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2.2.158 Fechnerian Psychophysics

Fechnerian psychophysics is based on the idea that an appropriately chosen measure of *local discriminability* in a continuous stimulus space (the degree with which an observer can discriminate a stimulus from its very close neighbors) can be used to compute "subjective" distances between very close stimuli; and that by appropriately integrating these small distances along paths connecting different stimuli one can compute "subjective" (*Fechnerian*) distances among all stimuli comprising the space. This idea, together with experimental procedures for measuring local discriminability, was originally proposed by Fechner (1860, 1877) for unidimensional stimulus continua (such as the space of fixed-frequency tones varying in intensity, or visually presented line segments varying in length). Historians often date the beginning of scientific psychology from Fechner's work.

1. Classical Theory (Refined)

If a stimulus x changes along a single physical dimension (an interval of reals), and $F(x)$ is the *local discriminability measure* at x (positive and continuous), then the *Fechnerian distance* between any two stimuli a, b is defined as

$$G(a,b) = \left| \int_a^b F(x)\,dx \right|. \tag{1}$$

The Fechnerian distance clearly satisfies the defining properties of a metric: $G(a,a) = 0$, $G(a,b) > 0$ for $a \neq b$, $G(a,b) = G(b,a)$, and $G(a,b) \leq G(a,x) + G(x,b)$.

Assuming that x is much greater than the *lower absolute threshold* $x_{\inf}$ of the stimulus continuum, so that the random variability in $x_{\inf}$ can be ignored, the distance $G(x, x_{\inf})$ can be called the *Fechnerian magnitude* of x (traditionally, "*sensation magnitude*").

Let $\xi_x(y)$ be the probability with which y is perceived to be greater than x, and assume that $\xi_x(x) = \frac{1}{2}$ (the latter can be ensured by an appropriate recalibration of the reference stimuli x). Then the *local discriminability measure* $F(x)$ is defined as

$$F(x) = C \frac{d\xi_x(y)}{dy}\Big|_{y=x+}, \tag{2}$$

where certain regularity assumptions ensure that the derivative exists and continuously changes with x; $C > 0$ is a proportionality coefficient allowed to be different for different stimulus continua.

For $\frac{1}{2} \leq p < 1$, let $W_p(x)$ denote the value of y at which $\xi_x(y) = p$; then $W_{\frac{1}{2}}(x) = x$. The difference $W_p(x) - x$ is called the *just noticeable difference* at x on level p, or

p-JND, for short. Putting $y = W_p(x)$ and using the identity $\xi_x[W_p(x)] = p$, Eqn. 2 can be written as

$$F(x) = C\frac{dp}{dW_p(x)}\Big|_{p=\frac{1}{2}+} = C\lim_{p\to\frac{1}{2}+}\frac{p-\frac{1}{2}}{W_p(x)-x}.\tag{3}$$

Since by Eqn 1

$$\lim_{p\to\frac{1}{2}+}\frac{G[a,W_p(a)]}{p-\frac{1}{2}} = \frac{dG[a,W_p(a)]}{dp}\Bigg|_{p=\frac{1}{2}+} = \frac{d\int_a^{W_p(a)}F(x)dx}{dp}\Bigg|_{p=\frac{1}{2}+},$$

one can use the chain rule and apply Eqn. 3 to $F[W_p(a)]\big|_{p=\frac{1}{2}+} = F(a)$ to show that

$$\lim_{p\to\frac{1}{2}+}\frac{G[a,W_p(a)]}{p-\frac{1}{2}} = C.\tag{4}$$

By fixing p at a value sufficiently close to $\frac{1}{2}$ one concludes that $G[a,W_p(a)]$ is approximately constant for all a, because of which $G(a,b)$ in Eqn. 1 is roughly proportional to the number of chained p-JNDs, $W_p(a)-a$, $W_p[W_p(a)]-W_p(a)$, etc., that fit between a and b (see Dzhafarov and Colonius, 1999, for details). The p-JNDs can be estimated from empirical data by a variety of techniques (see, e.g., Falmagne 1985). The use of the methods of _limits_ and _adjustments_ in Fechnerian psychophysics, as these provide JNDs on unknown probability levels, is predicated

on the assumption that these JNDs are roughly proportional to p-JNDs on a constant level p sufficiently close to $\frac{1}{2}$.

