Dissimilarity Cumulation as a Procedure Correcting for Violations of Triangle Inequality

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Abstract

Dissimilarity on a finite set is a function that assigns to every pair of points (stimuli) a nonengative number vanishing if and only if the two points are identical. For any two points in a finite set, the minimum of the length of all finite chains connecting these points is a quasimetric (i.e., it satisfies all metric axioms except for symmetry). The computation of the quasimetric from a dissimilarity can be viewed as a data-analytic procedure of "correcting" dissimilarities into (quasi)distances, an alternative to nonmetric Multidimensional Scaling: the procedure simply replaces the dissimilarity between any two points with the shortest length across all chains of points connecting them. It is shown in this paper that this procedure can be equivalently described as a series of recursive corrections for violations of the triangle inequality across all triples of points considered in an arbitrary order.

Keywords: asymmetric metric, dissimilarity, metric, Multidimensional Scaling, quasimetric, stimulus space, triangle inequality.

This note deals with the Dissimilarity Cumulation (DC) theory presented in Dzhafarov and Colonius (2007) and Dzhafarov (2008a, b). The central notions of this theory are those of a dissimilarity function on an arbitrary set and of the (generalized) metric it induces by means of the "dissimilarity cumulation" along finite chains of points. Like in the Finsler-geometric tradition from which the DC theory has descended (see, e.g., Shen, 2001), the notion of a metric in it falls between the notions of a conventional, symmetric metric and of a quasimetric (see Dzhafarov, submitted). The nuanced definitions, however, although briefly mentioned below, are not essential for the present paper as it deals with finite sets only. On a finite set, any nonnegative function of pairs of points that vanishes if and only if the two points are identical is a dissimilarity function (although a dissimilarity function on an arbitrary set is defined by two additional properties, mentioned below in footnote 2); and any quasimetric on a finite set is a metric in the sense of the DC theory (although on an arbitrary set a metric in this theory is defined as a quasimetric which is, in addition, "symmetric in the small," as explained below in footnote 3). Consequently in this paper we are able

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to use the relatively well-established term "quasimetric" in all contexts where the general DC theory would refer to a metric which may not be symmetric.[1]

Any data set being finite, the DC theory for finite sets can be viewed as a data-analytic tool alternative to nonmetric Multidimensional Scaling (MDS): rather than seeking a nonlinear transformation of a given set of dissimilarities into a metric of a prescribed kind (usually Euclidean), the DC replaces the dissimilarity value for each ordered pair of points with the shortest length ("cumulated dissimilarity") of a finite chain of points connecting the first element of the pair with the second one (Dzhafarov & Colonius, 2006). This shortest length as a function of pairs of points is a quasimetric, and it is referred to as the quasimetric induced by the dissimilarity function. The purpose of this note is to show that the procedure of computing quasidistances from dissimilarities can also be described in terms of a series of recursive corrections of the dissimilarity values for violations of the triangle inequality.

At each step of this recursive procedure one creates a sequence of ordered triads of points, arbitrary insofar as all triads are included, with repetitions allowed insofar as the sequence is finite, and with the sequence being generally different at different steps. A step is divided into substeps, each substep dealing with one oriented triangle (oriented because the dissimilarities assigned to their sides are generally asymmetric) and consisting in replacing the largest side of the triangle with the sum of two other sides if (and only if) they violate the triangle inequality. At the beginning of the first step the values of the sides of all triangles are the original dissimilarities, whereas each subsequent substep of the same or subsequent steps deals with the sides as computed by the end of the previous substep. The procedure terminates if at some step the triangle inequality is violated at no substep. (Many heuristic shortcuts for this procedure are available, but we are not concerned with computational effectiveness in this note.) The claim is that the procedure does terminate, and that when it terminates all dissimilarities have been replaced with the quasidistances induced by them. If symmetric distances are required, e.g., for the purpose of isometrically embedding a stimulus set into a low-dimensional Euclidean space (see Dzhafarov & Colonius, 2006), then the symmetrization can be achieved in accordance with the general theory of DC, as the sum of the quasidistances "to and fro."

