (a, b).

Proof. Since $g_\oplus$ is strictly monotonic, the contour $x \oplus y = a$, which is equivalent to $g_\oplus(x) + g_\oplus(y) = g_\oplus(a)$, defines a strictly decreasing relationship between x and y for any a. Hence, for any b, this contour cannot intersect straight lines $x=b$ and $y=b$ at more than one point each; due to the contour's symmetry with respect to the main bisector, the two intersection points are identical as unordered pairs (refer to Fig. 4). This proves the lemma for the first two systems of equations. For the third one, the unordered solution is always $(x, y)=(a, b)$. This completes the proof.

APPENDIX B: UNIQUENESS OF SIMPLE OPERATIONS UNDER STOCHASTIC INDEPENDENCE

In the theorems below the spectra of all r.v.'s are assumed to lie in the domains of the simple operations relating them to each other (the spectrum of a r.v. is the set of its possible values).

THEOREM B1. *For any given pair of r.v.'s* **A** *and* **B**, *there is at most one (unordered) pair of r.v.'s* $\mathbf{X} \perp \mathbf{Y}$, *such that*

$$\mathbf{X} \oplus \mathbf{Y} \overset{d}{=} \mathbf{A}$$
$$\mathbf{X} \otimes \mathbf{Y} \overset{d}{=} \mathbf{B},$$

provided that $g_{\otimes}$ *is strictly convex (strictly concave) with respect to* $g_{\oplus}$.

We first prove this theorem for a special subclass of discrete r.v.'s, referred to as *primitive*. The result is then generalized to arbitrary r.v.'s based on limit considerations, since any r.v. can be approximated by a weakly converging sequence of primitive r.v.'s. (The concept of weak convergence, which here coincides with that of convergence in distribution, is explained in most textbooks of probability; see, e.g., Loeve, 1963.)

DEFINITION B1. A r.v. **X** is called primitive if its spectrum S_x consists of a finite number of points $x_1 \leqslant \cdots \leqslant x_n$ each associated with a probability mass of $1/n$.

LEMMA B1. *Theorem B1 holds if the r.v.'s* **A** *and* **B** *are primitive.*

Proof. Let one solution $(\mathbf{X}, \mathbf{Y})$ exist. Obviously, both **X** and **Y** are primitive, and denoting their spectra by $S_X = x_1 \leqslant \cdots \leqslant x_n$ and $S_Y = y_1 \leqslant \cdots \leqslant y_m$, we observe that the spectra of **A** and **B** contain nm points each: $S_A = a_1 \leqslant \cdots \leqslant a_{nm}$, $S_B = b_1 \leqslant \cdots \leqslant b_{nm}$. For definiteness, assume that $(\mathbf{X}, \mathbf{Y})$ are always chosen so that $x_1 \leqslant y_1$. It is clear that

$$x_1 \oplus y_1 = a_1$$
$$x_1 \otimes y_1 = b_1,$$

hence, by Lemma A2, x_1 and y_1 are determined uniquely $(x_1 \leqslant y_1)$. Taking this as an induction basis, we assume that the first k points in S_X, $x_1 \leqslant \cdots \leqslant x_k$ $(k \geqslant 1)$ and the first l points in S_Y, $y_1 \leqslant \cdots \leqslant y_l$ $(l \geqslant 1)$ have been determined uniquely, and we show by induction that then either x_{k+1} or y_{l+1} (or both) are determined uniquely. This is sufficient to prove the lemma, because at the end of the induction chain one of the r.v.'s $(\mathbf{X}, \mathbf{Y})$ will be determined uniquely, and the other will then be determined from either of the distributional equations $g_{\oplus}(\mathbf{X}) + g_{\oplus}(\mathbf{Y}) \overset{d}{=} g_{\oplus}(\mathbf{A})$, $g_{\otimes}(\mathbf{X}) + g_{\otimes}(\mathbf{Y}) \overset{d}{=} g_{\otimes}(\mathbf{B})$, by deconvolution.

We begin the induction step by forming all pairwise combinations $x_i \oplus x_j$ and $x_i \otimes x_j$ among the values known by the induction hypothesis:

$$I_A = \{x_i \oplus x_j : i \leqslant k, j \leqslant l\} \subseteq S_A,$$
$$I_B = \{x_i \otimes x_j : i \leqslant k, j \leqslant l\} \subseteq S_B,$$

Let

$$p = \min\{i : a_i \in S_A, a_i \notin I_A\},$$
$$q = \min\{i : b_i \in S_B, b_i \notin I_B\};$$

that is, a_p and b_q are the smallest elements in S_A and S_B, respectively, that have not been "accounted for" in terms of the known values of S_X and S_Y. Obviously, either $x_1 \oplus y_{l+1} = a_p$, or $x_{k+1} \oplus y_1 = a_p$ (or both), because there can be no combination $x_i \oplus x_j \notin I_A$ that is less than both these values. Analogously, either $x_1 \otimes y_{l+1} = b_q$ or $x_{k+1} \otimes y_1 = b_q$ (or both). If

$$x_1 \oplus y_{l+1} = a_p$$
$$x_1 \otimes y_{l+1} = b_q,$$

then y_{l+1} is determined uniquely, and the induction step is complete. Analogously, if

$$x_{k+1} \oplus y_1 = a_p$$
$$x_{k+1} \otimes y_1 = b_q,$$

then x_{k+1} is determined uniquely, and the induction step is complete. These two possibilities are not, of course, mutually conflicting. Assume now that none of them takes place. Then

$$\text{either} \quad \begin{cases} x_1 \oplus y_{l+1} = a_p \\ x_{k+1} \otimes y_1 = b_q \end{cases} \quad \text{or} \quad \begin{cases} x_{k+1} \oplus y_1 = a_p \\ x_1 \otimes y_{l+1} = b_p \end{cases}$$

(or both)

Since each of the two equation pairs determines (x_{k+1}, y_{l+1}) uniquely, the induction step is complete if only one, but not another of these equation pairs holds. It remains to be proved, therefore, that if both these equation pairs hold, then their solutions $(\tilde{x}_{k+1}, \tilde{y}_{l+1})$ and $(\tilde{\tilde{x}}_{k+1}, \tilde{\tilde{y}}_{l+1})$ coincide. Indeed, if both

$$\begin{cases} x_1 \oplus \tilde{y}_{l+1} = a_p \\ \tilde{x}_{k+1} \otimes y_1 = b_q \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\tilde{x}}_{k+1} \oplus y_1 = a_p \\ x_1 \otimes \tilde{\tilde{y}}_{l+1} = b_q, \end{cases}$$

then there must exist $\tilde{p} > p$, $\tilde{q} > q$, $\tilde{\tilde{p}} > p$, and $\tilde{\tilde{q}} > q$, such that

$$\begin{cases} x_1 \otimes \tilde{y}_{l+1} = b_{\tilde{q}} \\ \tilde{x}_{k+1} \oplus y_1 = a_{\tilde{p}} \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\tilde{x}}_{k+1} \otimes y_1 = b_{\tilde{\tilde{q}}} \\ x_1 \oplus \tilde{\tilde{y}}_{l+1} = a_{\tilde{\tilde{p}}}. \end{cases}$$

