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Psychophysics without physics: extension of Fechnerian scaling from continuous to discrete and discrete-continuous stimulus spaces

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Abstract

The computation of subjective (Fechnerian) distances from discrimination probabilities involves cumulation of appropriately transformed psychometric increments along smooth arcs (in continuous stimulus spaces) or chains of stimuli (in discrete spaces). In a space where any two stimuli that are each other's points of subjective equality are given identical physical labels, psychometric increments are positive differences $\psi(\mathbf{x},\mathbf{y})-\psi(\mathbf{x},\mathbf{x})$ and $\psi(\mathbf{y},\mathbf{x})-\psi(\mathbf{x},\mathbf{x})$, where $\mathbf{x}\neq\mathbf{y}$ and ψ is the probability of judging two stimuli different. In continuous stimulus spaces the appropriate monotone transformation of these increments (called overall psychometric transformation) is determined uniquely in the vicinity of zero, and its extension to larger values of its argument is immaterial. In discrete stimulus spaces, however, Fechnerian distances critically depend on this extension. We show that if overall psychometric transformation is assumed (A) to be the same for a sufficiently rich class of discrete stimulus spaces, (B) to ensure the validity of the Second Main Theorem of Fechnerian Scaling in this class of spaces, and (C) to agree in the vicinity of zero with one of the possible transformations in continuous spaces, then this transformation can only be identity. This result is generalized to the broad class of "discrete-continuous" stimulus spaces, of which continuous and discrete spaces are proper subclasses.

Keywords: Discrimination probability; Subjective dissimilarity; Fechnerian scaling; Fechnerian distance; Self-dissimilarity; Regular minimality; Psychometric length; Same-different comparisons; Continuous stimulus space; Discrete stimulus space; Subjective distance

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1. Introduction

Fechnerian Scaling is a measurement procedure by which one can compute interstimulus distances (termed *Fechnerian distances* and interpreted as "subjective dissimilarities") from the probabilities with which "very similar" stimuli are discriminated from each other, in the "same-different" sense. This paper proposes a principled way of extending the theory of Fechnerian Scaling from continuous stimulus spaces to discrete ones. Discrete stimulus spaces are significant for two reasons: (A) stimulus spaces are discrete in a wide variety of applications; (B) discrete spaces serve as a universal tool for performing Fechnerian computations in stimulus spaces of any other kind: empirically, discrimination probabilities can only be estimated on a discrete (moreover, finite) subset of a stimulus space. We will also show how to extend the theory of Fechnerian Scaling to stimulus spaces comprised of continuous "chunks" isolated from each other. This kind of spaces contains discrete and continuous spaces as special cases.

The theory of Fechnerian Scaling was originally developed for stimulus spaces that form open connected regions of a Euclidean space Re^n (i.e., the set of n-dimensional vectors of real numbers endowed with the conventional topology).[1] This Euclidean version of the theory is presented in Dzhafarov and Colonius (1999, 2001) and Dzhafarov (2002a–d; 2003a–c).

Recently (Dzhafarov and Colonius, 2005) Fechnerian Scaling was extended to a much wider variety of "*smoothly connected*" stimulus spaces (intuitively, spaces in which one can "smoothly" move from one point to another). Such spaces are generically referred to as *continuous*. Unlike the Euclidean version in which the theory is based on physical properties of stimuli (such as the n-dimensionality of the space, its Euclidean topology, and the analytic properties of smooth arcs connecting points), the general theory of Fechnerian Scaling for continuous spaces is *purely psychological*: physical measurements of stimuli are only used to *identify* (assign labels to) stimuli, whereas all topological, analytic, and metric properties of a space are defined entirely in terms of the discrimination probabilities on a set of pairs of physical labels. In particular, all propositions of this theory are invariant under arbitrary identity-preserving transformations of the space (relabeling of stimuli).

Another recent extension of Fechnerian Scaling (Dzhafarov & Colonius, submitted) deals with stimulus spaces comprised of isolated from each other points. Such spaces are generically referred to as *discrete*. This theory is also purely psychological, in the same sense: all properties of stimuli (except for their identity)[2] are defined entirely in terms of discrimination probabilities.

The relationship between these two parts of the purely psychological theory of Fechnerian Scaling (continuous and discrete) has not been, however, made sufficiently clear. We know that both of them have vast areas of application, and that discrete stimulus spaces can be used for approximating continuous ones (because psychometric length of an arc is the limit for the lengths of discrete chains of stimuli). A certain critical choice made in the discrete theory, however, seems to be there more arbitrary than in the continuous theory: as explained below, we speak here of the choice of an appropriate transformation of psychometric increments, called the *overall psychometric transformation*. The issue becomes especially apparent when properties of continuous and discrete spaces are combined in more general spaces (introduced in Section 6). The present paper is aimed at filling in this gap.

[1]The term "Euclidean" therefore is not used here to imply the Euclidean or any other specific *metric*.

[2]As will be apparent from the discussion below, this exception should be understood with a caveat: physically different stimuli need not be psychologically distinct and may be, based on properties of discrimination probabilities, "lumped together"; however, a stimulus with a given physical identity cannot have more than one psychological identity.

The familiarity with our previous papers on Fechnerian Scaling (especially, Dzhafarov, 2002d; Dzhafarov and Colonius, 2005) would be helpful but is not strictly necessary.

The structure of the paper: In the first three sections following this introduction we briefly recapitulate some of the notions and facts pertaining to the purely psychological construction of Fechnerian Scaling (based on Dzhafarov and Colonius, 2005). Then we introduce a certain "principle of theory construction," according to which a theory of Fechnerian Scaling should not be restricting the class of possible stimulus spaces "too much," in some well-defined sense. We show that in the case of discrete spaces this principle is compatible only with the variant of their theory proposed in Dzhafarov & Colonius (submitted), while in the case of continuous spaces it is only compatible with the "cross-unbalanced" version of the theory as specified in Dzhafarov (2002d) and Dzhafarov and Colonius (2005). Finally, we show how to extend the Fechnerian theory to a broad class of stimulus spaces we call "discrete-continuous."

Notation: With minor variations we follow the notation conventions adopted in Dzhafarov and Colonius (2005).

Italics and Greek letters designate real-valued quantities, with the exception of indexing variables in the definition of discrete-continuous spaces, in Section 6.

Boldface lowercase letters $\mathbf{x}, \mathbf{y}, \mathbf{a}, \ldots$ always denote stimuli, or functions mapping into a set of stimuli, as in $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$.

Sets of stimuli are denoted by Gothic letters $\mathfrak{M}$, $\mathfrak{m}, \ldots$, and, with a few exceptions (intervals of reals in Section 5, Re and Re^+ for, respectively, the sets of reals and nonnegative reals), we use Gothic letters for numerical sets as well. Open symbol $\mathbb{D}$ is used to denote a certain set of stimulus spaces (Section 5).

Greek ι is reserved to represent "observation area" (as defined in Section 2), its value is always 1 or 2. This symbol, as well as its specific values (1 or 2) are used either as superscripts ($\boldsymbol{\Psi}^{(\iota)}$, $\boldsymbol{\Psi}^{(1)}$, $\boldsymbol{\Psi}^{(2)}$, etc., parenthesized to distinguish them from exponents) or as subscripts (G_ι, G_1, G_2, etc.). The difference between superscripts and subscripts in reference to "observation area" is purely decorative.

We use symbol $\to$ in four different meanings, clearly distinguishable by context: to designate mappings (as in $[a, b] \to \mathfrak{M}$), to designate convergence of real numbers (e.g., $h \to 0$), to designate convergence of stimuli (e.g., $\mathbf{x}_n \to \mathbf{x}$), and to schematically represent an arc or chain of stimuli connecting $\mathbf{a}$ to $\mathbf{b}$ (e.g., $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$).

2. General notions

First we briefly review some of the basic notions of Fechnerian Scaling that apply to both continuous and discrete stimulus spaces (as well as the discrete-continuous spaces introduced in Section 6). For details one should refer to Dzhafarov and Colonius (2005): here we mention only those aspects of the construction that are directly relevant to our present purposes.

2.1. Observation areas and discrimination probabilities

The main experimental paradigm for Fechnerian Scaling is that of the "same-different" judgments in response to pairwise presented stimuli. A discrimination probability function is a function

$$\psi^* : \mathfrak{M}_1^* \times \mathfrak{M}_2^* \to [0, 1]$$

interpreted as

$$\psi^*(\mathbf{x}, \mathbf{y}) = \Pr[\mathbf{x} \in \mathfrak{M}_1^* \text{ and } \mathbf{y} \in \mathfrak{M}_2^* \text{ are judged to be different}], \tag{1}$$

where $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ represent the sets of stimuli presented to the perceiver in two fixed and perceptually distinct *observation areas* (e.g., $\mathfrak{M}_1^*$ may contain all stimuli presented chronologically first or spatially "below," in which cases $\mathfrak{M}_2^*$ contains all stimuli presented, respectively, chronologically second and spatially "above").

The notion of two distinct observation areas is crucial for Fechnerian Scaling. Stimulus sets $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ are usually (but not necessarily) identical in all respects other than the observation area. The judgments "$\mathbf{x}$ and $\mathbf{y}$ are different" and "$\mathbf{x}$ and $\mathbf{y}$ are the same" are supposed to ignore the difference in the observation area. The perceivers may be asked, in addition, to ignore some other properties of stimuli in their judgments, or to pay attention to certain properties only. For a detailed discussion see Dzhafarov and Colonius (2005).

2.2. Psychological identity of stimuli

If, for $\mathbf{x}, \mathbf{x}' \in \mathfrak{M}_1^*$,

$$\psi^*(\mathbf{x}, \mathbf{y}) = \psi^*(\mathbf{x}', \mathbf{y}) \quad \text{for any } \mathbf{y} \in \mathfrak{M}_2^*,$$

we say that stimuli $\mathbf{x}$ and $\mathbf{x}'$ are *psychologically equal* (or indistinguishable). Analogously, $\mathbf{y}, \mathbf{y}' \in \mathfrak{M}_2^*$ are psychologically equal if

$$\psi^*(\mathbf{x}, \mathbf{y}) = \psi^*(\mathbf{x}, \mathbf{y}') \quad \text{for any } \mathbf{x} \in \mathfrak{M}_1^*.$$

We assume in the following that all stimuli forming an equivalence class of psychologically equal stimuli are labelled identically. Thus redefined stimulus spaces for the two observation areas are denoted by $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$, so that $\mathbf{x} \in \tilde{\mathfrak{M}}_1$ (or $\mathbf{y} \in \tilde{\mathfrak{M}}_2$) is in fact an equivalence class of physically distinct stimuli (unless the class is a singleton). We retain notation $\mathbf{x}, \mathbf{y}$ for elements of $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$, and we redefine the discrimination probability function as

$$\tilde{\psi} : \tilde{\mathfrak{M}}_1 \times \tilde{\mathfrak{M}}_2 \to [0, 1],$$

with $\tilde{\psi}(\mathbf{x},\mathbf{y})$ being equal to the value of ψ^* taken for any element of $\mathfrak{M}_1^*$ and any element of $\mathfrak{M}_2^*$ belonging to the equivalence classes $\mathbf{x}$ and $\mathbf{y}$, respectively.

2.3. Regular Minimality

Regular Minimality is the cornerstone principle of Fechnerian Scaling.

Axiom 1 (*Regular Minimality*). *There are functions* $\mathbf{h} : \tilde{\mathfrak{M}}_1 \to \tilde{\mathfrak{M}}_2$ *and* $\mathbf{g} : \tilde{\mathfrak{M}}_2 \to \tilde{\mathfrak{M}}_1$ *such that*

(i) $\tilde{\psi}(\mathbf{x},\mathbf{h}(\mathbf{x})) < \tilde{\psi}(\mathbf{x},\mathbf{y})$ *for all* $\mathbf{x}$ *and* $\mathbf{y} \neq \mathbf{h}(\mathbf{x})$,
(ii) $\tilde{\psi}(\mathbf{g}(\mathbf{y}),\mathbf{y}) < \tilde{\psi}(\mathbf{x},\mathbf{y})$ *for all* $\mathbf{y}$ *and* $\mathbf{x} \neq \mathbf{g}(\mathbf{y})$,
(iii) $\mathbf{h} \equiv \mathbf{g}^{-1}$.