Notation conventions. Let $\mathfrak{S}$ be a set of *stimuli* (*points*), denoted by boldface lowercase letters $\mathbf{x}, \mathbf{y}, \ldots$. A *chain*, denoted by boldface capitals, $\mathbf{X}, \mathbf{Y}, \ldots$, is a finite sequence of points. The set $\bigcup_{k=0}^{\infty} \mathfrak{S}^k$ of all chains with elements in $\mathfrak{S}$ is denoted by $\mathcal{S}$. It contains the empty chain and one-element chains (identified with their elements, so that $\mathbf{x} \in \mathfrak{S}$ is also the chain consisting of $\mathbf{x}$).

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[1]Note, however, that there is considerable variance in the use of metric-related terminology, e.g., the terms "distance" and "quasidistance." In the present paper, metric (quasimetric) always refers to a function, whereas distance (respectively, quasidistance) is understood to mean the value of this function. Thus, (quasi)metric = (quasi)metric function = (quasi)distance function, but one can only use "(quasi)distance" rather than "(quasi)metric" in conjunction with "from $\mathbf{a}$ to $\mathbf{b}$."

Concatenations of two or more chains are presented by concatenations of their symbols, $\mathbf{XY}$, $\mathbf{xYz}$, etc. Binary functions $\mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ (where $\mathbb{R}$ denotes real numbers) are presented as $D\mathbf{xy}$, $M\mathbf{xy}$, $G\mathbf{xy}$, ... Given a chain $\mathbf{X} = \mathbf{x}_1, \ldots \mathbf{x}_n$ and a binary (real-valued) function F, the notation $F\mathbf{X}$ stands for

$$\sum_{i=1}^{n-1} F\mathbf{x}_i \mathbf{x}_{i+1},$$

with the obvious convention that the quantity is zero if n is 1 (one-element chain) or 0 (empty chain).

1.   Preliminaries

As in this note we only deal with finite stimulus sets $\mathfrak{S}$, a *dissimilarity* is any real-valued function $D : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ such that $D\mathbf{aa} = 0$ (zero property) and $D\mathbf{ab} > 0$ (positivity) for any distinct $\mathbf{a}, \mathbf{b} \in \mathfrak{S}$.[2] A dissimilarity need not be symmetric and need not satisfy the triangle inequality.

A dissimilarity M on a finite set $\mathfrak{S}$ is a *quasimetric* if it satisfies the triangle inequality,
$$M\mathbf{ab} + M\mathbf{bc} \geq M\mathbf{ac}$$
for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{S}$.

Definition 1. Given a dissimilarity D on a finite set $\mathfrak{S}$, the *quasimetric G induced by D* is defined as

$$G\mathbf{ab} = \min_{\mathbf{X} \in \mathcal{S}} D\mathbf{aXb},$$

for all $\mathbf{a}, \mathbf{b} \in \mathfrak{S}$.

That G is a quasimetric is easy to prove (see, e.g., Dzhafarov & Colonius, 2006, 2007).[3]

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[2] The other two defining axioms of dissimilarity are the intrinsic uniform continuity and the chain property. Formulated in terms of sequences $\{\mathbf{a}_n\}$, $\{\mathbf{a}'_n,\}$ $\{\mathbf{b}_n\}$, $\{\mathbf{b}'_n\}$ in $\mathfrak{S}$ and sequences of chains $\{\mathbf{X}_n\}$ in $\mathcal{S}$, the intrinsic uniform continuity means that $D\mathbf{a}'_n\mathbf{b}'_n - D\mathbf{a}_n\mathbf{b}_n \to 0$ whenever $D\mathbf{a}_n\mathbf{a}'_n \to 0$ and $D\mathbf{b}_n\mathbf{b}'_n \to 0$; the chain property is that $D\mathbf{a}_n\mathbf{b}_n \to 0$ whenever $D\mathbf{a}_n\mathbf{X}_n\mathbf{b}_n \to 0$. On finite sets (more generally, on uniformly discrete sets, in which no dissimilarity between nonidentical points falls below some positive value) these two properties trivially follow from the positivity and zero properties.