Then we have, however, by grouping the equations differently,

$$\left.\begin{array}{l} \left.\begin{array}{l} x_1 \oplus \tilde{y}_{l+1} = a_p \\ x_1 \oplus \tilde{\tilde{y}}_{l+1} = a_{\tilde{\tilde{p}}} \geqslant a_p \end{array}\right\} \Rightarrow \tilde{\tilde{y}}_{l+1} \geqslant \tilde{y}_{l+1} \\ \left.\begin{array}{l} x_1 \otimes \tilde{\tilde{y}}_{l+1} = b_q \\ x_1 \otimes \tilde{y}_{l+1} = b_{\tilde{q}} \geqslant b_q \end{array}\right\} \Rightarrow \tilde{\tilde{y}}_{l+1} \leqslant \tilde{y}_{l+1} \end{array}\right\} \Rightarrow \tilde{\tilde{y}}_{l+1} = \tilde{y}_{l+1},$$

and

$$\left.\begin{array}{l} \left.\begin{array}{l} \tilde{\tilde{x}}_{k+1} \oplus y_1 = a_p \\ \tilde{x}_{k+1} \oplus y_1 = a_{\tilde{p}} \geqslant a_p \end{array}\right\} \Rightarrow \tilde{\tilde{x}}_{k+1} \leqslant \tilde{x}_{k+1} \\ \left.\begin{array}{l} \tilde{x}_{k+1} \otimes y_1 = b_q \\ \tilde{\tilde{x}}_{k+1} \otimes y_1 = b_{\tilde{\tilde{q}}} \geqslant b_q \end{array}\right\} \Rightarrow \tilde{\tilde{x}}_{k+1} \geqslant \tilde{x}_{k+1} \end{array}\right\} \Rightarrow \tilde{\tilde{x}}_{k+1} = \tilde{x}_{k+1}.$$

This completes the induction step and the proof of the lemma.

To generalize this uniqueness result to arbitrary r.v.'s (of which we are really interested in continuous ones only), we make use of topological considerations. The topological terms and facts used here and below are standard, their definitions and proofs being found is almost any textbook of topology or advanced calculus (see, e.g., Kelly, 1955). Let $\aleph$ be the class of all r.v.'s, $\aleph_{\mathrm{P}} \subset \aleph$ be the class of all primitive r.v.'s, and $\aleph_{\mathrm{P}}^n \subset \aleph_{\mathrm{P}}$ be the class of primitive r.v.'s whose spectra contain n elements. We introduce the following embedding relationship on $\aleph_{\mathrm{P}}$: r.v.'s $\mathbf{C} \in \aleph_{\mathrm{P}}^n$ and $\tilde{\mathbf{C}} \in \aleph_{\mathrm{P}}^{nk}$, $k = 1, 2, \ldots$, are equivalent if their ordered spectra are related as

$$\begin{array}{lcccc} S_C = c_1, & c_2, & \ldots, & c_n \\ S_{\tilde{C}} = \underbrace{c_1, c_1, \ldots, c_1}_{k \text{ times}} & \underbrace{c_2, c_2, \ldots, c_2}_{k \text{ times}} & \ldots, & \underbrace{c_n, c_n, \ldots, c_n}_{k \text{ times}} \end{array}$$

By such a refinement any set $\aleph_{\mathrm{P}}^n$ can be embedded as a subset in $\aleph_{\mathrm{P}}^{nk}$. Obviously, $\aleph_{\mathrm{P}}^n$ is closed (n-point spectra can weakly converge to n-point spectra only), and hence $\aleph_{\mathrm{P}}^n$ is compact in the sense of weak convergence (because, by Helly's theorem, $\aleph$ is compact in this sense; see Loeve, 1963, pp. 179–180). We will also need the obvious fact that the subspaces of $\aleph$ are Hausdorff.

Proof of Theorem B1. The distributional equations

$$\begin{array}{l} \mathbf{X} \oplus \mathbf{Y} \overset{\mathrm{d}}{=} \mathbf{A} \\ \mathbf{X} \otimes \mathbf{Y} \overset{\mathrm{d}}{=} \mathbf{B}, \end{array} \quad \mathbf{X} \perp \mathbf{Y},$$

define a continuous operator $\Omega: \aleph \times \aleph \to \aleph \times \aleph$ uniquely mapping pairs $(\mathbf{X}, \mathbf{Y})$ into pairs $(\mathbf{A}, \mathbf{B})$. We have to prove that Ω^{-1} maps $(\mathbf{A}, \mathbf{B})$ into $(\mathbf{X}, \mathbf{Y})$ uniquely, considering, of course, $(\mathbf{X}, \mathbf{Y})$ and $(\mathbf{Y}, \mathbf{X})$ indistinguishable. By Lemma B1, this is true if $(\mathbf{A}, \mathbf{B}) \in \Omega(\aleph_{\mathrm{P}} \times \aleph_{\mathrm{P}}) \subseteq \aleph_{\mathrm{P}} \times \aleph_{\mathrm{P}}$, that is, the restriction of Ω to $\aleph_{\mathrm{P}} \times \aleph_{\mathrm{P}} \to \Omega(\aleph_{\mathrm{P}} \times \aleph_{\mathrm{P}})$ is a bijection, obviously continuous. For an arbitrary $n > 1$, consider a sequence of embedded spaces $\aleph_{\mathrm{P}}^n \times \aleph_{\mathrm{P}}^n \subset \aleph_{\mathrm{P}}^{n^2} \times \aleph_{\mathrm{P}}^{n^2} \subset \cdots \subset \aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i} \subset \cdots$ and the corresponding sequence $\Omega(\aleph_{\mathrm{P}}^n \times \aleph_{\mathrm{P}}^n) \subset \Omega(\aleph_{\mathrm{P}}^{n^2} \times \aleph_{\mathrm{P}}^{n^2}) \subset \cdots \Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph^{n^i} P) \subset \cdots$. For any $i = 1, 2, \ldots$, the restriction of Ω to $\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i} \to \Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i})$ is a continuous bijective operator from a compact space $\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i}$ onto a Hausdorff space $\Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i})$. Hence, by a well-known theorem of topology (Kelly, 1955, p. 141), any such restriction of $\Omega: \aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i} \to \Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i})$ is a homeomorphism; that is, the restriction of Ω^{-1} to $\Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i}) \to \aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i}$ is a continuous bijection. Consider now the restriction of Ω^{-1} to $\Omega(\mathscr{I}_P) \to \mathscr{I}_P$, where

$$\mathscr{I}_P = \bigcup_1^\infty \aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i}, \qquad \Omega(\mathscr{I}_P) = \bigcup_1^\infty \Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i}).$$

Any element of $\Omega(\mathscr{I}_P)$ belongs to all spaces $\Omega(\aleph_{\mathrm{P}}^{n^i} \times \aleph_{\mathrm{P}}^{n^i})$ beginning with some value of i, and in all these spaces the corresponding restrictions of Ω^{-1} are continuous. As a result, the restriction of Ω^{-1} to $\Omega(\mathscr{I}_P) \to \mathscr{I}_P$ is continuous on $\Omega(\mathscr{I}_P)$.