For every $\mathbf{x} \in \tilde{\mathfrak{M}}_1$, $\mathbf{h}(\mathbf{x}) \in \tilde{\mathfrak{M}}_2$ is its *Point of Subjective Equality* (PSE). Analogously, $\mathbf{g}(\mathbf{y}) \in \tilde{\mathfrak{M}}_1$ is the PSE for $\mathbf{y} \in \tilde{\mathfrak{M}}_2$. The principle therefore says that $\mathbf{y}_0$ is the (unique) PSE for $\mathbf{x}_0$ if and only if $\mathbf{x}_0$ is the (unique) PSE for $\mathbf{y}_0$:

$$\mathbf{y}_0 = \arg\min_{\mathbf{y}\in\tilde{\mathfrak{M}}_2} \tilde{\psi}(\mathbf{x}_0,\mathbf{y}) \iff \mathbf{x}_0 = \arg\min_{\mathbf{x}\in\tilde{\mathfrak{M}}_1} \tilde{\psi}(\mathbf{x},\mathbf{y}_0).$$

The principle ensures that $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$ have the same cardinality, and $\mathbf{h}$ and $\mathbf{g}$ are bijections.

Empirical evidence for the principle of Regular Minimality is presented in Dzhafarov (2002d) and Dzhafarov and Colonius (2005).

2.4. Canonical relabeling

Canonical relabeling consists in assigning the same label to $\mathbf{x} \in \tilde{\mathfrak{M}}_1$ and $\mathbf{y} \in \tilde{\mathfrak{M}}_2$ whenever these stimuli are PSEs of each other. This is always possible due to Regular Minimality. For example, one can leave $\tilde{\mathfrak{M}}_1$ intact but relabel every $\mathbf{y} \in \tilde{\mathfrak{M}}_2$ into $\mathbf{g}(\mathbf{y})$. More generally, we introduce a "common" stimulus space $\mathfrak{M}$ (arbitrary in all respects except for being equipotent with $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$) and form bijective maps

$$\mathbf{f}_1 : \mathfrak{M} \to \tilde{\mathfrak{M}}_1, \mathbf{f}_2 : \mathfrak{M} \to \tilde{\mathfrak{M}}_2$$

so that $\mathbf{f}_1(\mathbf{a})$ and $\mathbf{f}_2(\mathbf{a})$ are mutual PSEs for every $\mathbf{a} \in \mathfrak{M}$. This allows us to redefine the discrimination probability function as

$$\psi : \mathfrak{M} \times \mathfrak{M} \to [0,1],$$

where

$$\psi(\mathbf{x},\mathbf{y}) = \tilde{\psi}(\mathbf{f}_1(\mathbf{x}),\mathbf{f}_2(\mathbf{y})). \tag{2}$$

The *canonical form of Regular Minimality* is

$$\psi(\mathbf{x},\mathbf{x}) < \begin{cases} \psi(\mathbf{x},\mathbf{y}), \\ \psi(\mathbf{y},\mathbf{x}) \end{cases} \tag{3}$$

for all $\mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$. As a result, quantities

$$\Psi^{(1)}(\mathbf{x},\mathbf{y}) = \psi(\mathbf{x},\mathbf{y}) - \psi(\mathbf{x},\mathbf{x}),$$
$$\Psi^{(2)}(\mathbf{x},\mathbf{y}) = \psi(\mathbf{y},\mathbf{x}) - \psi(\mathbf{x},\mathbf{x})$$

are always nonnegative and vanish only at $\mathbf{x} = \mathbf{y}$. They are called *psychometric increments*, respectively, of the *first kind* (or, in the second argument) and of the *second kind* (or, in the first argument).

Convention: In the following we will only deal with stimulus spaces canonically relabeled. In other words, the term "stimulus space" will always mean a "common" stimulus space $\mathfrak{M}$ endowed with a discrimination probability function $\psi(\mathbf{x},\mathbf{y})$ satisfying (3).

2.5. Nonconstant Self-Dissimilarity

There is ample empirical evidence (see Dzhafarov, 2002d; Dzhafarov & Colonius, 2005, submitted) that $\psi(\mathbf{x},\mathbf{x})$ is not generally the same for all $\mathbf{x}$. This property is called *Nonconstant Self-Dissimilarity* ($\psi(\mathbf{x},\mathbf{x})$ is not necessarily a constant across all $\mathbf{x} \in \mathfrak{M}$).[3] Note that the formulation of the property here assumes that function ψ is in a canonical form. Otherwise we would have to say instead that $\tilde{\psi}(\mathbf{x},\mathbf{y})$ is not necessarily the same for all PSE pairs $(\mathbf{x},\mathbf{y})$. The conjunction of Regular Minimality and Nonconstant Self-Dissimilarity turns out to have surprisingly restrictive consequences (Dzhafarov, 2002d, 2003a, b; Dzhafarov and Colonius, 2005).

2.6. Topology

Axiom 2 (*Convergence*). *As* $n \to \infty$,

$$\Psi^{(1)}(\mathbf{x},\mathbf{x}_n) \to 0 \iff \Psi^{(2)}(\mathbf{x},\mathbf{x}_n) \to 0.$$

The convergence in stimulus space $\mathfrak{M}$ is defined by

$$\mathbf{x}_n \to \mathbf{x} \iff \Psi^{(\iota)}(\mathbf{x},\mathbf{x}_n) \to 0, \quad \text{as } n \to \infty, \tag{4}$$

where ι may be 1 or 2. [4]This notion of convergence induces on $\mathfrak{M}$ a topology based on open balls

$$\mathfrak{B}(\mathbf{x},\varepsilon) = \{\mathbf{y} : \min\{\Psi^{(1)}(\mathbf{x},\mathbf{y}), \Psi^{(2)}(\mathbf{x},\mathbf{y})\} < \varepsilon\} \tag{5}$$

[3]Without formally involving modal logic, the expression "is not necessarily constant" means that (a) there are stimulus spaces in which one can find $\mathbf{x}$ and $\mathbf{y}$ with $\psi(\mathbf{x},\mathbf{x}) \neq \psi(\mathbf{y},\mathbf{y})$; but (b) stimulus spaces in which $\psi(\mathbf{x},\mathbf{x}) \equiv const$ may exist. To say, therefore that we assume Nonconstant Self-Dissimilarity is equivalent to saying that we do not assume Constant Self-Dissimilarity, a law according to which $\psi(\mathbf{x},\mathbf{x}) \equiv const$ in all stimulus spaces.

[4]Ali Ünlü, in a review of this paper, pointed out to us that we could define the convergence by $\mathbf{x}_n \to \mathbf{x} \iff \Psi^{(1)}(\mathbf{x},\mathbf{x}_n) \to 0$ and replace Axiom 2 with a weaker statement,

$$\Psi^{(2)}(\mathbf{x},\mathbf{x}_n) \to 0 \Longrightarrow \Psi^{(1)}(\mathbf{x},\mathbf{x}_n) \to 0.$$

The reverse implication then would have followed from Axiom 3, given below. We prefer the present version for purely esthetic reasons.

taken for all possible values of $\mathbf{x}$ and $\varepsilon > 0$ (for details see Dzhafarov and Colonius, 2005). It follows from the next axiom that discrimination probability function $\psi(\mathbf{x}, \mathbf{y})$ is continuous with respect to this topology.

Axiom 3 (*Intrinsic continuity*). *Discrimination probability function* ψ *is intrinsically continuous*:

$$(\mathbf{x}_n \to \mathbf{x}) \wedge (\mathbf{y}_n \to \mathbf{y}) \Rightarrow \psi(\mathbf{x}_n, \mathbf{y}_n) \to \psi(\mathbf{x}, \mathbf{y}).$$

Psychometric increments $\Psi^{(\iota)}(\mathbf{x}, \mathbf{y})$ ($\iota = 1, 2$) are then also continuous.

2.7. Overall psychometric transformation

An *arc*, denoted $\mathbf{x}(t)$ or $\mathbf{x}_{[a,b]}$, is defined as a homeomorphic (i.e., continuous with a continuous inverse) mapping of a real interval into stimulus space, $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$, where we allow $a = b$ (so that a single stimulus can be viewed as the graph of a "degenerate" arc). Arc $\mathbf{x}(t)$ is said to connect $\mathbf{x}(a) = \mathbf{a}$ to $\mathbf{x}(b) = \mathbf{b}$.

Note that we do not postulate here that any two points in $\mathfrak{M}$ can be connected by an arc (by definition this is true for spaces we call continuous, but not in general). It is even possible that no two distinct points in $\mathfrak{M}$ can be connected by an arc (as is the case in discrete spaces, considered later). Even then, however, the axiom and theorem stated below are formally valid (not violated).

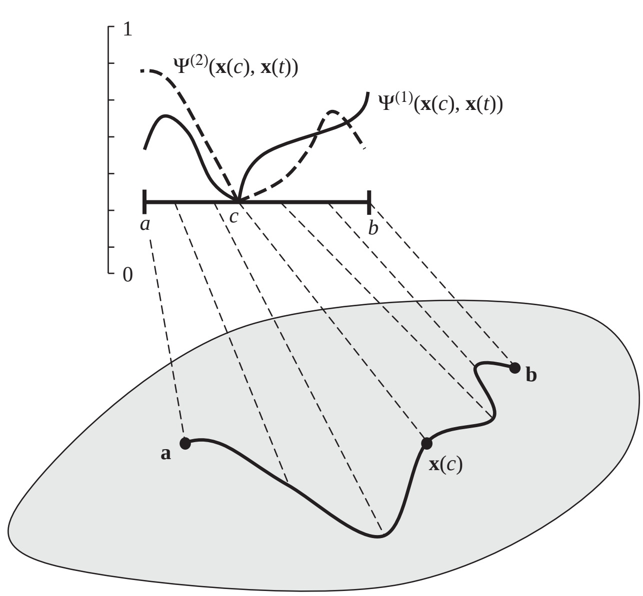

Fig. 1. A smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ shown in conjunction with psychometric increments $\Psi^{(1)}(\mathbf{x}(c), \mathbf{x}(t))$ (solid thick line) and $\Psi^{(2)}(\mathbf{x}(c), \mathbf{x}(t))$ (dashed thick line). Psychometric increments are continuously differentiable below c and above c, and they both increase as t slightly moves away from c in either direction.

Refer to Fig. 1. An arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ is called *smooth* if, for every $c \in [a, b]$,

(i) $\Psi^{(1)}(\mathbf{x}(c), \mathbf{x}(t))$ and $\Psi^{(2)}(\mathbf{x}(c), \mathbf{x}(t))$ are continuously differentiable in t on $[a, c) \cup (c, b]$; and
(ii) $\mathrm{d}\Psi^{(1)}(\mathbf{x}(c), \mathbf{x}(t))/\mathrm{d}t$ and $\mathrm{d}\Psi^{(2)}(\mathbf{x}(c), \mathbf{x}(t))/\mathrm{d}t$ are negative on $[c - \delta, c) \cap [a, b]$ and positive on $(c, c + \delta] \cap [a, b]$, for some $\delta > 0$.

A degenerate arc ($a = b$) can be formally viewed as a smooth arc.

Axiom 4 (*Comeasurability im Kleinen*). [5]*For any two nondegenerate smooth arcs* $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ *and* $\mathbf{y}(\tau) : [c, d] \to \mathfrak{M}$,

$$0 < \lim_{\alpha \to 0+} \frac{\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a + \alpha))}{\Psi^{(1)}(\mathbf{y}(c), \mathbf{y}(c + \alpha))} < \infty, \quad \iota = 1, 2.$$

The following consequence of this axiom is essentially an excerpt from what we call the *First Main Theorem of Fechnerian Scaling*, combined with some of its corollaries (Dzhafarov and Colonius, 2005).