[3] Outside the context of finite or uniformly discrete sets the min in the definition of G should be replaced with inf. G on an arbitrary set is not just a quasimetric, it is also symmetric in the small: i.e., $G\mathbf{a}_n\mathbf{b}_n \to 0$ implies $G\mathbf{b}_n\mathbf{a}_n \to 0$ for any sequences $\{\mathbf{a}_n\}$, $\{\mathbf{b}_n\}$ in $\mathfrak{S}$. This property being critical for topological, uniformity, and geometric considerations (whereas the global symmetry is relatively unimportant), Dzhafarov (submitted) proposed to use the term "metric" for a quasimetric which is symmetric in the small, and to use the term "symmetric metric" for the conventional metric.

The quantity

$$G^*\mathbf{ab} = G\mathbf{ab} + G\mathbf{ba}$$

is a (symmetric) metric on $\mathfrak{S}$. It can also be defined directly:

$$G^*\mathbf{ab} = \min_{\mathbf{X},\mathbf{Y}\in\mathcal{S}} D\mathbf{aXbYa} = \min_{\mathbf{X},\mathbf{Y}\in\mathcal{S}} D\mathbf{bXaYb}.$$

Let us illustrate these computations on a toy example. Let $\mathfrak{S}$ be $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$, and let the dissimilarity function D be

$$\begin{bmatrix}
\underline{D} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 3 & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 7 & 5 & 2 & 0
\end{bmatrix}.$$

To compute $G\mathbf{ab}$ one contemplates all possible chains $\mathbf{aXb}$ and finds out that $D\mathbf{aXb} \geq D\mathbf{ab}$ always. Hence $G\mathbf{ab} = D\mathbf{ab} = 1$. For $\mathbf{ad}$ the situation is different: it is easy to verify that the for any chain $\mathbf{X}$ (including the empty chain), $D\mathbf{aXd} \geq D\mathbf{abcd}$. So we put $G\mathbf{ad} = D\mathbf{abcd} = 3$. Proceeding in this manner for all ordered pairs from $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$ we arrive at the matrices below:

$$\begin{bmatrix}
\underline{\mathbf{X}} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & \cdot & \cdot & \mathbf{b} & \mathbf{bc} \\
\mathbf{b} & \cdot & \cdot & \cdot & \mathbf{c} \\
\mathbf{c} & \cdot & \cdot & \cdot & \cdot \\
\mathbf{d} & \mathbf{c} & \mathbf{c} & \cdot & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
\underline{G} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 2 & 3 \\
\mathbf{b} & 2 & 0 & 1 & 2 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 6 & 4 & 2 & 0
\end{bmatrix}.$$

The left matrix shows for every $\mathbf{x},\mathbf{y} \in \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$ a chain $\mathbf{X}$ such that $D\mathbf{xXy} = G\mathbf{xy}$ (the dots indicating the empty chain, when $D\mathbf{xy} = G\mathbf{xy}$). The matrix on the right shows the values of G, the quasimetric induced on $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$ by the dissimilarity D.

One of the obvious features of matrix G is that it is not symmetric, which makes it impossible to visualize $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$ as a configuration of points isometrically embedded in a low-dimensional Euclidean space. To achieve the latter we additively symmetrize the quasimetric as described above to obtain

$$\begin{bmatrix}
\underline{G^*} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 3 & 6 & 9 \\
\mathbf{b} & 3 & 0 & 3 & 6 \\
\mathbf{c} & 6 & 3 & 0 & 3 \\
\mathbf{d} & 9 & 6 & 3 & 0
\end{bmatrix}.$$