Suppose now that Ω^{-1} maps some pair of (nonprimitive) r.v.'s $(\mathbf{A}, \mathbf{B})$ into two distinct unordered pairs $(\mathbf{X}_1, \mathbf{Y}_1)$ and $(\mathbf{X}_2, \mathbf{Y}_2)$. Then in the small neighborhood of $(\mathbf{A}, \mathbf{B})$ one should be able to find two arbitrarily close members of $\Omega(\mathscr{I}_P)$ that map into two distinct members of $\mathscr{I}_P$, which contradicts the continuity of the restriction of Ω^{-1} to $\Omega(\mathscr{I}_P) \to \mathscr{I}_P$. This completes the proof.

THEOREM B2. *For any given pair of r.v.'s* $\mathbf{A}$ *and* $\mathbf{B}$, *there is at most one (unordered) pair of r.v.'s* $\mathbf{X} \perp \mathbf{Y}$, *such that*

$$\begin{array}{l} \mathbf{X} \oplus \mathbf{Y} \overset{\mathrm{d}}{=} \mathbf{A} \\ \max(\mathbf{X}, \mathbf{Y}) \overset{\mathrm{d}}{=} \mathbf{B}. \end{array}$$

Proof. One can verify step-by-step that the entire proof of Lemma B1 applies here with no non-trivial modifications, except that references to Lemma A2 should be replaced by those to Lemma A3. Hence the proof of Theorem B1 applies here in its entirety.

THEOREM B3. *For any given pair of r.v.'s* **A** *and* **B**, *there is at most one (unordered) pair of r.v.'s* $\mathbf{X} \perp \mathbf{Y}$, *such that*

$$\mathbf{X} \oplus \mathbf{Y} \overset{d}{=} \mathbf{A}$$

$$\min(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{B}.$$

Proof. It is easy to see that the induction chain in Lemma B1 could also begin at the maxima, rather than the minima of S_A and S_B (hence also of S_X and S_Y), and proceed in the decreasing, rather than increasing, order of indexation. With such an order-reversion, the proof of Lemma B1 (hence of Theorem B1) applies here with no non-trivial modifications.

The scheme of proof used in Lemma B1 does not work, however, if the two operations are min and max. Fortunately, this is the case when a proof of the uniqueness (that turns out to be of a weaker variety) can easily be obtained by elementary means. We need first one auxiliary result.

DEFINITION B2. Let $F_X(t), F_Y(t)$ be the distribution functions of r.v.'s **X**, **Y**. Point t is called a crossing point of the two distribution functions if for all sufficiently small $\varepsilon > 0$, $[F_X(t+\varepsilon) - F_Y(t-\varepsilon)][F_X(t-\varepsilon) - F_Y(t+\varepsilon)] \leqslant 0$ and either $F_X(t+\varepsilon) - F_Y(t+\varepsilon) \neq 0$ or $F_X(t-\varepsilon) - F_Y(t-\varepsilon) \neq 0$, Let $\{\cdots < t_{-1} < t_0 < t_1 < \cdots\}$ be the sequence of all such crossing points. Functions $G_1(t), G_2(t)$, right-continuous, are called cross-over rearrangements of the distribution functions $F_X(t), F_Y(t)$ if in any interval $[t_i, t_{i+1})$, either $G_1(t) \equiv F_X(t)$, $G_2(t) \equiv F_Y(t)$, or $G_1(t) \equiv F_Y(t)$, $G_2(t) \equiv F_X(t)$. [Obviously, cross-over rearrangements of distribution functions are themselves distribution functions, because they are increasing and right-continuous.] By abuse of language, we will say that r.v.'s with distribution functions $G_1(t)$, $G_2(t)$ are cross-over rearrangements of the r.v.'s **X**, **Y** with distribution functions $F_X(t)$, $F_Y(t)$.

LEMMA B2. *If, for any real t at which the values of distribution functions $F_X(t)$, $F_Y(t)$ exist, these values are determined uniquely as an unordered pair (i.e., up to their interchange), then $F_X(t)$ and $F_Y(t)$ are determined uniquely up to cross-over rearrangements.*

Proof. Observe that $F_X(t)$ and $F_Y(t)$ are non-decreasing and that in any interval (t_i, t_{i+1}) between two crossing points, either $F_X(t) > F_Y(t)$ or $F_X(t) < F_Y(t)$. Therefore, if the entire segments of $F_X(t)$ and $F_Y(t)$ on (t_i, t_{i+1}) are interchanged, the functions will remain non-decreasing. However, any partial interchange in (t_i, t_{i+1}) will necessarily create a point of decrease in one of the functions. This completes the proof.

THEOREM B4. *For any given pair of r.v.'s* **A** *and* **B**, *there is at most one (up to cross-over rearrangements) pair of r.v.'s* $\mathbf{X} \perp \mathbf{Y}$, *such that*

$$\max(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{A}$$

$$\min(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{B}.$$

Proof. This system of distributional equations is equivalent to the following numerical equations relating the distribution functions of the r.v.'s involved:

$$F_X(t)\, F_Y(t) = F_A(t)$$

$$[1 - F_X(t)][1 - F_Y(t)] = 1 - F_B(t).$$

Their solutions, for any given t, are unique up to an interchange of $F_X(t)$ and $F_Y(t)$ values, and the proof is completed by applying Lemma B2.

APPENDIX C: UNIQUENESS OF SIMPLE OPERATIONS UNDER PERFECT POSITIVE INTERDEPENDENCE

DEFINITION C1. The quantile function $Q(p)$ $(0 \leqslant p \leqslant 1)$ for a r.v. **X** is defined as $\inf\{t\colon F(t) > p\}$, where $F(t)$ is the distribution function for **X**. [This makes quantile functions right-continuous, with a countable number of discontinuities of the first kind.] The definition of cross-over rearrangements of the quantile functions $Q_X(p)$, $Q_Y(p)$ for r.v.'s **X**, **Y** is analogous to Definition B2. [Obviously, cross-over rearrangements of quantile functions correspond to cross-over rearrangements of the corresponding distribution functions. We may continue, therefore, to speak of the cross-over rearrangements of r.v.'s without mentioning either quantile or distribution functions.]

LEMMA C1. *If, for any p $(0 \leqslant p \leqslant 1)$ at which the values of quantile functions $Q_X(p)$, $Q_Y(p)$ exist, these values are determined uniquely as an unordered pair (i.e., up to their interchange), then $Q_X(p)$ and $Q_Y(p)$ are determined uniquely up to cross-over rearrangements.*

Proof. The same as for Lemma B2.

THEOREM C1. *If $g_\otimes$ is strictly convex (strictly concave) with respect to $g_\oplus$, then for any given pair of r.v.'s* **A** *and* **B**, *there is at most one (up to cross-over rearrangements) pair of r.v.'s* $\mathbf{X} \parallel \mathbf{Y}$, *such that*

$$\mathbf{X} \oplus \mathbf{Y} \overset{d}{=} \mathbf{A}$$

$$\mathbf{X} \otimes \mathbf{Y} \overset{d}{=} \mathbf{B}.$$

Proof. The distributional equations can immediately be rewritten as numerical ones with quantile functions,

$$\begin{aligned} Q_X(p) \oplus Q_Y(p) &= Q_A(p) \\ Q_X(p) \otimes Q_Y(p) &= Q_B(p), \end{aligned} \quad 0 \leqslant p \leqslant 1,$$

whose solutions, by Lemma A2, are unique up to an interchange of $Q_X(p)$ and $Q_Y(p)$ values. The proof is completed by applying Lemma C1.