Theorem 1. *There is a function* $\Phi(h) : [0, 1] \to \mathrm{Re}^+$ (*called overall psychometric transformation*, Re^+ *being the set of nonnegative reals*) *such that, for any nondegenerate smooth arc* $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ *and any* $t \in [a, b)$,

$$0 < \lim_{\alpha \to 0+} \frac{\Phi[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t + \alpha))]}{\alpha} < \infty, \quad \iota = 1, 2.$$

Function $\Phi(h)$ *is*

(i) *regularly varying at* $h = 0$ *with a positive exponent*, ξ;[6]
(ii) *continuous and increasing*[7] *on some interval* $[0, \varepsilon)$, *with* $\Phi(0) = 0$;

$\Phi(h)$ *is determined asymptotically uniquely* (*as* $h \to 0+$), *that is, any other function* $\Phi^*(h)$ *with the same properties satisfies*

$$\lim_{h \to 0+} \frac{\Phi^*(h)}{\Phi(h)} = k > 0.$$

2.8. Sets versus spaces

At this point it may be useful to point out that both in the foregoing and subsequent text we follow the

[5]"Im kleinen" means "in the small," in the vicinity of zero. In Dzhafarov and Colonius (2005) this axiom is numbered 5, following the axiom of arc-connectedness (which defines continuous spaces, discussed below).

[6]Regular variation at zero with exponent $\xi > 0$ means that

$$\lim_{h \to 0+} \frac{\Phi(h)}{h^{\xi} L(h)} = 1$$

for some function $L(h)$ such that for every $\lambda > 0$, $L(\lambda h)/L(h) \to 1$ as $h \to 0+$. Examples of $\Phi(h)$ include h^{ξ}, $h^{\xi} \log \frac{1}{h}$, $h^{\xi} + h^{2\xi}$, etc. See Dzhafarov (2002a) for details.

[7]"Increasing" in this paper always means "strictly increasing."

established mathematical "tradition" of referring to spaces by their sets alone, without explicitly mentioning their space-forming structure (in our case, the discrimination probability function). Thus, in Sections 2.2 and 2.3 we speak of "stimulus spaces" $\tilde{\mathfrak{M}}_1, \tilde{\mathfrak{M}}_2$ whereas it would have been more correct to speak of a single stimulus space $(\tilde{\mathfrak{M}}_1, \tilde{\mathfrak{M}}_2, \tilde{\psi})$, consisting of two sets and one function. Following a canonical transformation (Section 2.4) it would have been more correct to use the term "stimulus space" for $(\mathfrak{M}, \psi)$ rather than for $\mathfrak{M}$. It should be kept in mind, therefore, that such expressions as "space of auditory tones" or "space of letters" mentioned below refer not to the sets of physical entities per se but rather to these sets taken together with discrimination probability functions. One and the same set of stimuli presented to two different observers will generally create two different stimulus spaces (possibly differently relabeled to achieve canonical forms).

3. Continuous stimulus spaces

The most obvious examples of continuous stimulus spaces include the unidimensional continua of Fechner's original theory (with stimuli identified by their intensity or extent) and multi-attribute spaces of colors, tones, or parametrized shapes. In the construction presented in Dzhafarov and Colonius (2005) the "continuity" of such spaces is defined in terms of the discrimination probabilities $\psi(\mathbf{x}, \mathbf{y})$, without resorting to the physical properties of stimuli $\mathbf{x}, \mathbf{y}$.

Formally, space $\mathfrak{M}$ is called continuous if it is *arc-connected*, that is, if any two points in it can be connected by an arc. The axioms of Fechnerian Scaling for continuous spaces[8] ensure that $\mathfrak{M}$ is also *smoothly connected*, which means that any two points in $\mathfrak{M}$ can be connected by a *piecewise smooth arc* (an arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ whose domain can be partitioned into a finite number of intervals upon each of which $\mathbf{x}(t)$ is smooth).

In reference to Fig. 2, let $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ be a piecewise smooth arc. Consider

$$L^{(\iota)}[\mathbf{x}_{[a,b]}] = \lim_{\Delta(\mathfrak{I}) \to 0} \sum_{j=1}^{k} \Gamma^{(\iota)}(\mathbf{x}(t_{j-1}), \mathbf{x}(t_j)), \qquad (6)$$

where the limit is taken over all possible segmentations $\mathfrak{I}$ of $[a, b]$,

$$\mathfrak{I} = \{a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b\},$$

[8]These axioms include the four given in the previous section and some additional assumptions that we do not discuss in this paper (because they involve a variety of notions we do not use in the present development). The reader should consult Dzhafarov & Colonius (2005) for details.

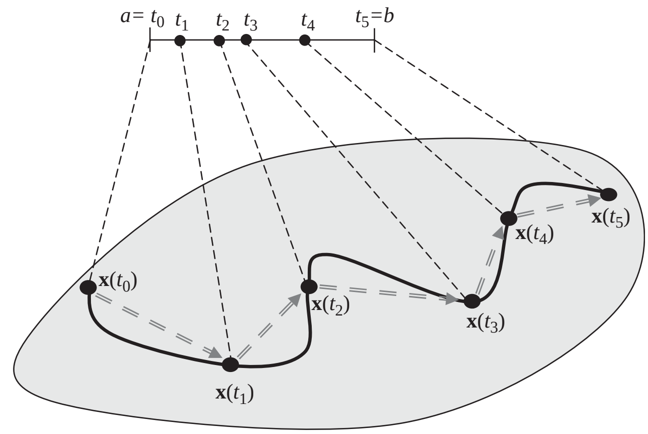

Fig. 2. A piecewise smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$. An arrow leading from $\mathbf{x}(t_{j-1})$ to $\mathbf{x}(t_j)$ indicates the value of $\Gamma^{(\iota)}(\mathbf{x}(t_{j-1}), \mathbf{x}(t_j)) = \Phi[\Psi^{(\iota)}(\mathbf{x}(t_{j-1}), \mathbf{x}(t_j))]$. As segmentation of $[a, b]$ gets progressively finer, the sum of these values gets progressively closer to the psychometric length of the arc.

$\Delta(\mathfrak{I})$ denotes $\max_j [t_j - t_{j-1}]$, and

$$\Gamma^{(\iota)}(\mathbf{x}, \mathbf{y}) = \Phi[\Psi^{(\iota)}(\mathbf{x}, \mathbf{y})], \quad \iota = 1, 2. \qquad (7)$$

That is, overall psychometric transformation Φ is applied to psychometric increments, of both kinds. It is shown in Dzhafarov and Colonius (2005) that limit $L^{(\iota)}[\mathbf{x}_{[a,b]}]$ exists as a finite nonnegative quantity, vanishing only at degenerate arcs ($a = b$). This quantity is taken to be the *psychometric length* of arc $\mathbf{x}(t)$ of the ιth kind ($\iota = 1, 2$). Positive diffeomorphic reparametrizations of $\mathbf{x}_{[a,b]}$ (i.e., arcs $\mathbf{y}(\tau) : [c, d] \to \mathfrak{M}$ such that $\mathbf{y}(\tau(t)) = \mathbf{x}(t)$, where $\tau(t)$ is a bijective mapping $[a, b] \to [c, d]$ with $\tau'(t) > 0$) are all equivalent, in the sense that they do not change its psychometric length.

When confusion is excluded by context, arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ connecting $\mathbf{a}$ to $\mathbf{b}$ can be written as $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$. Then "the same arc" traversed "in the opposite direction" (i.e., any arc $\mathbf{y}(\tau) : [c, d] \to \mathfrak{M}$ such that $\mathbf{y}(\tau(t)) = \mathbf{x}(t)$, where $\tau(t)$ is a bijective mapping $[a, b] \to [c, d]$ with $\tau'(t) < 0$) can be written as $\mathbf{b} \to \mathbf{x} \to \mathbf{a}$. Psychometric lengths $L^{(\iota)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}]$ and $L^{(\iota)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]$ are generally different quantities ($\iota = 1, 2$).

The infimum $G_\iota(\mathbf{a}, \mathbf{b})$ of the psychometric lengths (of the ιth kind) taken over all piecewise smooth arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ is shown to be an *oriented metric* (of the ιth kind): it is nonnegative, vanishing at $\mathbf{a} = \mathbf{b}$ only, and it satisfies the triangle inequality. Discrimination probabilities ψ determine the oriented Fechnerian distances uniquely, up to multiplication by arbitrary $k > 0$.[9]

[9]Scaling constant k is associated with the choice of overall psychometric transformation Φ (see the uniqueness part of Theorem 1). In order not to have to mention this trivial rescaling every time, we will assume henceforth that for some fixed $\mathbf{a}, \mathbf{b}$ in $\mathfrak{M}$, $G_1(\mathbf{a}, \mathbf{b})$ or $G_2(\mathbf{a}, \mathbf{b})$ is equated to 1. This makes the oriented Fechnerian distances within a given space unique. Equivalently, this forces $k = 1$ in the uniqueness part of Theorem 1.

The following two results (Dzhafarov and Colonius, 2005) play the central role in the subsequent development.

Theorem 2 (*Second Main Theorem for Continuous Spaces*). *For any* $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$,

(E1) $L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] = L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] + L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}]$, *for all piecewise smooth arcs* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ *and* $\mathbf{b} \to \mathbf{y} \to \mathbf{a}$;
(E2) $G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a})$.

The second statement of this theorem is, of course, an immediate consequence of the first. Due to this theorem, *overall Fechnerian distance* $G(\mathbf{a}, \mathbf{b})$ can be defined as

$$G(\mathbf{a}, \mathbf{b}) = G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a}). \quad (8)$$

Clearly, $G(\mathbf{a}, \mathbf{b})$ is a well-defined metric: nonnegative, vanishing at $\mathbf{a} = \mathbf{b}$ only, symmetrical, and satisfying the triangle inequality. The significance of (8) is in the fact that $G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a})$ has the natural interpretation of the "to and fro" distance between stimuli **a** and **b** belonging to the first observation area; while $G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a})$ is interpreted as the "to and fro" distance between the same two stimuli[10] within the second observation area. The numerical equality of these distances means that (following a canonical relabeling) we can simply speak of the overall Fechnerian distance between **a** and **b**, without mentioning their observation area.

In the next theorem ξ has the same meaning as in Theorem 1 (the exponent of regular variation for Φ).

Theorem 3. *Nonconstant Self-Dissimilarity is only consistent with one of two possibilities*:[11]

(P1) $\xi = 1$ *and* $\lim \frac{\Phi(h)}{h} = 1$ *as* $h \to 0+$;[12]
(P2) $\xi \geqslant 1$ *and* $\liminf \frac{\Phi(h)}{h} = 0$ *as* $h \to 0+$.[13]

In case P1, *for any piecewise smooth arc* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$,

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}). \end{aligned}$$

In case P2, *for any piecewise smooth arc* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$,

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] = 0. \end{aligned}$$

The latter equality is also implied by Constant Self-Dissimilarity.

It is easy to verify that the statements of Theorem 2 are satisfied in either case. Note that possibility P2 implies $G_1(\mathbf{a}, \mathbf{b}) = G_2(\mathbf{b}, \mathbf{a})$ (and the same is implied by Constant Self-Dissimilarity).

Empirical analysis presented in Dzhafarov and Colonius (2005) strongly supports Nonconstant Self-Dissimilarity and strongly favors possibility P1 (the so-called "*cross-unbalanced*" case) over possibility P2 (the "*cross-balanced*" case).

In view of the subsequent development (especially, Theorems 4, 5, and 8), it is useful to note the following immediate consequence of Theorem 3.