Figure 1 shows the isometric embedding (by means of metric MDS) of $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$ endowed with G^* into a Euclidean space: clearly a unidimensional configuration.
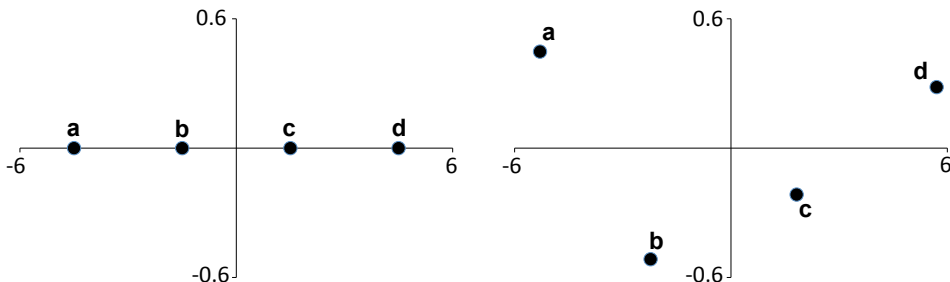
Figure 1: Embeddings of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ of our toy example into a 2D Euclidean space. Left: embedding (by means of metric MDS) that preserves the values of the symmetric distances G^* induced by the dissimilarity function D. Right: embedding (by means of nonmetric MDS) that preserves the order of the additively symmetrized dissimilarities D^*. The stress value is zero in both cases.

For comparison, the figure also shows a nonmetric MDS performed on additively symmetrized initial dissimilarities, $D^*\mathbf{ab} = D\mathbf{ab} + D\mathbf{ba}$.

Clearly, there is no need to try different chains $\mathbf{X}$ for every pair of points if the dissimilarity D is known to be a quasimetric: in this case the triangle inequality implies

$$G\mathbf{ab} = \min_{\mathbf{X} \in \mathcal{S}} \mathbf{aXb} = D\mathbf{ab}.$$

This means that one can always begin by looking at the ordered triads of points only, and determining whether any of them violates the triangle inequality. If there are no violations, then $D \equiv G$. If violations are observed, however, a direct application of the definition of G (and G^*) means that one now has to deal with all possible chains of points rather than with the ordered triads only. Details aside, this is what is done by the software packages designed to perform Fechnerian Scaling, the main psychophysical application of DC.[4]

This procedural discrepancy (all finite chains versus just triads) raises the question of whether the computation of G from D when the two do not coincide can be recast entirely in terms of "correcting" the dissimilarities for the violations of the triangle inequality in ordered triads, i.e., replacing $D\mathbf{ac}$ with $D\mathbf{ab} + D\mathbf{bc}$ in every $\mathbf{abc}$ with $D\mathbf{ac} > D\mathbf{ab} + D\mathbf{bc}$. The answer turns out to be affirmative, provided that the procedure may apply to one and the same triangle repeatedly, with D being gradually redefined until it becomes G.[5] The procedure and the proof that it works are presented next.

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[4]This includes the R-language package "fechner" described in Ünlü, Kiefer, & Dzhafarov, 2009 and available at CRAN; a program FSCAMDS available at http://www1.psych.purdue.edu/~ehtibar/Links.html; and a Matlab toolbox available at http://www.psychologie.uni-oldenburg.de/stefan.rach/31856.html

[5]To prevent misunderstanding, the issue here is not one of computational effectiveness: we are not concerned with whether repeated "corrections" of triangles take less or more time than computations of chain lengths. Rather we pose a question of principle: can one deal with dissimilarities in terms of the triangle inequality alone? Put differently, we are interested in

2. Correcting for Violations of Triangle Inequality

Let $\mathfrak{S}$ contain $k \geq 3$ distinct elements. The cases with $k = 0, 1, 2$ are trivial and need not be considered. Let $\mathfrak{S}_3$ denote the set of $k\,(k-1)\,(k-2)$ ordered triads of pairwise distinct points of $\mathfrak{S}$. The elements of $\mathfrak{S}_3$ will be referred to as ordered triads, or triangles, the pairwise distinctness being implied tacitly.

For $n = 0, 1, \ldots$, let $\mathbf{T}^{(n)}$ denote a sequence of t_n elements of $\mathfrak{S}_3$ which contains the entire $\mathfrak{S}_3$. For each n, we index the triads in $\mathbf{T}^{(n)}$ by double indices $(n, 1), (n, 2), \ldots, (n, t_n)$, and we order all such pairs, for $n = 0, 1, \ldots$, lexicographically: $(n, i)'$, the successor of (n, i), is $(n, i+1)$ if $i < t_n$, so that the (n, i)'th triad is in $\mathbf{T}^{(n)}$, and $(n, t_n)'$ is $(n+1, 1)$, so that the (n, t_n)'th triad is the first one in $\mathbf{T}^{(n+1)}$. In the following we will tacitly assume that the double-indexed sequences $\mathbf{T}^{(n)}$ for $n = 0, 1, \ldots$ are fixed. As stated below in Corollary 1 to Theorem 1, however, one's choice of these sequences is inconsequential for the end result of one's computations.