THEOREM C2. *For any given pair of r.v.'s* **A** *and* **B**, *there is at most one (up to cross-over rearrangements) pair of r.v.'s* $\mathbf{X} \parallel \mathbf{Y}$, *such that*

$$\mathbf{X} \oplus \mathbf{Y} \overset{d}{=} \mathbf{A}$$

$$\max(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{B}.$$

The same is true for systems

$$\begin{cases} \mathbf{X} \oplus \mathbf{Y} \overset{d}{=} \mathbf{A} \\ \min(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{B}, \end{cases} \quad \text{and} \quad \begin{cases} \max(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{A} \\ \min(\mathbf{X}, \mathbf{Y}) \overset{d}{=} \mathbf{B}. \end{cases}$$

Proof. The same as for Theorem C1, except that uniqueness up to interchanges is now justified by Lemma A3.

APPENDIX D: EXISTENCE OF DECOMPOSITIONS UNDER PERFECT POSITIVE INTERDEPENDENCE

Here and in the next appendix we will need some auxiliary algebraic results.

LEMMA D1. *Any four reals* $c_{11}, c_{22}, c_{12}, c_{21}$ *such that* $c_{11} + c_{22} = c_{12} + c_{21}$ *can be additively decomposed as* $c_{ij} = a_i + b_j$ $(i = 1, 2;\ j = 1, 2)$, *all decomposing* $\{a_1, a_2, b_1, b_2\}$*-quadruples being obtained by arbitrarily choosing* a_1 *and putting*

$$a_2 = c_{21} - c_{11} + a_1$$

$$b_1 = c_{11} - a_1$$

$$b_2 = c_{12} - a_1.$$

If $c_{11}, c_{22}, c_{12}, c_{21}$ *are all non-negative (non-positive), then there exist non-negative (non-positive) decomposing* $\{a_1, a_2, b_1, b_2\}$*-quadruples: all such quadruples are obtained by restricting* a_1 *to the closed interval* $[\max\{c_{11} - c_{21}, 0\}, \min\{c_{11}, c_{12}\}]$ *(if non-negative) or the closed interval* $[\max\{c_{11}, c_{12}\}, \min\{c_{11} - c_{21}, 0\}]$ *(if non-positive).*

Proof. The system determinant of the linear equations (with respect to a_1, a_2, b_1, b_2)

$$c_{11} = a_1 + b_1$$

$$c_{12} = a_1 + b_2$$

$$c_{21} = a_2 + b_1$$

$$c_{22} = a_2 + b_2$$

is zero, and the equations are mutually consistent because $c_{11} + c_{22} = c_{12} + c_{21}$. All real solutions of this system, therefore, can be presented as

$$a_1 \text{ is arbitrary}$$

$$a_2 = c_{21} - c_{11} - a_1$$

$$b_1 = c_{11} - a_1$$

$$b_2 = c_{12} - a_1.$$

If $c_{11}, c_{22}, c_{12}, c_{21}$ are all non-negative, then by subjecting the four right-hand expressions above to nonnegativity constraints, and solving all the inequalities for a_1, one gets

$$a_1 \geqslant 0$$

$$a_1 \geqslant c_{11} - c_{21}$$

$$a_1 \leqslant c_{11}$$

$$a_1 \leqslant c_{12},$$

which is equivalent to $\max\{c_{11} - c_{21}, 0\} \leqslant a_1 \leqslant \min\{c_{11}, c_{12}\}$. This proves that all non-negative solutions that exist are obtained as stated. To prove that they do exist, one has to show that $\max\{c_{11} - c_{21}, 0\} \leqslant \min\{c_{11}, c_{12}\}$. This inequality is equivalent to the system

$$c_{11} \geqslant 0$$

$$c_{12} \geqslant 0$$

$$c_{11} \geqslant c_{11} - c_{21}$$

$$c_{12} \geqslant c_{11} - c_{21},$$

in which the first three inequalities hold trivially (because c_{12}, c_{21}, c_{11} are all non-negative), and the fourth inequality holds because $(c_{12} + c_{21}) - c_{11} = c_{22} \geqslant 0$. The non-positive case is considered analogously. This completes the proof.

LEMMA D2. *If* $c_{11} + c_{22} = c_{12} + c_{21}$, $\tilde{c}_{11} + \tilde{c}_{22} = \tilde{c}_{12} + \tilde{c}_{21}$, *and* $\tilde{c}_{ij} \geqslant c_{ij}$ $(i = 1, 2;\ j = 1, 2)$, *then the decomposing quadruples* $\{a_1, a_2, b_1, b_2\}$ *and* $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ *of Lemma D1 can always be chosen so that* $\tilde{a}_i \geqslant a_i$, $\tilde{b}_i \geqslant b_i$ $(i = 1, 2)$.

Proof. By Lemma D1, we have to prove that one can always choose $\tilde{a}_1 \geqslant a_1$ so that

$$\begin{aligned} \tilde{c}_{21} - \tilde{c}_{11} + \tilde{a}_1 &\geqslant c_{21} - c_{11} + a_1 \\ \tilde{c}_{11} - \tilde{a}_1 &\geqslant c_{11} - a_1 \\ \tilde{c}_{12} - \tilde{a}_1 &\geqslant c_{12} - a_1, \end{aligned}$$

and that, under non-negativity or non-positivity constraints, both $\tilde{a}_1$ and a_1 belong to the appropriate intervals. It is easy to see that the inequalities above are satisfied iff

$$\begin{aligned} &\max\{(\tilde{c}_{11} - c_{11}) - (\tilde{c}_{21} - c_{21}), 0\} \\ &\qquad \leqslant \tilde{a}_1 - a_1 \leqslant \min\{\tilde{c}_{11} - c_{11}, \tilde{c}_{12} - c_{12}\}, \end{aligned}$$

the interval being non-empty because

$$\begin{aligned} &\tilde{c}_{11} - c_{11} \geqslant 0 \\ &\tilde{c}_{12} - c_{12} \geqslant 0 \\ &\tilde{c}_{11} - c_{11} \geqslant (\tilde{c}_{11} - c_{11}) - (\tilde{c}_{21} - c_{21}) \\ &\tilde{c}_{12} - c_{12} \geqslant (\tilde{c}_{12} - c_{12}) - (\tilde{c}_{22} - c_{22}) \\ &\qquad\qquad = (\tilde{c}_{11} - c_{11}) - (\tilde{c}_{21} - c_{21}). \end{aligned}$$

In the non-negativity case we sum the two double-inequalities

$$\begin{aligned} \max\{(\tilde{c}_{11} - c_{11}) - (\tilde{c}_{21} - c_{21}), 0\} \leqslant \tilde{a}_1 - a_1 & \\ \leqslant \min\{\tilde{c}_{11} - c_{11}, \tilde{c}_{12} - c_{12}\} & \\ \max\{c_{11} - c_{21}, 0\} \leqslant a_1 \leqslant \min\{c_{11}, c_{12}\} & \end{aligned}$$

and observe that

$$\begin{aligned} &\tilde{a}_1 \leqslant \min\{\tilde{c}_{11} - c_{11}, \tilde{c}_{12} - c_{12}\} + \min\{c_{11}, c_{12}\} \\ &\quad \leqslant \min\{\tilde{c}_{11}, \tilde{c}_{12}\} \\ &\tilde{a}_1 \geqslant \max\{c_{11} - c_{21}, 0\} \\ &\quad + \max\{(\tilde{c}_{11} - c_{11}) - (\tilde{c}_{21} - c_{21}), 0\} \\ &\quad \geqslant \max\{\tilde{c}_{11} - \tilde{c}_{21}, 0\}, \end{aligned}$$

that is, $\tilde{a}_1 \in [\max\{\tilde{c}_{11} - \tilde{c}_{21}, 0\}, \min\{\tilde{c}_{11}, \tilde{c}_{12}\}]$, as required by Lemma D1. The non-positivity case is considered analogously. This completes the proof.