Corollary 1. *If overall psychometric transformation* Φ *is identity*, $\Phi(h) = h$, *then*

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}). \end{aligned}$$

Proof. Under Nonconstant Self-Dissimilarity, $\Phi(h) = h$ agrees with P1 and not with P2. Under Constant Self-Dissimilarity, the differences of the $L^{(i)}$'s equal zero, and so does $\psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a})$. □

4. Discrete stimulus spaces

Examples of discrete spaces are numerous: letters of alphabet, schematic faces, brands of a consumer product, categories of lung dysfunctions depicted in X-ray films, etc. (each of these sets taken with a discrimination probability function). The judgment "**x** is the same as **y**" may have different meanings depending on context and procedure. Thus, each stimulus may have a fixed physical realization (say, a Morse code for a letter, auditorily presented) in which case "**x** is the same as **y**" may signify overall identity (the same code) or identity in a particular respect (say, the code of the same length). In other applications each stimulus may be represented by a finite or infinite number of samples, or realizations, treated as replications of this stimulus (e.g., each lung dysfunction may be represented by a series of X-ray films exhibiting this dysfunction, or each author can be represented by a series of handwritings), and in such cases "**x** is the same as **y**" means that the two samples being presented belong to the same category, have the same source, or the same significance. All these differences are immaterial for the present theory, insofar as the resulting

[10] One must not forget that the stimulus space here is assumed to be in a canonical form, because of which "one and the same stimulus" **a** in the first and second observation areas is in fact a pair of identically labeled mutual PSEs, **x** and **y**, in the pre-canonical spaces $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$, respectively. **x** and **y** may very well be physically distinct stimuli (more precisely, non-identical equivalence classes of stimuli in the original spaces $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$).

[11] In reference to footnote 3, this means that any other possibility would force Constant Self-Dissimilarity.

[12] The limit could be "more generally" written as $k > 0$, but it would be a matter of trivial normalization to set it equal to 1 (see footnote 9).

[13] If $\xi > 1$, $\lim \Phi(h)/h = 0$.

product is a set of stimuli endowed with a function of stimulus pairs satisfying the axioms given in Section 2.

Definition 1. Point $\mathbf{x}$ in stimulus space $\mathfrak{M}$ is called *isolated* if

$$\inf_{\mathbf{y}\in\mathfrak{M}\setminus\{\mathbf{x}\}} \min\{\Psi^{(1)}(\mathbf{x},\mathbf{y}),\Psi^{(2)}(\mathbf{x},\mathbf{y})\}>0.$$

Space $\mathfrak{M}$ is *discrete* if it consists of isolated points.

Observe that this definition is purely psychological: it is given entirely in terms of discrimination probabilities, with no reference to physical properties of the stimuli involved.

Observe also that with respect to the topology introduced in Section 2.6, based on open balls

$$\mathfrak{B}(\mathbf{x},\varepsilon)=\{\mathbf{y}:\min\{\Psi^{(1)}(\mathbf{x},\mathbf{y}),\Psi^{(2)}(\mathbf{x},\mathbf{y})\}<\varepsilon\},$$

stimulus space $\mathfrak{M}$ which is discrete in the sense of Definition 1 is also discrete in the conventional topological sense: every singleton $\{\mathbf{x}\}$ in $\mathfrak{M}$ is an open set. Because of this our Axioms 2 and 3 are satisfied trivially.

As the only continuous mapping $\mathbf{x}(t):[a,b]\to\mathfrak{M}$ with $\mathfrak{M}$ being topologically discrete is $\mathbf{x}(t)=\mathbf{x}_0$ (a constant), it follows that in a discrete $\mathfrak{M}$ the only arcs $\mathbf{x}(t):[a,b]\to\mathfrak{M}$ (homeomorphisms) are degenerate arcs ($a=b$). Because of this Axiom 4 is also satisfied trivially.

We conclude that in discrete stimulus spaces the only axiom whose validity has to be postulated (in empirical applications, checked) is Regular Minimality. It is easy to see that Regular Minimality implies that every finite $\mathfrak{M}$ is discrete.[14]

A *chain* connecting $\mathbf{a}$ to $\mathbf{b}$ is any finite sequence ($\mathbf{a}=\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k=\mathbf{b}$), where the elements need not be distinct. By analogy with continuous spaces, and especially in view of the construction depicted in Fig. 2, it is natural to define the *psychometric length* (of the $\imath$th kind) of a chain as

$$L^{(\imath)}[(\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k)]=\sum\Gamma^{(\imath)}(\mathbf{x}_i,\mathbf{x}_{i+1}),\quad \imath=1,2,\tag{9}$$

where $\Gamma^{(\imath)}(\mathbf{x},\mathbf{y})=\Phi[\Psi^{(\imath)}(\mathbf{x},\mathbf{y})]$, $\Phi(h)$ being a function continuous and increasing in the vicinity of $h=0$. Denoting a chain connecting $\mathbf{a}$ to $\mathbf{b}$ by $\mathbf{a}\to\mathbf{x}\to\mathbf{b}$,

$$G_\imath(\mathbf{a},\mathbf{b})=\inf_{\text{all }\mathbf{a}\to x\to\mathbf{b}} L^{(\imath)}[\mathbf{a}\to x\to\mathbf{b}],\ \imath=1,2,\tag{10}$$

is the *oriented Fechnerian distance* of the $\imath$th kind. It is trivial to show that $G_\imath(\mathbf{a},\mathbf{b})$ satisfies all properties of a metric except for symmetry.

[14]We can always assume that $\mathfrak{M}$ contains at least two elements, but formally, a single-element space is discrete because the infimum in Definition 1 is sought over an empty set. Single-element $\mathfrak{M}$ can also be shown to be formally continuous. (If $\mathfrak{M}$ is empty itself, to be meticulous, it can be viewed as discrete, continuous, or possessing any other structure.)

Here, however, the analogy with the continuous theory ends. Overall psychometric transformation Φ in the continuous theory is specified asymptotically uniquely. The choice (or "variant") of its extension beyond an arbitrarily small vicinity of zero is immaterial for the validity of Theorem 2 and for the ensuing definition of the overall Fechnerian distance, $G(\mathbf{a},\mathbf{b})$ in (8). In the case of a discrete stimulus space, however, Φ should generally be defined on the entire interval $[0,1]$, and different choices of Φ result in different (not multiplicatively transformable into each other) values of $L^{(\imath)}[(\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k)]$ in (9). Consequently, they result in different (not multiplicatively transformable into each other) oriented Fechnerian distances $G_\imath(\mathbf{a},\mathbf{b})$ ($\imath=1,2$). Moreover, with an arbitrary extension of Φ the analogue of Theorem 2 for discrete spaces will not generally hold.

In Dzhafarov and Colonius (submitted) Φ is implicitly chosen to be identity, so that $\Gamma^{(\imath)}(\mathbf{x},\mathbf{y})=\Psi^{(\imath)}(\mathbf{x},\mathbf{y})$.[15] In reference to Theorem 3, clearly, $\Phi\equiv$ *identity* agrees im kleinen (in the vicinity of zero) with possibility P1 but not with possibility P2. Moreover, we have the following analogue of Corollary 1.

Theorem 4. *If overall psychometric transformation Φ is identity, then for any $\mathbf{a},\mathbf{b}\in\mathfrak{M}$ and any chain $\mathbf{a}\to\mathbf{x}\to\mathbf{b}$,*

$$\begin{aligned}&L^{(1)}[\mathbf{a}\to\mathbf{x}\to\mathbf{b}]-L^{(2)}[\mathbf{b}\to\mathbf{x}\to\mathbf{a}]\\&\quad=L^{(2)}[\mathbf{a}\to\mathbf{x}\to\mathbf{b}]-L^{(1)}[\mathbf{b}\to\mathbf{x}\to\mathbf{a}]\\&\quad=\psi(\mathbf{b},\mathbf{b})-\psi(\mathbf{a},\mathbf{a}).\end{aligned}$$

It immediately follows that the discrete-space analogue of Theorem 2 then holds in its entirety. In the formulation below, if $\mathbf{a}\to\mathbf{x}\to\mathbf{b}$ is $(\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k)$, then $\mathbf{b}\to\mathbf{x}\to\mathbf{a}$ ("the same chain in the opposite direction") is $(\mathbf{y}_1,\mathbf{y}_2,\dots,\mathbf{y}_k)$ with $\mathbf{y}_i=\mathbf{x}_{k-i+1}$.

Theorem 5 (*Main Theorem for Discrete Spaces*). *If overall psychometric transformation Φ is identity, then for any $\mathbf{a},\mathbf{b}\in\mathfrak{M}$,*

(E1) $L^{(1)}[\mathbf{a}\to\mathbf{x}\to\mathbf{b}]+L^{(1)}[\mathbf{b}\to\mathbf{y}\to\mathbf{a}]=L^{(2)}[\mathbf{b}\to\mathbf{x}\to\mathbf{a}]+L^{(2)}[\mathbf{a}\to\mathbf{y}\to\mathbf{b}]$, *for all chains* $\mathbf{a}\to\mathbf{x}\to\mathbf{b}$ *and* $\mathbf{b}\to\mathbf{y}\to\mathbf{a}$;
(E2) $G_1(\mathbf{a},\mathbf{b})+G_1(\mathbf{b},\mathbf{a})=G_2(\mathbf{a},\mathbf{b})+G_2(\mathbf{b},\mathbf{a})$.

As a result, overall Fechnerian distance $G(\mathbf{a},\mathbf{b})$ can be defined as in (8). In some cases it is convenient to concatenate the chains $\mathbf{a}\to\mathbf{x}\to\mathbf{b}$ and $\mathbf{b}\to\mathbf{y}\to\mathbf{a}$ into *closed loops* $\mathbf{a}\to\mathbf{x}\to\mathbf{b}\to\mathbf{y}\to\mathbf{a}$ (when traversed in the opposite direction, $\mathbf{a}\to\mathbf{y}\to\mathbf{b}\to\mathbf{x}\to\mathbf{a}$) and present Equation E1 in the form

$$L^{(1)}[\mathbf{a}\to\mathbf{x}\to\mathbf{b}\to\mathbf{y}\to\mathbf{a}]=L^{(2)}[\mathbf{a}\to\mathbf{y}\to\mathbf{b}\to\mathbf{x}\to\mathbf{a}].\tag{11}$$

[15]To remind, putting $\Phi(h)=kh$ and $\Gamma^{(\imath)}(\mathbf{x},\mathbf{y})=k\Psi^{(\imath)}(\mathbf{x},\mathbf{y})$ would be only superficially more general. See footnote 9.

It is easy to see that putting $\Phi \equiv$ *identity* is not a necessary condition for satisfying Equations $E_1 - E_2$ in Theorem 5. Consider, for example, a discrete space on which $\psi(\mathbf{x}, \mathbf{y}) = \psi_0$ for all $\mathbf{x} \neq \mathbf{y}$, and $\psi(\mathbf{x}, \mathbf{x}) = \psi_{\mathbf{x}}$, a variable quantity less than ψ_0. Clearly, in this space Equations $E_1 - E_2$ are satisfied for any choice of (increasing) function Φ. Given any finite stimulus space, psychometric increments $\Psi^{(1)}(\mathbf{x}, \mathbf{y}), \Psi^{(2)}(\mathbf{x}, \mathbf{y})$ attain altogether only finite number of values, $0 = \Psi_0 < \Psi_1 < \cdots < \Psi_N$: clearly, if $\Phi(\Psi_i) = \Psi_i$ for all i, Equations $E_1 - E_2$ are satisfied irrespective of how Φ is interpolated between Ψ_i and Ψ_{i+1} or extrapolated above Ψ_N. If a stimulus space consists of three points, given virtually any function Φ one can choose discrimination probabilities for the nine ordered pairs so that Equations $E_1 - E_2$ are satisfied.

In all such examples, however, the possibility of using nonidentity (non-proportionality) Φ capitalizes on what can be intuitively seen as "numerical accidents." In reference to the examples just given, the values of $\psi(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \neq \mathbf{y}$ in general need not equal a constant; the gaps between successive values for $\Psi^{(1)}(\mathbf{x}, \mathbf{y})$ and $\Psi^{(2)}(\mathbf{x}, \mathbf{y})$ may be different for different $\psi(\mathbf{x}, \mathbf{y})$; function Φ chosen for a particular set of nine probabilities in a three-point configuration will not be suitable for another set of nine probabilities. In other words, even though overall psychometric transformation Φ can sometimes be chosen to be nonidentity, for some stimulus spaces, it cannot be so chosen without ceasing to be suitable for some other stimulus spaces, those involving different stimulus sets or different discrimination probability functions defined on the same set.