Definition 2. Let $\mathfrak{S}$ be endowed with a dissimilarity function D. The dissimilarity function $M^{(n,i)}$ for $n = 0, 1, \ldots$ and $i = 1, 2, \ldots, t_n$ is defined by induction as follows. $M^{(0,i)} \equiv D$ for $i = 1, 2, \ldots, t_0$. Let $M^{(n,i)}$ be defined for some $(n, i) \geq (0, t_0)$. Then $M^{(n,i)'}\mathbf{xy} = M^{(n,i)}\mathbf{xy}$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{S}$ except, possibly, for $M^{(n,i)'}\mathbf{ac}$ in the $(n, i)'$th triad $\mathbf{abc}$, which is "corrected" as

$$M^{(n,i)'}\mathbf{ac} = \min\{M^{(n,i)}\mathbf{ac}, M^{(n,i)}\mathbf{ab} + M^{(n,i)}\mathbf{bc}\}.$$

This completes the definition of $M^{(n,i)}$ (referred to as the *corrected dissimilarity function*) for $n = 0, 1, \ldots$ and $i = 1, 2, \ldots, t_n$.

If $M^{(n,i)}$ for some (n, i) is a quasimetric on $\mathfrak{S}$, then we call $M \equiv M^{(n,i)}$ the *terminal corrected dissimilarity function*.

Lemma 1. $M^{(n,i)}$ *is the terminal corrected dissimilarity function for some* $i \leq t_n$ *if and only if* $M^{(n+1,t_{n+1})} \equiv M^{(n,t_n)}$.

Proof. The "only if" part is obvious: $M^{(n,i)}$ being a quasimetric, $M^{(m,j)} \equiv M^{(n,i)}$ for any $(m, j) \geq (n, i)$. To proof the "if" part, it follows from Definition 2 that for any $\mathbf{x}, \mathbf{y} \in \mathfrak{S}$, if $(m, j) \geq (n, i)$ then $M^{(m,j)}\mathbf{xy} \leq M^{(n,i)}\mathbf{xy}$. Hence $M^{(n+1,t_{n+1})} \equiv M^{(n,t_n)}$ implies $M^{(n+1,1)} \equiv M^{(n,t_n)}$, whence $M^{(n,i)}$ contains no violations of the triangle inequality. $\square$

Lemma 2. *For any* $n = 0, 1, \ldots$, *any* $i = 1, 2, \ldots, t_n$, *and any* $\mathbf{a}, \mathbf{b} \in \mathfrak{S}$, *there is a chain* $\mathbf{aXb}$ *such that*

$$M^{(n,i)}\mathbf{ab} = D\mathbf{aXb}.$$

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whether dissimilarities on finite sets can be viewed as "imperfect" quasidistances which can be transformed into true quasidistances by "correcting" them, one by one and recursively, for every violation of the triangle inequality.

Proof. By induction on the lexicographically ordered (n, i). The statement is clearly true for $n = 0$. Let it be true for all double indices up to and including $(n, i) \geq (0, t_0)$. Then, if $\mathbf{abc}$ is the $(n, i)'$th triad, the statement must hold for $M^{(n,i)'}\mathbf{ac}$ whether it equals $M^{(n,i)}\mathbf{ac}$ or $M^{(n,i)}\mathbf{ab} + M^{(n,i)}\mathbf{bc}$; and it must hold for any other $(\mathbf{x}, \mathbf{y})$ because then $M^{(n,i)'}\mathbf{xy} = M^{(n,i)}\mathbf{xy}$. $\qquad\square$

Theorem 1. *A terminal corrected dissimilarity function M exists. It coincides with the quasimetric G induced by the initial dissimilarity function D.*