LEMMA D3. *Any four* $c_{11}, c_{22}, c_{12}, c_{21} \in I \subseteq \mathrm{Re}$ *such that* $\max\{c_{11}, c_{22}\} = \max\{c_{12}, c_{21}\}$ *can be decomposed as* $c_{ij} = \max\{a_i, b_j\}$ $(i = 1, 2; j = 1, 2)$, *with* $a_1, a_2, b_1, b_2 \in I$. *In particular, the* $\{a_1, a_2, b_1, b_2\}$*-quadruples can be obtained as*

$$\begin{cases} a_1 = c_{11} = c_{12} \\ b_1 = c_{21} \\ b_2 = c_{22} \\ a_2 \leqslant b_1, b_2 \quad (a_2 \in I), \end{cases} \qquad \begin{cases} b_1 = c_{11} = c_{21} \\ a_1 = c_{12} \\ a_2 = c_{22} \\ b_2 \leqslant a_1, a_2 \quad (b_2 \in I), \end{cases}$$

$$\begin{cases} b_2 = c_{22} = c_{12} \\ a_1 = c_{11} \\ a_2 = c_{21} \\ b_1 \leqslant a_1, a_2 \quad (b_1 \in I), \end{cases} \qquad \begin{cases} a_2 = c_{22} = c_{21} \\ b_1 = c_{11} \\ b_2 = c_{12} \\ a_1 \leqslant b_1, b_2 \quad (a_1 \in I), \end{cases}$$

corresponding to, respectively,

$$\begin{cases} \max\{c_{11}, c_{22}\} = c_{11} \\ \max\{c_{12}, c_{21}\} = c_{12}, \end{cases} \qquad \begin{cases} \max\{c_{11}, c_{22}\} = c_{11} \\ \max\{c_{12}, c_{21}\} = c_{21}, \end{cases}$$

$$\begin{cases} \max\{c_{11}, c_{22}\} = c_{22} \\ \max\{c_{12}, c_{21}\} = c_{12}, \end{cases} \qquad \begin{cases} \max\{c_{11}, c_{22}\} = c_{22} \\ \max\{c_{12}, c_{21}\} = c_{21}. \end{cases}$$

The theorem also holds for max being replaced with min everywhere, provided that $\leqslant$ *is replaced with* $\geqslant$ *in the last lines of the expressions for* a_1, a_2, b_1, b_2.

Proof. Obtained by direct verification.

LEMMA D4. *If* $\max\{c_{11}, c_{22}\} = \max\{c_{12}, c_{21}\}$, $\max\{\tilde{c}_{11}, \tilde{c}_{22}\} = \max\{\tilde{c}_{12}, \tilde{c}_{21}\}$, *and* $\tilde{c}_{ij} \geqslant \tilde{c}_{ij}$ $(i = 1, 2;\ j = 1, 2)$, *then the decomposing quadruples* $\{a_1, a_2, b_1, b_2\}$ *and* $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ *of Lemma D3 can always be chosen so that* $\tilde{a}_i \geqslant a_i$, $\tilde{b}_i \geqslant b_i$ $(i = 1, 2)$. *The theorem also holds for max being replaced with min.*

Proof. Without loss of generality, assume that

$$\begin{aligned} \max\{c_{11}, c_{22}\} &= c_{11} \\ \max\{c_{12}, c_{21}\} &= c_{12} \end{aligned}$$

and consider the four possibilities

$$\underset{(1)}{\begin{cases} \max\{\tilde{c}_{11}, \tilde{c}_{22}\} = \tilde{c}_{11} \\ \max\{\tilde{c}_{12}, \tilde{c}_{21}\} = \tilde{c}_{12}, \end{cases}} \qquad \underset{(2)}{\begin{cases} \max\{\tilde{c}_{11}, \tilde{c}_{22}\} = \tilde{c}_{11} \\ \max\{\tilde{c}_{12}, \tilde{c}_{21}\} = \tilde{c}_{21}, \end{cases}}$$

$$\underset{(3)}{\begin{cases} \max\{\tilde{c}_{11}, \tilde{c}_{22}\} = \tilde{c}_{22} \\ \max\{\tilde{c}_{12}, \tilde{c}_{21}\} = \tilde{c}_{12}, \end{cases}} \qquad \underset{(4)}{\begin{cases} \max\{\tilde{c}_{11}, \tilde{c}_{22}\} = \tilde{c}_{22} \\ \max\{\tilde{c}_{12}, \tilde{c}_{21}\} = \tilde{c}_{21}. \end{cases}}$$

In case (1), by Lemma D3, one can put

$$\begin{cases} a_1 = c_{11} = c_{12} \\ b_1 = c_{21} \\ b_2 = c_{22} \\ a_2 \leqslant b_1, b_2 \quad (a_2 \in I), \end{cases} \qquad \begin{cases} \tilde{a}_1 = \tilde{c}_{11} = \tilde{c}_{12} \\ \tilde{b}_1 = \tilde{c}_{21} \\ \tilde{b}_2 = \tilde{c}_{22} \\ \tilde{a}_2 \leqslant \tilde{b}_1, \tilde{b}_2 \quad (a_2 \in I), \end{cases}$$

and the statement of the theorem holds trivially: $a_1 \leqslant \tilde{a}_1$, $b_1 \leqslant \tilde{b}_1$, $b_2 \leqslant \tilde{b}_2$, and one can always choose $\tilde{a}_2$ between a_2 and $\min\{\tilde{b}_1, \tilde{b}_2\} \geqslant \min\{b_1, b_2\}$. In case (2), by Lemma D3, one can put

$$\begin{cases} a_1 = c_{11} = c_{12} \\ b_1 = c_{21} \\ b_2 = c_{22} \\ a_2 \leqslant b_1, b_2 \quad (a_2 \in I), \end{cases} \qquad \begin{cases} \tilde{b}_1 = \tilde{c}_{11} = \tilde{c}_{21} \\ \tilde{a}_1 = \tilde{c}_{21} \\ \tilde{a}_2 = \tilde{c}_{22} \\ \tilde{b}_2 \leqslant \tilde{a}_1, \tilde{a}_2 \quad (b_2 \in I), \end{cases}$$

and observe that $a_1 \leqslant \tilde{a}_1$ (because $c_{12} \leqslant \tilde{c}_{12}$), $b_1 \leqslant \tilde{b}_1$ (because $c_{21} \leqslant \tilde{c}_{21}$), and $a_2 \leqslant \tilde{a}_2$ (because $a_2 \leqslant b_2 = c_{22} \leqslant \tilde{c}_{22} = \tilde{a}_2$). Obviously, $b_2 \leqslant \tilde{a}_2$, and also $b_2 \leqslant \tilde{a}_1$ because otherwise we would have $c_{12} \geqslant c_{22} > \tilde{c}_{12}$; hence $b_2 \leqslant \min\{\tilde{a}_1, \tilde{a}_2\}$, and one can always choose $\tilde{b}_2$ between these values. The remaining cases are considered analogously, and the theorem is transferred to min trivially. This completes the proof.