5. Extending continuous theory to discrete spaces

The preceding observation suggests the following guiding desiderata in constructing an analogue of the continuous theory for discrete spaces.

Desiderata. Overall psychometric transformation $\Phi : [0,1] \to \mathrm{Re}^+$ should be chosen so that

(D1) $\Phi(h)$ be one and the same function for a sufficiently rich set of discrete stimulus spaces;
(D2) $\Phi(h)$ be continuous and increasing on some interval $[0, \varepsilon)$, with $\Phi(0) = 0$;[16]
(D3) Equation E1 (hence also Equation E_2) in Theorem 5 be satisfied.

[16] This desideratum merely replicates property (ii) in Theorem 1. It would be natural to require more: that Φ also agree with property (i) in Theorem 1 and with one of the two possibilities in Theorem 3. It turns out, however, that with our approach such an agreement can be deduced rather than required.

We have to explicate now the notion of a "sufficiently rich set" of discrete stimulus spaces. The examples of "numerical accidents" given in the previous section prompt the following approach: in a sufficiently rich set of discrete stimulus spaces discrimination probabilities associated with different pairs of stimuli do not constrain each other "too tightly."

We can assume that every stimulus space contains at least two elements, and we consider all pairs of distinct elements $(\mathbf{a}, \mathbf{b})$ taken across all discrete stimulus spaces $\mathfrak{M}$ (all in canonical forms) within some collection of spaces $\mathbb{D}$. Recall that different spaces $\mathfrak{M}$ in $\mathbb{D}$ may but need not involve different sets of stimuli: for a given set of stimuli, we have potentially as many spaces $\mathfrak{M}$ as we have different observers and observation conditions leading to different sets of discrimination probabilities. A pair of distinct stimuli $(\mathbf{a}, \mathbf{b})$ in $\mathfrak{M} \in \mathbb{D}$ is associated with four positive psychometric increments, $\Psi^{(1)}(\mathbf{a}, \mathbf{b})$, $\Psi^{(1)}(\mathbf{b}, \mathbf{a})$, $\Psi^{(2)}(\mathbf{a}, \mathbf{b})$, $\Psi^{(2)}(\mathbf{b}, \mathbf{a})$, and it follows from their definition that

$$\Psi^{(1)}(\mathbf{a}, \mathbf{b}) + \Psi^{(1)}(\mathbf{b}, \mathbf{a}) = \Psi^{(2)}(\mathbf{a}, \mathbf{b}) + \Psi^{(2)}(\mathbf{b}, \mathbf{a}). \quad (12)$$

We denote

$$T^{(1)}(\mathbf{a}, \mathbf{b}) = (\Psi^{(1)}(\mathbf{a}, \mathbf{b}), \Psi^{(1)}(\mathbf{b}, \mathbf{a}), \Psi^{(2)}(\mathbf{a}, \mathbf{b})),$$
$$T^{(2)}(\mathbf{a}, \mathbf{b}) = (\Psi^{(2)}(\mathbf{a}, \mathbf{b}), \Psi^{(2)}(\mathbf{b}, \mathbf{a}), \Psi^{(1)}(\mathbf{a}, \mathbf{b})). \quad (13)$$

We say that a collection of spaces $\mathbb{D}$ is sufficiently rich if the three psychometric increments in $T^{(\iota)}(\mathbf{a}, \mathbf{b})$, $\iota = 1, 2$, do not interdetermine each other. Intuitively, we expect that if $T^{(\iota)}(\mathbf{a}, \mathbf{b}) = (x, y, z)$ for some pair $(\mathbf{a}, \mathbf{b})$ in some set of stimuli presented (pairwise) to some observer, then for any (x', y', z) taken sufficiently close to (x, y, z), it should be possible to find a generally different pair $(\mathbf{a}', \mathbf{b}')$ in a generally different stimulus set presented to a generally different observer, for which $T^{(\iota)}(\mathbf{a}', \mathbf{b}') = (x', y', z)$ (with the same z). In addition, we assume that for any $x \in (0, 1]$ one can find in $\mathbb{D}$ a psychometric increment (of the first or of the second kind) whose value is x. In other words, we assume that in $\mathbb{D}$ the union of set $\mathfrak{S}^{(1)}$ of all possible values for $\Psi^{(1)}(\mathbf{a}, \mathbf{b})$ and of set $\mathfrak{S}^{(2)}$ of all possible values for $\Psi^{(2)}(\mathbf{a}, \mathbf{b})$ is the entire interval $(0, 1]$ (no gaps).

Definition 2. A set of discrete stimulus spaces $\mathbb{D}$ (with all spaces in canonical forms) is called sufficiently rich if the following two conditions are satisfied:

(i) denoting by $\mathfrak{T}^{(\iota)}$ the set of all possible values for $T^{(\iota)}(\mathbf{a}, \mathbf{b})$ in $\mathbb{D}$ $(\iota = 1, 2)$, if $(x, y, z) \in \mathfrak{T}^{(\iota)}$, then any $(x', y', z) \in \mathfrak{T}^{(\iota)}$, with (x', y') taken in a sufficiently small neighborhood of (x, y) open with respect to $(0, 1] \times (0, 1]$;
(ii) denoting by $\mathfrak{S}^{(\iota)}$ the set of all possible values for $\Psi^{(\iota)}(\mathbf{a}, \mathbf{b})$ in $\mathbb{D}$ $(\iota = 1, 2)$, $\mathfrak{S}^{(1)} \cup \mathfrak{S}^{(2)} = (0, 1]$.

A few remarks are due.

1. Unless $z=1$, fixing the value of z in $(x',y',z)\in\mathfrak{D}^{(\imath)}$ does not imply that any of the four discrimination probabilities $\psi(\mathbf{a},\mathbf{b})$, $\psi(\mathbf{b},\mathbf{a})$, $\psi(\mathbf{a},\mathbf{a})$, $\psi(\mathbf{b},\mathbf{b})$ remains fixed. As (x',y') varies within a neighborhood of (x,y), all four probabilities may covary, subject to holding constant $\psi(\mathbf{b},\mathbf{a})-\psi(\mathbf{a},\mathbf{a})=\Psi^{(2)}(\mathbf{a},\mathbf{b})$ (if $\imath=1$) or $\psi(\mathbf{a},\mathbf{b})-\psi(\mathbf{a},\mathbf{a})=\Psi^{(1)}(\mathbf{a},\mathbf{b})$ (if $\imath=2$).
2. Our definition of $\mathbb{D}$ implies Nonconstant Self-Dissimilarity (see Section 2.5): if it were the case that $\psi(\mathbf{a},\mathbf{a})=\psi(\mathbf{b},\mathbf{b})$ across all spaces in $\mathbb{D}$, then we would always have $\Psi^{(1)}(\mathbf{a},\mathbf{b})=\Psi^{(2)}(\mathbf{b},\mathbf{a})$. This would mean that $y=z$ in both $T^{(1)}(\mathbf{a},\mathbf{b})$ and $T^{(2)}(\mathbf{a},\mathbf{b})$, contrary to the requirement that y can vary with z fixed. The definition does not, however, imply asymmetry: (i) and (ii) can hold even if $\psi(\mathbf{a},\mathbf{b})=\psi(\mathbf{b},\mathbf{a})$ across all spaces in $\mathbb{D}$ (a desirable property as it allows for the practice of averaging across observation areas, i.e., replacing $\psi(\mathbf{a},\mathbf{b})$ with $(\psi(\mathbf{a},\mathbf{b})+\psi(\mathbf{b},\mathbf{a}))/2$, $\sqrt{\psi(\mathbf{a},\mathbf{b})\psi(\mathbf{b},\mathbf{a})}$, etc.). This dovetails with the greater theoretical importance we attach to Nonconstant Self-Dissimilarity than to asymmetry (Dzhafarov, 2002d, 2003a, b; Dzhafarov and Colonius, 2005).
3. The openness of the neighborhood in (i) is understood with respect to $(0,1]\times(0,1]$ rather than Re × Re. Thus, $(\frac{1}{2}-\delta,\frac{1}{2}+\delta)\times(1-\eta,1]$ (δ,η being small positive numbers) is an open neighborhood of $(x=\frac{1}{2},y=1)$. With no loss of generality, an open neighborhood of (x,y) can always be taken to be a rectangle, $I^{(\imath)}_{xyz}\times J^{(\imath)}_{xyz}$,
$$I^{(\imath)}_{xyz}=(x-\delta,x+\delta)\cap(0,1],$$
$$J^{(\imath)}_{xyz}=(y-\eta,y+\eta)\cap(0,1] \tag{14}$$
for some positive δ,η (that generally depend on x,y,z, and $\imath$). Property (ii) implies, obviously,
$$\bigcup_{\imath=1,2}\ \bigcup_{(x,y,z)\in\mathfrak{D}^{(\imath)}} I^{(\imath)}_{xyz}=\mathfrak{S}^{(1)}\cup\mathfrak{S}^{(2)}=(0,1]. \tag{15}$$
4. It is critical for our analysis that $\mathfrak{S}^{(1)}\cup\mathfrak{S}^{(2)}$ is an interval, and that it has 0 as its left endpoint (exclusive). The right endpoint, however, could very well be replaced with any $0<M<1$, inclusive or exclusive. Moreover, the theory does not make critical use of the restriction $M\leqslant 1$. This is important in view of the possibility, mentioned in the Conclusion, of applying Fechnerian Scaling to transformed probabilities (in which case M may conceivably be ∞).

For the theorem below, we will need the following simple topological property of intervals.

Lemma 1. *Let* $\mathfrak{C}$ *be a cover of* $[\alpha,\beta)$ $(\beta\leqslant\infty)$ *by finite intervals open with respect to* $[\alpha,\beta)$. *Then* $[\alpha,\beta)$ *is covered by a sequence of intervals* I_n *in* $\mathfrak{C}$ $(n=1,2,\dots)$ *such that* $I_n\cap I_{n+1}\neq\emptyset$ *and* $\sup I_n$ *form a nondecreasing sequence converging to* β. *The analogous property holds for* $(\alpha,\beta]$ $(\alpha\geqslant-\infty)$, *with* $\inf I_n$ *forming a nonincreasing sequence converging to* α.

Proof. Consider sequence $\alpha=\alpha_0<\alpha_1\leqslant\cdots\leqslant\alpha_n\leqslant\cdots$ defined by
$$\alpha_n=\sup_{I\in\mathfrak{C}}\{\sup I: I\ni\alpha_{n-1}\},\quad n=1,2,\dots$$
If $\alpha_n\to\beta'<\beta$, then for any $\mathfrak{C}\ni I\ni\beta'$, $I\ni\alpha_{n-1}$ for a sufficiently large n, whence, by definition of α_n, $\sup I\leqslant\alpha_n$. But $\alpha_n\leqslant\beta'<\sup I$. So, $\alpha_n\to\beta$. For $n=1,2,\dots$, take any $\mathfrak{C}\ni I\ni\alpha_n$ and put $I=I_{2n}$. For $n=1,2,3,\dots$, take for I_{2n-1} any $\mathfrak{C}\ni I\ni\alpha_{n-1}$ such that $\sup I\geqslant\sup I_{2n-2}$ (if $n>1$) and $I\cap I_{2n}\neq\emptyset$ (exists by definition of α_n). The resulting sequence satisfies the requirements of the lemma. The proof for $(\alpha,\beta]$ is analogous. □

The reader should also recall certain facts about
$$f(x)+g(y)=h(x+y),\quad (x,y)\in I\times J, \tag{16}$$
a Pexider functional equation restricted to rectangle $I\times J$ (Aczèl, 1987).[17]

Lemma 2. *Let* (16) *hold for* I,J *open in* Re.