Proof. Deny the existence of M, and then, by Lemma 1, $M^{(n+1,t_{n+1})}$ and $M^{(n,t_n)}$ do not coincide for all $n = 0, 1, \ldots$. Since $\mathfrak{S}^2$ is finite, there should exist a pair of nonidentical $\mathbf{a}, \mathbf{b} \in \mathfrak{S}$ and an infinite sequence of positive integers $n_1 < n_2 < \ldots$ for which

$$D\mathbf{ab} \neq M^{(n_1, t_{n_1})}\mathbf{ab} \neq M^{(n_2, t_{n_2})}\mathbf{ab} \neq \ldots.$$

It follows then from the definition of $M^{(n,i)}$ that

$$D\mathbf{ab} > M^{(n_1, t_{n_1})}\mathbf{ab} > M^{(n_2, t_{n_2})}\mathbf{ab} > \ldots.$$

This implies, by Lemma 2, that there is an infinite sequence of chains $\mathbf{X}_{n_1}, \mathbf{X}_{n_2}, \ldots$ such that

$$M^{(n_i, t_{n_i})}\mathbf{ab} = D\mathbf{a}\mathbf{X}_{n_i}\mathbf{b}, \; i = 1, 2, \ldots,$$

and

$$D\mathbf{ab} > D\mathbf{a}\mathbf{X}_{n_1}\mathbf{b} > D\mathbf{a}\mathbf{X}_{n_2}\mathbf{b} > \ldots.$$

But this is clearly impossible, as the set of chains in a finite set whose lengths are below a given value is finite. This proves the existence of M. To prove that $M \equiv G$, observe that for any $\mathbf{a}, \mathbf{b} \in \mathfrak{S}$ and any chain $\mathbf{X} \in \mathcal{S}$,

$$D\mathbf{a}\mathbf{X}\mathbf{b} \geq M\mathbf{a}\mathbf{X}\mathbf{b}.$$

Since the triangle inequality holds for M, we have

$$M\mathbf{ab} \leq M\mathbf{a}\mathbf{X}\mathbf{b},$$

whence

$$M\mathbf{ab} \leq D\mathbf{a}\mathbf{X}\mathbf{b}.$$

Combined with Lemma 2 this means that

$$M\mathbf{ab} = \min_{\mathbf{X} \in \mathcal{S}} D\mathbf{a}\mathbf{X}\mathbf{b},$$

and the latter quantity is $G\mathbf{ab}$ by definition. $\qquad\square$

Corollary 1. *The terminal corrected dissimilarity function M does not depend on the choice of the sequences $\mathbf{T}^{(n)}$ for $n = 0, 1, \ldots$.*

Proof. Immediately follows from the fact that $M \equiv G$ and G is uniquely determined by D (as is apparent from Definition 1). □

This concludes the analysis. We have seen that dissimilarities on finite sets can be viewed as "imperfect" quasidistances, and the dissimilarity cumulation procedure can be recast as a transformation of these dissimilarities into true quasidistances by means of a series of recursive corrections for the violations of the triangle inequality. [6]

Let us illustrate the correction procedure on the toy example of the previous section, with $\mathfrak{S} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and the dissimilarity function

$$
\begin{bmatrix}
\underline{D} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 3 & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 7 & 5 & 2 & 0
\end{bmatrix}.
$$

Let us form a single sequence of all ordered triads to be used as $\mathbf{T}^{(n)}$ for $n = 0, 1, \ldots$ (see footnote 6). This can be done, e.g., by cycling through the first element (4 values), subcycling through the last element (3 values), and sub-subcycling through the middle element (2 values), in the alphabetic order. This yields
$$\mathbf{T} = \{\mathbf{acb}, \mathbf{adb}, \mathbf{abc}, \mathbf{adc}, \ldots, \mathbf{dac}, \mathbf{dbc}\}.$$

Putting $\mathbf{T}^{(1)} = \mathbf{T}$ and testing the triangles in the order shown, the first violation of the triangle inequality occurs in the triangle #3, $\mathbf{abc}$:

$$3 = D\mathbf{ac} > D\mathbf{ab} + D\mathbf{bc} = 1 + 1.$$

We "correct" the value of $D\mathbf{ac}$ therefore by replacing 3 with 2 (shown in parentheses in matrix $M^{(1,3)}$ below, the superscript indicating the third triangle in