THEOREM D1. *Let $\oplus$ be an addition-like operation on $I \subseteq \mathrm{Re}$, let $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ be r.v.'s with spectra in I, and let*

$$\mathbf{T}_{11} \oplus \mathbf{T}_{22} \stackrel{\mathrm{d}}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \quad (\mathbf{T}_{11} \,\|\, \mathbf{T}_{22}, \mathbf{T}_{12} \,\|\, \mathbf{T}_{21}).$$

Then $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ can be decomposed as

$$\mathbf{T}_{ij} \stackrel{\mathrm{d}}{=} \mathbf{A}_i \oplus \mathbf{B}_j \quad (\mathbf{A}_i \,\|\, \mathbf{B}_j; \mathbf{i} = 1, 2, \mathbf{j} = 1, 2),$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ are r.v.'s with spectra in I. The quantile functions of $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ are all continuous at every quantile level at which the quantile functions of $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ are all continuous.

Proof. The assumptions of the theorem can be presented as

$$g_\oplus[Q_{11}(p)] + g_\oplus[Q_{22}(p)] \equiv g_\oplus[Q_{12}(p)] + g_\oplus[Q_{21}(p)], \quad 0 \leqslant p \leqslant 1,$$

where $Q_{ij}(p) \in I$ are the quantille functions of the r.v.'s $\mathbf{T}_{ij}$ $(i = 1, 2,\ j = 1, 2)$. We have to prove that there exist quantille functions $A_1(p)$, $A_2(p)$, $B_1(p)$, $B_2(p) \in I$ such that

$$\left.\begin{aligned} g_\oplus[Q_{11}(p)] &\equiv g_\oplus[A_1(p)] + g_\oplus[B_1(p)] \\ g_\oplus[Q_{12}(p)] &\equiv g_\oplus[A_1(p)] + g_\oplus[B_2(p)] \\ g_\oplus[Q_{21}(p)] &\equiv g_\oplus[A_2(p)] + g_\oplus[B_1(p)] \\ g_\oplus[Q_{22}(p)] &\equiv g_\oplus[A_2(p)] + g_\oplus[B_2(p)], \end{aligned}\right\} \quad 0 \leqslant p \leqslant 1.$$

Obviously, the only two constraints imposed on $g_\oplus[A_1(p)]$, $g_\oplus[A_2(p)]$, $g_\oplus[B_1(p)]$, $g_\oplus[B_2(p)]$ are that they are non-decreasing and that their values belong to $g_\oplus(I)$. By Lemma A1, $g_\oplus(I)$ can only be $(-\infty, 0)$, $(0, \infty)$, or $(-\infty, \infty)$. In any of these three cases Lemmas D1 and D2 guarantee the existence of additive decompositions of $g_\oplus[Q_{ij}(p)]$-functions (non-strictly) increasing with p while remaining within $g_\oplus(I)$. To prove the continuity part of the theorem, observe that by Lemma D2, for any $p < \tilde{p}$,

$$\begin{aligned} \max\{&(g_\oplus[Q_{11}(\tilde{p})] - g_\oplus[Q_{11}(p)]) \\ &- (g_\oplus[Q_{21}(\tilde{p})] - g_\oplus[Q_{21}(p)]), 0\} \\ &\leqslant g_\oplus[A_1(\tilde{p})] - g_\oplus[A_1(p)] \\ &\leqslant \min\{g_\oplus[Q_{11}(\tilde{p})] - g_\oplus[Q_{11}(p)], \\ &\qquad g_\oplus[Q_{12}(\tilde{p})] - g_\oplus[Q_{12}(p)]\}. \end{aligned}$$

If all four $g_\oplus[Q_{ij}(p)]$-functions are continuous at some point $\bar{p}$ between p and $\tilde{p}$, then both the left-hand and the right-hand expressions vanish as $p \to \bar{p}$ or $\tilde{p} \to \bar{p}$, forcing $g_\oplus[A_1(p)]$ to be continuous at $\bar{p}$. The functions

$$\begin{aligned} g_\oplus[A_2(p)] &\equiv g_\oplus[Q_{21}(p)] - g_\oplus[Q_{11}(p)] + g_\oplus[A_1(p)] \\ g_\oplus[B_1(p)] &\equiv g_\oplus[Q_{11}(p)] - g_\oplus[A_1(p)] \\ g_\oplus[B_2(p)] &\equiv g_\oplus[Q_{12}(p)] - g_\oplus[A_1(p)], \end{aligned}$$

are then continuous at $\bar{p}$, too. The proof is complete.

THEOREM D2. *Let $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ be r.v.'s with spectra in I, and let*

$$\max\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \stackrel{\mathrm{d}}{=} \max\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \,\|\, \mathbf{T}_{22}, \mathbf{T}_{12} \,\|\, \mathbf{T}_{21}).$$

Then $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ can be decomposed as

$$\mathbf{T}_{ij} \stackrel{\mathrm{d}}{=} \max\{\mathbf{A}_i, \mathbf{B}_j\} \quad (\mathbf{A}_i \,\|\, \mathbf{B}_j; i = 1, 2, j = 1, 2),$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ are r.v.'s with spectra in I. $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ can be chosen so that their quantile functions are all continuous at every quantile level at which the quantile functions of $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ are all continuous. The theorem also holds for max being replaced with min.