(A) *If* $f(x)$ *is continuous at least at one point in* I, *then, for some* a,b,k,
$$f(x)=a+kx,\quad x\in I,$$
$$g(y)=b+ky,\quad y\in J.$$

(B) *If* $f(x)=a+kx$ *on a subinterval of* I, *then the same is true for entire* I.

(C) *If* $f(x)=a+kx$ *on* $I=(\alpha,\beta)$, $\beta<\infty$, *then* $f(\beta)=a+k\beta$.

Proof. For (A) see Aczèl (1987, Chapter 5). (B) immediately follows from (A). To prove (C), observe first that, due to (A), $f(x)=a+kx$ on I implies $g(y)=b+ky$ on J. Then, for some $\inf J<y<\sup J$ and a sufficiently small $\delta>0$,
$$\begin{aligned}f(\beta)+(b+ky)&=f(\beta)+g(y)=h(\beta+y)\\&=f(\beta-\delta)+g(y+\delta)\\&=(a+k(\beta-\delta))+(b+k(y+\delta)),\end{aligned}$$
and the proof obtains by simple algebra. □

[17]The use of the Pexider functional equation in establishing our main result was prompted by a suggestion made by Jun Zhang (personal communication, November 16, 2004). Although the suggestion was made in the context of a different approach, we owe to Jun Zhang our gratitude.

We are ready now to prove our main result.[18]

Theorem 6. *Desiderata* D1–D3 *are satisfied in* $\mathbb{D}$ *if and only if overall psychometric transformation* Φ *is identity.*

Proof. The "if" part is simple: identity function clearly satisfies D1 and D2, and it satisfies D3 by Theorem 5 (which is valid for any discrete stimulus space).

To prove the "only if" part, we construct two sequences

$$\{(x_n, y_n, z_n), \iota_n\}_{n=0,-1,-2,\ldots}, \quad \{(x_n, y_n, z_n), \iota_n\}_{n=0,1,2,\ldots},$$

termed, respectively, the left sequence and the right sequence, with $\iota_n = 1$ or 2, and $(x_n, y_n, z_n) \in \mathfrak{T}^{(\iota_n)}$. By Remark 3 to Definition 2, each $((x_n, y_n, z_n), \iota_n)$ is associated with open rectangle $I_n \times J_n$ (denoting $I_n = I^{(\iota_n)}_{x_n y_n z_n}$, $J_n = J^{(\iota_n)}_{x_n y_n z_n}$, as defined in (14)). The initial element, $((x_0, y_0, z_0), \iota_0)$, is chosen so that $x_0 < \varepsilon$, where ε is defined as in D2. By Definition 2(ii), such an element should exist: otherwise $\mathfrak{S}^{(1)} \cup \mathfrak{S}^{(2)}$ would not include values below ε.

Having chosen (x_0, y_0, z_0) and ι_0, we construct the right sequence so that $\sup I_n \to 1$ as $n \to \infty$, and

$$\sup I_{n-1} < 1 \Longrightarrow \inf I_n < \sup I_{n-1} \leqslant \sup I_n, \quad n = 1, 2, \ldots .$$

By Lemma 1, this is always possible because, due to (15), intervals $I^{(\iota)}_{xyz}$ provide an open cover for $[x_0, 1]$.[19] If $\sup I_{n-1} = 1$ and $I_{n-1} \ni 1$, the sequence stops at $((x_{n-1}, y_{n-1}, z_{n-1}), \iota_{n-1})$ (formally, all subsequent elements replicate it). If $\sup I_{n-1} = 1$ and $I_{n-1} \not\ni 1$, then the sequence stops at $((x_n, y_n, z_n), \iota_n)$, chosen so that $x_n = 1$. This should always be possible, because $\mathfrak{S}^{(1)} \cup \mathfrak{S}^{(2)}$ includes 1.

The left sequence is constructed so that $\inf I_n \to 0$ as $n \to -\infty$, and

$$\inf I_{n+1} > 0 \Longrightarrow \inf I_n \leqslant \inf I_{n+1} < \sup I_n, \quad n = -1, -2, \ldots .$$

By Lemma 1, this is always possible because, due to (15), intervals $I^{(\iota)}_{xyz}$ provide an open cover for $(0, x_0]$. If $\inf I_{n+1} = 0$ (0 is never included), the sequence stops at $((x_{n+1}, y_{n+1}, z_{n+1}), \iota_{n+1})$.

For definiteness, let $\iota_0 = 1$, that is, $(x_0, y_0, z_0) \in \mathfrak{T}^{(1)}$. By D3, for any $(x, y) \in I_0 \times J_0$ one can find $\mathbf{a}, \mathbf{b} \in \mathfrak{M} \in \mathbb{D}$ such that

$$L^{(1)}[(\mathbf{a}, \mathbf{b}, \mathbf{a})] = \Phi(x) + \Phi(y) = \Phi(z_0) + \Phi(x + y - z_0) = L^{(2)}[(\mathbf{a}, \mathbf{b}, \mathbf{a})].$$

z_0 being fixed, we have

$$\Phi(x) + \Phi(y) = R_0(x + y), \quad (x, y) \in I_0 \times J_0,$$

which is a restricted Pexider equation. Due to *D*2, Φ is continuous at $x_0 < \varepsilon$, whence, by Lemma 2(A),

$$\Phi(x) = m + kx, \quad x \in I_0.$$

We take this as our induction basis for extending this solution to interval $[0, 1]$ and specifying m and k in the process.

Dealing with the left sequence first, let this solution be established on $I_0 \cup \cdots \cup I_n$ for some $n \leqslant 0$,

$$\Phi(x) = m + kx, \quad x \in I_0 \cup \cdots \cup I_n, n \leqslant 0.$$

If $\inf I_n > 0$, then this solution holds on $(I_0 \cup \cdots \cup I_n) \cap I_{n-1} = (\inf I_n, \sup I_{n-1})$, whence, by Lemma 2(B), it extends to entire I_{n-1}. If $\inf I_n = 0$, the process stops. By this induction step we conclude that

$$\Phi(x) = m + kx, \quad x \in (0, \sup I_0).$$

Moreover, D2 implies that $m = 0$, because Φ is continuous on $[0, \varepsilon)$ and $\Phi(0) = 0$; and $k > 0$, because Φ is increasing on the same interval. We have, therefore,

$$\Phi(x) = kx, \quad x \in [0, \sup I_0), \quad k > 0.$$

Turning now to the right sequence, let this solution be established on $I_0 \cup \cdots \cup I_n$ for some $n \geqslant 0$,

$$\Phi(x) = kx, \quad x \in I_0 \cup \cdots \cup I_n, \quad n \geqslant 0.$$

If $\sup I_n < 1$, then this solution holds on $(I_0 \cup \cdots \cup I_n) \cap I_{n+1} = (\inf I_{n+1}, \sup I_n)$, whence it extends to entire $I_{n+1} \backslash \{\sup I_{n+1}\}$ by Lemma 2(B). If $\sup I_n = 1$ and $I_n \not\ni 1$, then the solution extends to $x = 1$ by Lemma 2(C). Finally, if $I_n \ni 1$, the process stops. By this induction step we conclude that

$$\Phi(x) = kx, \quad x \in (\inf I_0, 1], \quad k > 0$$

and combining this with the result for the left sequence,

$$\Phi(x) = kx, \quad x \in [0, 1], k > 0.$$

Finally, by trivial rescaling (see footnote 9 and Theorem 3, possibility P1), we can always put $k = 1$ to obtain $\Phi(h) = h$. $\square$

Identity Φ clearly satisfies properties (i) and (ii) in Theorem 1, and it agrees with possibility P1 (but not P2) of Theorem 3. We state this important result formally.

Corollary 2. *Desiderata* D1–D3 *agree with possibility* P1 *and exclude possibility* P2 *in Theorem* 3.

It is worth observing that the exclusion of possibility P2 by Theorem 6 is only based on the form of function Φ in the vicinity of zero: lim inf $\Phi(h)/h$ is 1 and not 0 for $\Phi(h) = h$. As stated in Theorem 3, however, possibility P2 in the case of continuous spaces also includes the statement of "cross-balancedness":

$$L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] = L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]$$

[18] The proof of this result has greatly benefited from comments made by Ali Ünlü in his review of an earlier draft of the paper.

[19] Due to the compactness of $[x_0, 1]$ we could, in fact, speak of the right sequence as being finite. In view of Remark 4 to Definition 2, however, we prefer a proof that would apply to any M replacing 1, exclusive or inclusive.

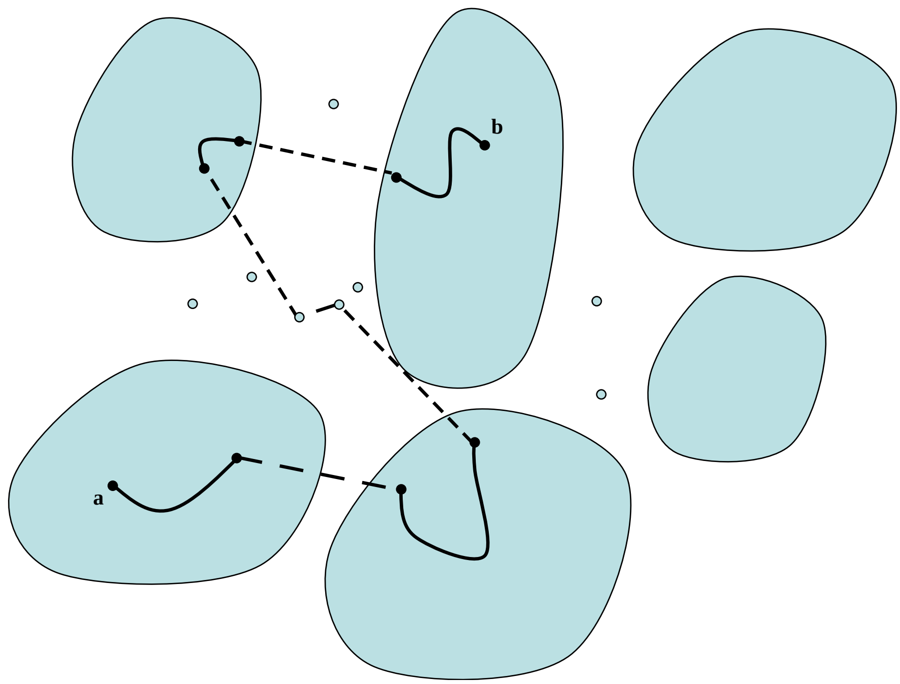

Fig. 3. Schematic representation of a space containing continuous components isolated from each other. Some of the components may be isolated points. The set of the components need not be finite or countable. The solid lines connected by dashed lines represent the graph of a chain-of-arcs connecting **a** to **b**.

for any piecewise smooth arc $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$. It is easy to show that in the case of discrete spaces this identity too would have to be excluded, as not compatible with Nonconstant Self-Dissimilarity. Indeed, by taking any two-element chain $(\mathbf{a}, \mathbf{b})$, the identity above implies

$$\Phi[\psi(\mathbf{a}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a})] = \Phi[\psi(\mathbf{a}, \mathbf{b}) - \psi(\mathbf{b}, \mathbf{b})],$$

and consequently (Φ being increasing) $\psi(\mathbf{a}, \mathbf{a}) = \psi(\mathbf{b}, \mathbf{b})$.

6. Spaces with isolated continuous components

We now consider a class of spaces of which both continuous spaces and discrete spaces are proper subclasses. A schematic representation of such a "*discrete-continuous*" space is given in Fig. 3. Intuitively, one can expect a space to be discrete-continuous if, for example, it is comprised of stimuli one characteristic whereof is discrete while other characteristics vary continuously for each value of the discrete one (e.g., a finite set of simple shapes varying in color, or letters of alphabet with continuously deformed graphical elements).

In the definition to follow it is tacitly assumed that, as any other stimulus space, a discrete-continuous space satisfies Axioms 1–4 given in Section 2. In particular, it satisfies Regular Minimality and can therefore be put in a canonical form.