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[6]The possibilities that the sequence $\mathbf{T}^{(n)}$ of the triads may contain several replications of one and the same triad, and that $\mathbf{T}^{(n)}$ and $\mathbf{T}^{(m)}$ may be different for $m \neq n$, are allowed for the sake of generality only. It is probably always more convenient to use a fixed permutation of the elements of $\mathfrak{S}_3$. Moreover, the requirement that $\mathbf{T}^{(n)}$ should include the entire $\mathfrak{S}_3$ does not mean that all triads have to be examined in every sequence (on every step). It can be seen from the procedure and the proofs that if the triangle inequality is found to be violated in a triad $\mathbf{abc}$, then one may skip all subsequent triads $\mathbf{axc}$ in the same sequence, or even terminate the step altogether and move on to the next one. These observations may help to make the procedure more efficient (but see footnote 5).

the sequence $\mathbf{T}^{(1)} = \mathbf{T}$).

$$
\begin{bmatrix}
\underline{D} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 3 & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 7 & 5 & 2 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\underline{M^{(1,3)}} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & (2) & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 7 & 5 & 2 & 0
\end{bmatrix}
$$

$$
\Rightarrow
\begin{bmatrix}
\underline{M^{(1,20)}} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 2 & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & (6) & 5 & 2 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\underline{M^{(1,22)}} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 2 & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 6 & (4) & 2 & 0
\end{bmatrix} .
$$

The next violation occurs in the triangle #20, $\mathbf{dca}$:

$$
7 = M^{(1,19)}\mathbf{da} > M^{(1,19)}\mathbf{dc} + M^{(1,19)}\mathbf{ca} = 2 + 4,
$$

where $M^{(1,19)} \equiv M^{(1,18)} \equiv \ldots \equiv M^{(1,3)}$. So we correct $M^{(1,19)}\mathbf{da}$ from 7 to 6, as shown in matrix $M^{(1,20)}$. We deal analogously with the third violation of the triangle inequality, in the triangle #22, $\mathbf{dcb}$:

$$
5 = M^{(1,21)}\mathbf{db} > M^{(1,21)}\mathbf{dc} + M^{(1,21)}\mathbf{cb} = 2 + 2,
$$

where $M^{(1,21)} \equiv M^{(1,20)} \equiv M^{(1,19)}$. At the remaining two substeps before the first sequence of the triangles has been exhausted, $M^{(1,23)}$ and $M^{(1,24)}$, no values change, so $M^{(1,24)} \equiv M^{(1,23)} \equiv M^{(1,22)}$ is the matrix submitted to the second step of the procedure.

At the second step, using $\mathbf{T}^{(2)} = \mathbf{T}$, the first and only violation occurs at the triangle #5, $\mathbf{abd}$:

$$
4 = M^{(2,4)}\mathbf{ad} > M^{(2,4)}\mathbf{ab} + M^{(2,4)}\mathbf{bd} = 1 + 2,
$$

where $M^{(2,4)} \equiv \ldots \equiv M^{(2,1)} \equiv M^{(1,24)}$. So the final "correction" occurs in matrix $M^{(2,5)}$, as shown below.

$$
\begin{bmatrix}
\underline{M^{(1,24)}} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 2 & 4 \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 6 & 4 & 2 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\underline{M^{(2,5)}} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\mathbf{a} & 0 & 1 & 2 & (3) \\
\mathbf{b} & 2 & 0 & 1 & 3 \\
\mathbf{c} & 4 & 2 & 0 & 1 \\
\mathbf{d} & 6 & 4 & 2 & 0
\end{bmatrix} .
$$

It is easy to verify that $M^{(2,5)}$ is a quasimetric on $\mathfrak{S} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, so that $M^{(2,6)}$ and higher-indexed corrected dissimilarities remain identical to $M^{(2,5)}$. The latter therefore is the terminal corrected dissimilarity M, and its comparison with the computations made in the previous section shows that it coincides with G, the quasimetric induced by the initial dissimilarity function D.

Acknowledgement

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