Proof. The assumptions of the theorem can be presented as

$$\max\{Q_{11}(p), Q_{22}(p)\} \equiv \max\{Q_{12}(p), Q_{21}(p)\}, \quad 0 \leqslant p \leqslant 1,$$

where $Q_{ij}(p) \in I$ are the quantile functions of the r.v.'s $\mathbf{T}_{ij}$ ($i=1, 2$, $j=1, 2$). We have to prove that there exist quantile (i.e., non-decreasing) functions $A_1(p)$, $A_2(p)$, $B_1(p)$, $B_2(p) \in I$, such that

$$\left.\begin{array}{l} Q_{11}(p) \equiv \max\{A_1(p), B_1(p)\} \\ Q_{12}(p) \equiv \max\{A_1(p), B_2(p)\} \\ Q_{21}(p) \equiv \max\{A_2(p), B_1(p)\} \\ Q_{22}(p) \equiv \max\{A_2(p), B_2(p)\}, \end{array}\right. \quad 0 \leqslant p \leqslant 1.$$

In this formulation, the proof immediately follows from Lemmas D3 and D4. The continuity statement immediately follows from the fact that if all $Q_{ij}(p)$-functions are continuous at some point p, then at least one of the four patterns

$$\begin{cases} \max\{Q_{11}(p), Q_{22}(p)\} \equiv Q_{11}(p) \\ \max\{Q_{12}(p), Q_{21}(p)\} \equiv Q_{12}(p), \end{cases}$$

$$\begin{cases} \max\{Q_{11}(p), Q_{22}(p)\} \equiv Q_{11}(p) \\ \max\{Q_{12}(p), Q_{21}(p)\} \equiv Q_{21}(p), \end{cases}$$

$$\begin{cases} \max\{Q_{11}(p), Q_{22}(p)\} \equiv Q_{22}(p) \\ \max\{Q_{12}(p), Q_{21}(p)\} \equiv Q_{12}(p), \end{cases}$$

$$\begin{cases} \max\{Q_{11}(p), Q_{22}(p)\} \equiv Q_{22}(p) \\ \max\{Q_{12}(p), Q_{12}(p)\} \equiv Q_{21}(p), \end{cases}$$

must hold in some neighborhood of p once it holds at p. The proof for min being the same, the theorem is proved.

APPENDIX E: EXISTENCE OF DECOMPOSITIONS UNDER INDEPENDENCE

THEOREM E1. *Let $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ be r.v.'s with spectra in I, and let*

$$\max\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \stackrel{\mathrm{d}}{=} \max\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}).$$

Then $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ can be decomposed as

$$\mathbf{T}_{ij} \stackrel{\mathrm{d}}{=} \max\{\mathbf{A}_i, \mathbf{B}_j\} \quad (\mathbf{A}_i \perp \mathbf{B}_j;\ i=1, 2, j=1, 2),$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ are r.v.'s (right-complete, but not necessarily left-complete) with spectra in I; however, at least one of the pairs $(\mathbf{A}_1, \mathbf{A}_2)$ or $(\mathbf{B}_1, \mathbf{B}_2)$ is a pair of complete r.v.'s. The distribution functions of $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ are all continuous at every point of I at which the distribution functions of $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ are all continuous.

Proof. The assumptions of the theorem can be presented as

$$\log F_{11}(t) + \log F_{22}(t)\} \equiv \log F_{12}(t) + \log F_{21}(t), \quad t \in I,$$

where $F_{ij}(t)$ are the distribution functions of the r.v.'s $\mathbf{T}_{ij}$ ($i=1, 2, j=1, 2$). We have to prove that there exist distribution functions $A_1(t)$, $A_2(t)$, $B_1(t)$, $B_2(t)$, $t \in I$, generally left-incomplete, such that

$$\left.\begin{array}{l} \log F_{11}(t) \equiv \log A_1(t) + \log B_1(t) \\ \log F_{12}(t) \equiv \log A_1(t) + \log B_2(t) \\ \log F_{21}(t) \equiv \log A_2(t) + \log B_1(t) \\ \log F_{22}(t) \equiv \log A_2(t) + \log B_2(t) \end{array}\right. \quad t \in I.$$

The only constraints imposed on a log-distribution function on an interval I are that this function is non-decreasing and non-positive, that it converges to 0 as $t \to \sup I$ (if it is right-complete), and that it converges to $-\infty$ as $t \to \inf I$ (if it is left-complete). It immediately follows from Lemmas D1 and D2 that non-decreasing and non-positive functions $\log A_1(t)$, $\log A_2(t)$, $\log B_1(t)$, $\log B_2(t)$ can indeed be found that satisfy the decomposition identities above. Specifically, $\log A_1(t)$ is a non-decreasing function such that

$$\max\{\log F_{11}(t), \log F_{12}(t)\} \leqslant \log A_1(t) \leqslant \min\{\log F_{11}(t) - \log F_{21}(t), 0\},$$

whereas

$$\log A_2(t) = \log F_{21}(t) - \log F_{11}(t) + \log A_1(t)$$
$$\log B_1(t) = \log F_{11}(t) - \log A_1(t)$$
$$\log B_2(t) = \log F_{12}(t) - \log A_1(t).$$

Clearly, since $\log F_{ij}(t)$-functions vanish as $t \to \sup I$, $\log A_1(t)$, $\log A_2(t)$, $\log B_1(t)$, $\log B_2(t)$ vanish, too, which means that $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$ are all right-complete. As $t \to \inf I$, however, even though $\log F_{ij}(t)$-functions tend to $-\infty$, it cannot be excluded that some of the functions $\log A_1(t)$, $\log A_2(t)$, $\log B_1(t)$, $\log B_2(t)$ converge to finite values. At the same time, if both $\log A_1(t)$ and $\log A_2(t)$ have finite limits, then both $\log B_1(t)$ and $\log B_2(t)$ tend to $-\infty$, and conversely, if both $\log B_1(t)$ and $\log B_2(t)$ have finite limits, then both $\log A_1(t)$ and $\log A_2(t)$ tend to $-\infty$. This immediately follows from algebraic identities

$$\begin{aligned}\log B_1(t) &\equiv \log F_{11}(t) - \log A_1(t)\\ &\equiv \log F_{21}(t) - \log A_2(t)\\ \log B_2(t) &\equiv \log F_{12}(t) - \log A_1(t)\\ &\equiv \log F_{22}(t) - \log A_2(t),\\ \log A_1(t) &\equiv \log F_{11}(t) - \log B_1(t)\\ &\equiv \log F_{12}(t) - \log B_2(t)\\ \log A_2(t) &\equiv \log F_{21}(t) - \log B_1(t)\\ &\equiv \log F_{22}(t) - \log B_2(t),\end{aligned}$$

proving that either $(\mathbf{A}_1, \mathbf{A}_2)$ or $(\mathbf{B}_1, \mathbf{B}_2)$ is a pair of complete r.v.'s. The continuity part of the theorem is proved as in Theorem D1. The proof is complete.

THEOREM E2. *Let* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *be r.v.'s with spectra in I, and let*

$$\min\{\mathbf{T}_{11}, \mathbf{T}_{22}\} \stackrel{d}{=} \min\{\mathbf{T}_{12}, \mathbf{T}_{21}\} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}).$$

Then $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *can be decomposed as*

$$\mathbf{T}_{ij} \stackrel{d}{=} \min\{\mathbf{A}_i, \mathbf{B}_j\} \quad (\mathbf{A}_i \perp \mathbf{B}_j;\ i=1,2,\ j=1,2),$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *are r.v.'s (left-complete, but not necessarily right-complete) with spectra in I; however, at least one of the pairs* $(\mathbf{A}_1, \mathbf{A}_2)$ *or* $(\mathbf{B}_1, \mathbf{B}_2)$ *is a pair of complete r.v.'s. The distribution functions of* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *are all continuous at every point of I at which the distribution functions of* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *are all continuous.*