Definition 3. Stimulus space $\mathfrak{M}$ (in a canonical form) is discrete-continuous if

$$\mathfrak{M} = \bigcup_{\omega \in \mathfrak{H}} \mathfrak{m}_\omega,$$

where $\mathfrak{H}$ is any indexing set, $\omega \to \mathfrak{m}_\omega$ is bijective, and for any $\omega \in \mathfrak{H}$,

(i) $\mathfrak{m}_\omega$ is arc-connected and satisfies all axioms for continuous spaces;[20]
(ii) on denoting $\mathfrak{M}_{\backslash\omega} = \bigcup_{\omega' \in \mathfrak{H} \backslash \{\omega\}} \mathfrak{m}_{\omega'}$,

$$\inf_{\substack{\mathbf{x} \in \mathfrak{m}_\omega \\ \mathbf{y} \in \mathfrak{M}_{\backslash\omega}}} \min\{\Psi^{(1)}(\mathbf{x}, \mathbf{y}), \Psi^{(2)}(\mathbf{x}, \mathbf{y})\} > 0.$$

For any $\omega \in \mathfrak{H}$, $\mathfrak{m}_\omega$ is referred to as an (isolated continuous) component of $\mathfrak{M}$.[21]

Any discrete space formally satisfies all the requirements of this definition because a single isolated stimulus is formally an arc-connected component. Discrete space therefore is a special case of discrete-continuous space. That continuous space is another special case (with $\mathfrak{H}$ a singleton) is obvious.

[20] See footnote 8. We do not need a complete list of axioms for the present development. It will suffice to note that all results presented in Section 3 apply to any component $\mathfrak{m}_\omega$.

[21] This definition and related aspects of the subsequent development have greatly benefited from comments made by Ali Ünlü in his review of an earlier draft of the paper.

Theorem 7. *Continuous components of discrete-continuous space $\mathfrak{M}$ are disjoint open sets. No two points belonging to different components can be connected by an arc.*

Proof. That $\mathfrak{m}_\omega, \mathfrak{m}_{\omega'}$ with $\omega \neq \omega'$ do not intersect is obvious from property (ii) of Definition 3.

The openness of $\mathfrak{m}_\omega$ is understood with respect to the topology based on open balls

$$\mathfrak{B}(\mathbf{x}, \varepsilon) = \{\mathbf{y} : \min\{\Psi^{(1)}(\mathbf{x}, \mathbf{y}), \Psi^{(2)}(\mathbf{x}, \mathbf{y})\} < \varepsilon\}$$

(Section 2.6). It is proved by denoting the infimum in (ii) by r_ω and observing that $\mathfrak{B}(\mathbf{x}, r_\omega) \subset \mathfrak{m}_\omega$ for any $\mathbf{x} \in \mathfrak{m}_\omega$.

Finally, assume that $\mathbf{a} \in \mathfrak{m}_\omega$ and $\mathbf{b} \in \mathfrak{m}_{\omega'}$ $(\omega \neq \omega')$ could be connected by arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$. Consider set

$$S_\omega = \{t \in [a, b] : t' \leqslant t \Longrightarrow \mathbf{x}(t') \in \mathfrak{m}_\omega\}.$$

It is nonempty (because $a \in S_\omega$) and $s = \sup S_\omega \leqslant b$. If $\mathbf{x}(s) \in \mathfrak{m}_\omega$, then $s < b$, and every interval $[s, b')$, $b' < b$, contains a point s' at which $\mathbf{x}(s') \in \mathfrak{M}_{\backslash\omega}$, as defined in (ii). Making $b' \to s+$ will induce $s' \to s+$ and, by continuity of $\mathbf{x}(t)$, $\mathbf{x}(s') \in \mathfrak{M}_{\backslash\omega} \to \mathbf{x}(s) \in \mathfrak{m}_\omega$. This contradicts (ii). If $\mathbf{x}(s) \in \mathfrak{M}_{\backslash\omega}$, then $s > a$ and $\mathbf{x}(s) \in \mathfrak{m}_{\omega''}$, for some $\omega'' \neq \omega$. We arrive at a contradiction by considering $s' \to s-$ leading to $\mathbf{x}(s') \in \mathfrak{m}_\omega \to \mathbf{x}(s) \in \mathfrak{m}_{\omega''}$. □

The notions of a chain for discrete spaces and an arc for continuous ones, in discrete-continuous spaces are generalized into the notion of a *chain-of-arcs* (see Figs. 3 and 4).

Definition 4. A chain-of-arcs in a discrete-continuous space $(\bigcup_{\omega \in \mathfrak{H}} \mathfrak{m}_\omega, \psi)$ is a finite sequence of arcs

$$(\mathbf{x}^{(i)}(t) : [a_i, b_i] \to \mathfrak{m}_{\omega_i})_{i=1,\ldots,k},$$

where $\omega_i \neq \omega_{i+1}$ $(i = 1, \ldots, k-1)$ but $i \to \omega_i$ is not necessarily one-to-one. A chain-of-arcs is said to connect $\mathbf{a} = \mathbf{x}^{(1)}(a_1) \in \mathfrak{m}_{\omega_1}$ to $\mathbf{b} = \mathbf{x}^{(k)}(b_k) \in \mathfrak{m}_{\omega_k}$. A chain-of-arcs is called allowable if all its arcs are piecewise smooth.

As in the case of arcs and chains, when no confusion is possible a chain-of-arcs can be written as

$$[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] = (\mathbf{a}_i \to \mathbf{x}^{(i)} \to \mathbf{b}_i)_{i=1,\ldots,k}.$$

"The same" chain-of-arcs but traversed "in the opposite direction" then can be denoted by

$$[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] = (\mathbf{b}_{k-i+1} \to \mathbf{x}^{(k-i+1)} \to \mathbf{a}_{k-i+1})_{i=1,\ldots,k}.$$

Definition 5. Given an allowable chain-of-arcs $(\mathbf{x}^{(i)}(t) : [a_i, b_i] \to \mathfrak{m}_{\omega_i})_{i=1,\ldots,k}$ connecting $\mathbf{a}$ to $\mathbf{b}$ (and representable therefore as $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$), its psychometric length of the ιth kind $(\iota = 1, 2)$ is

$$L^{(\iota)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] = \sum_{i=1}^{k} L^{(\iota)}[\mathbf{x}^{(i)}_{[a_i, b_i]}] + \sum_{i=2}^{k} \Gamma^{(\iota)}(\mathbf{b}_{i-1}, \mathbf{a}_i),$$

where $\mathbf{a}_i = \mathbf{x}^{(i)}(a_i)$, $\mathbf{b}_i = \mathbf{x}^{(i)}(b_i)$, $L^{(\iota)}[\mathbf{x}^{(i)}_{[a_i, b_i]}]$ is defined by (6), and $\Gamma^{(\iota)}(\mathbf{b}_{i-1}, \mathbf{a}_i) = \Phi[\Psi^{(\iota)}(\mathbf{b}_{i-1}, \mathbf{a}_i)]$, Φ being overall psychometric transformation.

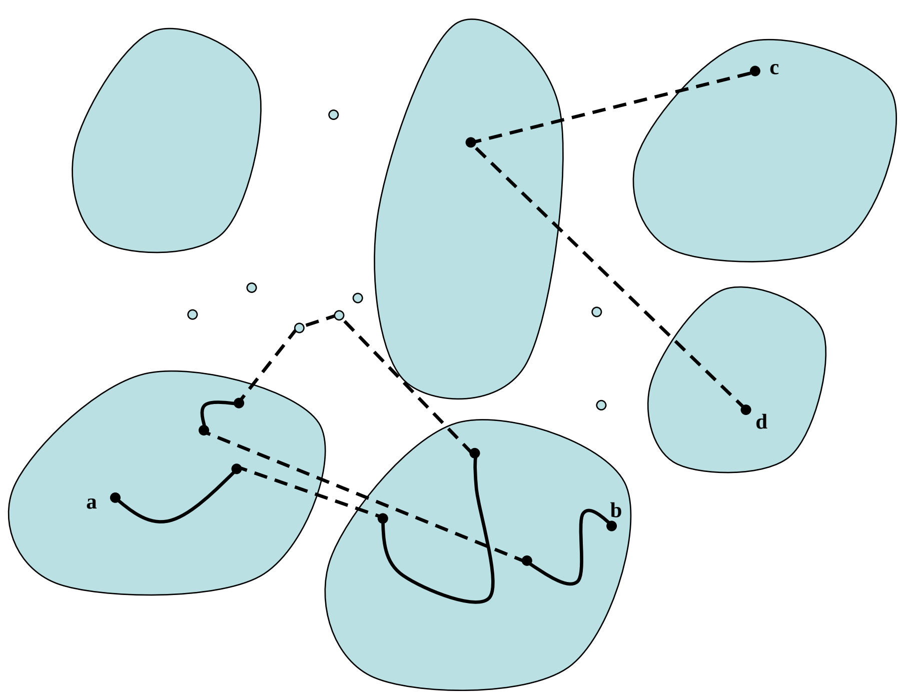

Fig. 4. A continuous component may contain more than one arc of a chain-of-arcs (as in the one connecting **a** to **b**). An arc within a component may be degenerate, single-point (as in the chain-of-arcs connecting **c** to **d**). But a "jump" cannot be made within a component.

In other words, the psychometric length of a chain-of-arcs is simply the sum of the lengths of its arcs (defined as in the continuous theory) and of the lengths of the two-element chains, or "jumps," connecting the successive arcs (defined as in the discrete theory).

Since Axiom 4 is assumed to hold for entire space $\mathfrak{M}$, it applies to any two nondegenerate smooth arcs $\mathbf{x}(t) : [a, b] \to \mathfrak{m}_\omega$ and $\mathbf{y}(\tau) : [c, d] \to \mathfrak{m}_{\omega'}$, whether or not $\omega = \omega'$. If $\mathfrak{M}$ contains some nondegenerate components (i.e., $\mathfrak{M}$ is not entirely discrete), then, by Theorem 1, overall psychometric transformation $\Phi(h)$ is determined for entire $\mathfrak{M}$, but only in an arbitrarily small vicinity of $h = 0$ and only asymptotically uniquely. This is all one needs to uniquely (see footnote 9) compute the lengths of all arcs $\mathbf{x}^{(i)}_{[a_i,b_i]}$ in a chain-of-arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$. The computation of "jumps" $\Phi[\Psi^{(\iota)}(\mathbf{b}_{i-1}, \mathbf{a}_i)]$, however, generally requires that Φ be extended from arbitrarily small $[0, \varepsilon)$ to $[0, 1]$. We face the same problem as for the discrete stimulus spaces.

This time, however, the solution is readily available: since discrete spaces form a proper subclass of the discrete-continuous spaces, it is natural to stipulate that a "sufficiently rich class" of the latter should include the sufficiently rich class of the former. Then, being guided by Desiderata D1–D3 of the previous section, the "only if" part of Theorem 6 constrains the possibilities for overall psychometric transformation to just one, $\Phi \equiv$ *identity*. It remains to verify the discrete-continuous counterpart of the "if" part of Theorem 6: Desiderata D1 and D2 being satisfied trivially, we have to focus on D3 and show that, with $\Phi \equiv$ *identity*,

(a) the generalization of Equation E1 in Theorems 2 and 5 does hold for all discrete-continuous spaces,
(b) oriented metrics $G_1(\mathbf{a}, \mathbf{b})$ and $G_2(\mathbf{a}, \mathbf{b})$ can be defined as in the continuous and discrete theories,
(c) and consequently, the generalization of Equation E_2 in Theorems 2 and 5 also holds for all discrete-continuous spaces.

Theorem 8. *If overall psychometric transformation* Φ *for discrete-continuous space* $\mathfrak{M}$ *is identity, then for any* $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$ *and any allowable chain-of-arcs* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$,

$$\begin{aligned} L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ = L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}). \end{aligned}$$

Consequently,

$$\begin{aligned} L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] \\ = L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] + L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}]. \end{aligned}$$

for all allowable chains-of-arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ *and* $\mathbf{b} \to \mathbf{y} \to \mathbf{a}$.