Proof. The proof is the same as that of Theorem E1, except that instead of distribution functions it is conducted in terms of log-survival functions $\log[1-F_{ij}(t)]$ $(i=1,2,\ j=1,2)$, constrained by

$$\begin{aligned}&\log[1-F_{11}(t)] + \log[1-F_{22}(t)]\\ &\quad\equiv \log[1-F_{12}(t)] + \log[1-F_{21}(t)],\end{aligned}$$

and decomposed as

$$\left.\begin{aligned}\log[1-F_{11}(t)] &\equiv \log[1-A_1(t)] + \log[1-B_1(t)]\\ \log[1-F_{12}(t)] &\equiv \log[1-A_1(t)] + \log[1-B_2(t)]\\ \log[1-F_{21}(t)] &\equiv \log[1-A_2(t)] + \log[1-B_1(t)]\\ \log[1-F_{22}(t)] &\equiv \log[1-A_2(t)] + \log[1-B_2(t)],\end{aligned}\right\} t \in I.$$

DEFINITION E1. Let a certain interval $I \subseteq \mathrm{Re}$ be fixed, and let all r.v.'s mentioned below have their spectra in I. A r.v. $\mathbf{X}$ is called an independent additive component of a r.v. $\mathbf{U}$, in which case we write $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{U}$, if $\mathbf{U} \stackrel{d}{=} \mathbf{X} + \mathbf{Y}$ $(\mathbf{X} \perp \mathbf{Y})$, for some r.v. $\mathbf{Y}$. [If $\mathbf{X}$ is degenerate, i.e., $\mathbf{X} = x \in I$ with probability 1, then $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{U}$, for any $\mathbf{U}$; also, for any constant $x \in I$, $\mathbf{U} - x \subseteq^{\text{i.a.}} \mathbf{U}$, provided the spectrum of $\mathbf{U} - x$ is in I.] A r.v. $\mathbf{X}$ is called *irreducible* if it is non-degenerate and cannot be presented as $\mathbf{X} \stackrel{d}{=} \mathbf{X}_1 + \mathbf{X}_2$ $(\mathbf{X}_1 \perp \mathbf{X}_2)$ unless either $\mathbf{X}_1$ or $\mathbf{X}_2$ is degenerate. [An important fact about irreducible r.v.'s is that if $I = \mathrm{Re}$, or I is a closed interval in Re, then the space of all irreducible r.v.'s is everywhere dense (in the sense of weak convergence) in the space of all r.v.'s (Linnik & Ostrovskii, 1972, p. 97, Rusza & Szekely, 1988, p. 161).] Finally, a r.v. $\mathbf{X}$ is called *prime* if $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{U} + \mathbf{V}$ $(\mathbf{U} \perp \mathbf{V})$ implies $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{U}$ or $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{V}$.

LEMMA E1. *In any of the three intervals* $(-\infty, 0)$, $(0, \infty)$, *or* $(-\infty, \infty)$, *one can find a r.v.* $\mathbf{X}$ *that is irreducible (in this interval) but is not prime (in this interval).*

Proof. The lemma is proved by examples discussed or referred to in the main text. Examples for the interval $(-\infty, 0)$ are trivially obtained from those for the interval $(0, \infty)$, by multiplying positive r.v.'s with -1. For the interval $(-\infty, \infty)$ the statement of the lemma also follows from a general non-constructive theorem by Rusza & Szekely (1988, pp. 128–129) that says that no r.v. is prime in $(-\infty, \infty)$.

THEOREM E3. *For any addition-like operation* $\oplus$ *on* $I \subseteq \mathrm{Re}$, *one can find four r.v.'s* $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ *with spectra in I, such that even though*

$$\mathbf{T}_{11} \oplus \mathbf{T}_{22} \stackrel{d}{=} \mathbf{T}_{12} \oplus \mathbf{T}_{21} \quad (\mathbf{T}_{11} \perp \mathbf{T}_{22}, \mathbf{T}_{12} \perp \mathbf{T}_{21}),$$

they cannot be decomposed as

$$\mathbf{T}_{ij} \stackrel{d}{=} \mathbf{A}_i \oplus \mathbf{B}_j \quad (\mathbf{A}_i \perp \mathbf{B}_j;\ i=1,2,\ j=1,2),$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ *are r.v.'s with spectra in I.*

Proof. By Lemma A1, $g_{\oplus}(\mathbf{I})$ is one of the three intervals, $(-\infty, 0)$, $(0, \infty)$, or $(-\infty, \infty)$. Then, by Lemma E1, one can find an irreducible $\mathbf{X}$ in $g_{\oplus}(\mathbf{I})$ that is not prime. This means that one can find r.v.'s $\mathbf{Y}, \mathbf{U}, \mathbf{V}$ with spectra in $g_{\oplus}(\mathbf{I})$, such that

$$\mathbf{X} + \mathbf{Y} \stackrel{d}{=} \mathbf{U} + \mathbf{V} \quad (\mathbf{X} \perp \mathbf{Y}, \mathbf{U} \perp \mathbf{V}),$$

but neither $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{U}$ nor $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{V}$. Putting $\mathbf{X} = g_{\oplus}(\mathbf{T}_{11})$, $\mathbf{Y} = g_{\oplus}(\mathbf{T}_{22})$, $\mathbf{U} = g_{\oplus}(\mathbf{T}_{12})$, $\mathbf{V} = g_{\oplus}(\mathbf{T}_{21})$, assume that, contrary to what the theorem says, there are r.v.'s $g_{\oplus}(\mathbf{A}_1)$, $g_{\oplus}(\mathbf{A}_2)$, $g_{\oplus}(\mathbf{B}_1)$, $g_{\oplus}(\mathbf{B}_2)$ in $g_{\oplus}(\mathbf{I})$, such that

$$g_{\oplus}(\mathbf{T}_{ij}) \stackrel{d}{=} g_{\oplus}(\mathbf{A}_i) + g_{\oplus}(\mathbf{B}_j) \quad (\mathbf{A}_i \perp \mathbf{B}_j;\ i=1,2,\ j=1,2).$$

Since $g_{\oplus}(\mathbf{T}_{11})$ is irreducible, either $g_{\oplus}(\mathbf{A}_1)$ or $g_{\oplus}(\mathbf{B}_1)$ in the decomposition

$$g_{\oplus}(\mathbf{T}_{11}) \stackrel{d}{=} g_{\oplus}(\mathbf{A}_1) + g_{\oplus}(\mathbf{B}_1) \quad (\mathbf{A}_1 \perp \mathbf{B}_1)$$

must be degenerate: say, $g_{\oplus}(\mathbf{A}_1) = a$ (with probability 1), in which case $g_{\oplus}(\mathbf{B}_1) = g_{\oplus}(\mathbf{T}_{11}) - a$. Then the decomposition of $g_{\oplus}(\mathbf{T}_{21})$ can be presented as

$$g_{\oplus}(\mathbf{T}_{21}) \stackrel{d}{=} g_{\oplus}(\mathbf{A}_2) + [g_{\oplus}(\mathbf{T}_{11}) - a] \quad (\mathbf{A}_2 \perp \mathbf{T}_{11}),$$

from which it follows that

$$\mathbf{X} = g_{\oplus}(\mathbf{T}_{11}) \subseteq^{\text{i.a.}} g_{\oplus}(\mathbf{T}_{21}) = \mathbf{V}.$$

As this contradicts the assumption that neither $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{U}$ nor $\mathbf{X} \subseteq^{\text{i.a.}} \mathbf{V}$, the theorem is proved.

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