Proof. Let $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ be $(\mathbf{x}^{(i)}(t) : [a_i, b_i] \to \mathfrak{m}_{\omega_i})_{i=1,\dots,k}$. For every arc $\mathbf{x}^{(i)}(t) : [a_i, b_i] \to \mathfrak{m}_{\omega_i}$ connecting $\mathbf{a}_i$ to $\mathbf{b}_i$ $(i = 1, \dots, k)$ we have, by Corollary 1,

$$\begin{aligned} L^{(1)}[\mathbf{a}_i \to \mathbf{x}^{(i)} \to \mathbf{b}_i] - L^{(2)}[\mathbf{b}_i \to \mathbf{x}^{(i)} \to \mathbf{a}_i] \\ = L^{(2)}[\mathbf{a}_i \to \mathbf{x}^{(i)} \to \mathbf{b}_i] - L^{(1)}[\mathbf{b}_i \to \mathbf{x}^{(i)} \to \mathbf{a}_i] \\ = \psi(\mathbf{b}_i, \mathbf{b}_i) - \psi(\mathbf{a}_i, \mathbf{a}_i). \end{aligned}$$

For any jump $(\mathbf{b}_{i-1}, \mathbf{a}_i)$ $(i = 2, \dots, k)$ we have, by Theorem 4,

$$\begin{aligned} L^{(1)}[(\mathbf{b}_{i-1}, \mathbf{a}_i)] - L^{(2)}[(\mathbf{a}_i, \mathbf{b}_{i-1})] \\ = L^{(2)}[(\mathbf{b}_{i-1}, \mathbf{a}_i)] - L^{(1)}[(\mathbf{a}_i, \mathbf{b}_{i-1})] \\ = \psi(\mathbf{a}_i, \mathbf{a}_i) - \psi(\mathbf{b}_{i-1}, \mathbf{b}_{i-1}). \end{aligned}$$

The first statement of the theorem obtains by adding all these equations together, and the second follows trivially. □

To complete the construction, we define, as before,

$$G_\iota(\mathbf{a}, \mathbf{b}) = \inf_{\text{all allowable } \mathbf{a} \to x \to \mathbf{b}} L^{(\iota)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}], \quad \iota = 1, 2.$$

Theorem 9. $G_1(\mathbf{a}, \mathbf{b})$ *and* $G_2(\mathbf{a}, \mathbf{b})$ *are oriented metrics, and* $G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a})$.

Proof. That $G_\iota(\mathbf{a}, \mathbf{b})$ $(\iota = 1, 2)$ are nonnegative, vanish at $\mathbf{a} = \mathbf{b}$, and satisfy the triangle inequality is obvious.[22] To show that $G_\iota(\mathbf{a}, \mathbf{b}) > 0$ for $\mathbf{a} \neq \mathbf{b}$, observe that if $\mathbf{a} \in \mathfrak{m}_\omega$, $\mathbf{b} \in \mathfrak{m}_{\omega'}$ $(\omega \neq \omega')$, then any chain-of-arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ should contain a "jump" from $\mathfrak{m}_\omega$ to another component. Hence $G_\iota(\mathbf{a}, \mathbf{b})$ cannot fall below

$$\inf_{\substack{\mathbf{x} \in \mathfrak{m}_\omega \\ \mathbf{y} \in \mathfrak{M} \setminus \omega}} \{\Psi^{(\iota)}(\mathbf{x}, \mathbf{y})\},$$

which is positive by Definition 3. If $\mathbf{a}, \mathbf{b} \in \mathfrak{m}_\omega$, then the length of any $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ that "jumps" outside $\mathfrak{m}_\omega$ (before eventually returning to it) cannot fall below the infimum displayed above; while the length of any $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ that remains inside $\mathfrak{m}_\omega$ cannot fall below the infimum of the lengths of all such "internal" arcs (which is positive, as guaranteed by the continuous theory). $G_\iota(\mathbf{a}, \mathbf{b})$ therefore cannot fall below the smaller of these two quantities.

The equality of $G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a})$ and $G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a})$ immediately follows from Theorem 8. □

The overall Fechnerian distance can now be defined as before,

$$G(\mathbf{a}, \mathbf{b}) = G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a}).$$

[22] The latter due to the fact that any allowable chain-of-arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ when continued by any allowable chain-of arcs $\mathbf{b} \to \mathbf{y} \to \mathbf{c}$ forms an allowable chain-of-arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{y} \to \mathbf{c}$ (all arcs of the first chain followed by all arcs of the second).

7. Conclusion

There are two principal degrees of freedom in constructing Fechnerian Scaling: the choice of increasing continuous functions φ and Φ in increments

$$\Phi(\varphi[\psi(\mathbf{x},\mathbf{y})] - \varphi[\psi(\mathbf{x},\mathbf{x})]),$$
$$\Phi(\varphi[\psi(\mathbf{y},\mathbf{x})] - \varphi[\psi(\mathbf{x},\mathbf{x})]).$$

The logic of Fechnerian Scaling is compatible with first transforming (by φ) all discrimination probabilities and then transforming (by Φ) the increments of these transformed probabilities (both transformations being increasing and continuous). In our previous publications we (usually tacitly) assumed $\varphi \equiv$ *identity*, but as pointed out in Dzhafarov and Colonius (2005), as well as in Section 5 of the present paper (Remark 4 following Definition 2), all our results would have remained valid if we chose some other φ. The problem of determining φ in a principled way is tied in Dzhafarov and Colonius (2005) to one's choice of a response bias model. With some caveats, $\varphi \equiv$ *identity* corresponds to linear models of response bias (e.g., Luce, 1963), whereas if one opted, say, for the usual "equal-variance normal-normal" - version of signal detectability model (e.g., Green and Swets, 1966), φ would have to be determined from

$$\psi(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varphi[\psi(\mathbf{x},\mathbf{y})]} e^{-z^2/2}\,\mathrm{d}z.$$

As we do not know which model of response bias should be preferred, the question of determining φ remains essentially unsolved, and our use of "raw probabilities" ($\varphi \equiv$ *identity*) is justified essentially by simplicity considerations only. There are other ways of approaching the problem of determining φ, related to the "uncertainty blobs" introduced in Dzhafarov (2003b), but their discussion is far beyond the scope of this paper.

The problem of choosing Φ turns out to be more tractable. The idea of Fechnerian Scaling (as the name indicates) was derived from Fechner's (1860) original theory, or more precisely, from an interpretation of Fechner's theory in terms of discrimination probabilities which was proposed in Dzhafarov and Colonius (1999) and Dzhafarov (2001). Stated in our present terms, Fechner's idea was to compute the subjective distance between real-valued stimuli a and b by integrating over interval (a,b) the limit quantity[23]

$$\lim_{\alpha\to 0+} \frac{\psi(x, x+\alpha) - \psi(x,x)}{\alpha}.$$

Formally, this corresponds to putting $\Phi \equiv$ *identity*.

[23]Here, $\psi(x,y) = \Pr[$'x is different from y'$]$ is assumed to be computed from $\gamma(x,y) = \Pr[$'y is greater than x'$]$ by means of a transformation described in Dzhafarov and Colonius (2005) and Dzhafarov (2002b).

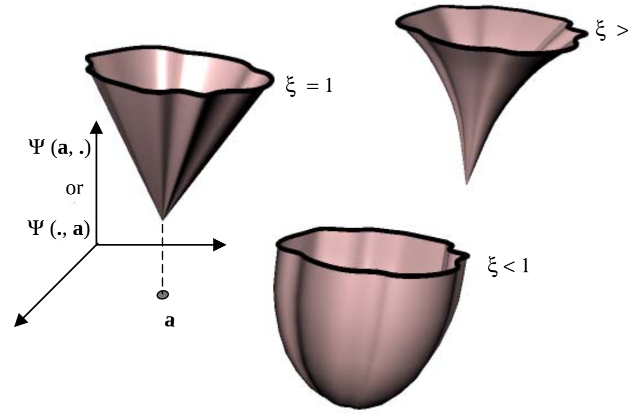

Fig. 5. "Rounded" ($\xi < 1$), "pencil-sharp" ($\xi = 1$), and "needle-sharp" ($\xi > 1$) forms of psychometric functions $\mathbf{x} \to \psi(\mathbf{a},\mathbf{x})$ (or $\mathbf{x} \to \psi(\mathbf{x},\mathbf{a})$) shown in a very small vicinity of $\mathbf{x} = \mathbf{a}$ (two-dimensional Euclidean case). ξ is the exponent of regular variation for overall psychometric transformation Φ.

On the initial stages of our development of Fechnerian Scaling (for multidimensional Euclidean spaces) it was not obvious at all that this choice was generalizable. In reference to Theorem 1, it was not obvious that

$$\lim_{\alpha\to 0+} \frac{\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))}{\alpha}$$

had to or even could be a positive quantity. In fact, it seemed plausible that psychometric functions $\mathbf{x} \to \psi(\mathbf{a},\mathbf{x})$ and $\mathbf{x} \to \psi(\mathbf{x},\mathbf{a})$ should be "rounded" at $\mathbf{x} = \mathbf{a}$ (see Fig. 5), which would make this limit vanish.[24] To make it positive one would have to transform $\Psi^{(\iota)}(\mathbf{x}(t_{j-1}), \mathbf{x}(t_j))$ by Φ with $\xi < 1$ (compare with Theorem 3).[25] The introduction in our theory of the two distinct observation areas and the formulation of Regular Minimality and Nonconstant Self-Dissimilarity in Dzhafarov (2002d) has proved this intuition to be wrong: at $\mathbf{x} = \mathbf{a}$ psychometric functions $\mathbf{x} \to \psi(\mathbf{a},\mathbf{x})$ and $\mathbf{x} \to \psi(\mathbf{x},\mathbf{a})$ can never be "rounded," they are (referring to Fig. 5) either "pencil-sharp" (possibility P1 in Theorem 3) or "needle-sharp"(possibility P2). A related finding in Dzhafarov (2002d) was the unexpected relationship

$$L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}]$$
$$= L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] + L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}],$$

$$G_1(\mathbf{a},\mathbf{b}) + G_1(\mathbf{b},\mathbf{a}) = G_2(\mathbf{a},\mathbf{b}) + G_2(\mathbf{b},\mathbf{a}),$$

[24]In other words, in this case $\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))$ would be of a greater order of infinitesimality than α. A parabolic roundness at $\mathbf{x} = \mathbf{a}$, e.g., would mean $\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))$ asymptotically proportional to α^2.

[25]To prevent confusion, this and related issues were discussed in our previous publications in terms of what we called *psychometric order*, $\mu = 1/\xi$, with the corresponding reversal of "<1" and ">1".

which allowed us to introduce overall Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ as the final outcome of Fechnerian Scaling, in place of four oriented distances $G_1(\mathbf{a}, \mathbf{b})$, $G_1(\mathbf{b}, \mathbf{a})$, $G_2(\mathbf{a}, \mathbf{b})$ and $G_2(\mathbf{b}, \mathbf{a})$. We see now that the only way for these equations to hold true in a "sufficiently rich" set of discrete-continuous stimulus spaces under the assumption that all such spaces share the same overall psychometric transformation Φ, is to have $\Phi \equiv identity$. In particular, $\Phi \equiv identity$ is the only choice for purely continuous and purely discrete spaces, for both belong to the class of discrete-continuous spaces. In the continuous case, $\Phi \equiv identity$ implies the exclusion of the "needle-sharp" psychometric functions (possibility P2 in Theorem 3).

As mentioned earlier, empirical data presented in Dzhafarov and Colonius (2005) exclude possibility P2. Since these experimental data are confined to specific kinds of stimuli, however, the generality of this conclusion is arguable. Now we have an additional, theoretical support for this conclusion: only possibility P1 is extendable to a "sufficiently rich class" of discrete and discrete-continuous stimulus spaces with the preservation of the Second Main Theorem of Fechnerian Scaling.

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