2. Psychophysical Laws

If, with some choice of the physical measure for x, $d\xi_x(y)/dy$ at $y = x$ is proportional to $1/x$ (_Weber's law_), then Eqns. 1 and 2 yield $G(a,b) = C\log(a/b)$ (_Fechner's psychophysical law_). In spite of the enormous importance attached to Weber's and Fechner's laws in the history of psychology, the logic of Fechnerian psychophysics by no means implies them and is not contingent on their validity. Fechnerian psychophysics is equally consistent, for example, with such often found empirical approximations as $F(x) \quad 1/(x + \delta)$ (_modified Weber's law_) or $F(x) \quad 1/x^\alpha$ (called the _near miss to Weber's law_ if α is just below 1, and the _Fullerton-Cattell law_ if $\alpha \quad 0.5$). These two forms of $F(x)$ imply, respectively, $G(a,b) = C\log[(a + \delta)/(b + \delta)]$ and $G(a,b) = C\left(b^{1-\alpha} - a^{1-\alpha}\right)$ as substitutes for Fechner's law.

3. Criticisms of Classical Theory

3.1. *Infinitesimals versus finite differences*

The account in Sect. 1 is a refined interpretation of Fechner's original theory. It is supported by Fechner's writings, but being voluminous and rather lacking in

mathematical rigor, these writings lend themselves to alternative interpretations. According to one of them, Fechner believed in "true" irreducible JNDs, $W_*(x) - x$, and simultaneously maintained that $G[x, W_*(x)] = \text{const}$ ("***Fechner's postulate***"), and that Eqn. 1 holds with $F(x) = C/[W_*(x) - x]$ (compare with Eqn. 3). These propositions, however, are mutually inconsistent (Luce and Edwards 1958). Fechner knew of this criticism, and his 1877 rejoinder indicates that the interpretation it is based on is not correct. Whatever the historical truth, it is a sound scientific strategy to reserve the term "Fechnerian psychophysics" for the interpretation that preserves the essence of Fechner's approach while freeing it from logical contradictions.

3.2. *Fechnerian distances and subjective matching*

Consider four stimuli, say, tones with amplitudes and frequencies (A_1, f), (A_2, f), (A_1, f'), (A_2, f'), and let (A_1, f) and (A_1, f') be *matched in loudness*, and the same be true for (A_2, f) and (A_2, f'). Then the "loudness-wise" Fechnerian distance between (A_1, f) and (A_2, f) should equal that between (A_1, f') and (A_2, f'), and one could expect that the numbers of chained amplitude JNDs fitting between the first two and the last two tones are roughly equal (under the assumptions stipulated in Sect. 1). Riesz (1933) found this prediction wrong: for instance, the number of amplitude JNDs between two 4000-Hz tones may exceed that between the corresponding 200-Hz tones by a factor of four. This may be thought to invalidate the Fechnerian theory, as the difference seems too large to be attributed to the

approximation error associated with counting finite JNDs (on unknown probability levels). The counter-argument is as follows. If the amplitudes coupled with frequency f and those coupled with frequency f can be treated as *separate* stimulus continua, then the values of C in Eqn. 3 (and hence the approximate proportionality coefficients between Fechnerian distances and JND counts) may be different for f and f. If, on the other hand, the amplitude-frequency combinations should be treated as comprising a single two-dimensional stimulus space (see Sect. 5), then it is no longer justifiable to measure the Fechnerian distance between (A_1, f) and (A_2, f) along the straight line connecting them. For a detailed treatment of subjectively unidimensional (e.g., "loudness-wise") discriminations among multidimensional stimuli see Dzhafarov and Colonius (1999).

3.3. *Fechnerian magnitudes versus "directly" estimated magnitudes*

For certain stimulus continua, called __prothetic__ (intensity, length, duration, etc.), the Fechnerian magnitudes $G(x, x_{\text{inf}})$ computed from discrimination data are typically found nonlinearly related to magnitudes obtained by "__direct scaling__" methods (Stevens 1975). For instance, if Weber's law holds on a continuum of x, implying $G(x, x_{\text{inf}}) = C \log(x / x_{\text{inf}})$, the observer's numerical estimates of x are often found closely adhering to $S(x) = K x^{\alpha} = \exp[\lambda G(x, x_{\text{inf}})]$ (__Stevens's psychophysical law__). Many have agreed with Stevens that $S(x)$, and not $G(x, x_{\text{inf}})$, is the "true" measure of the "sensory magnitude" of x. As no independent definition of sensory

magnitude is available, however, the argument is largely semantic. The classical Fechnerian theory does not predict that all judgments of stimuli interpretable in terms of "how large", "how high", etc., must be proportional to Fechnerian magnitudes (although this happens to be the case for some *metathetic* continua, such as frequencies of constant-amplitude tones). A meaningful question to ask within the framework of Fechnerian psychophysics is whether one can propose a plausible mechanism, process, or computation by which an observer arrives at the $S(x)$-measures of prothetic stimuli based on their Fechnerian magnitudes $G(x, x_{\text{inf}})$. A naive but refinable example of such a proposal can be found in Ekman (1964).

4. Probability-distance hypothesis

According to Eqns. 1 and 2, Fechnerian distances $G(a,b)$ are computed from discrimination probabilities $\xi_x(y)$ taken in arbitrarily small vicinities of $y = x$. A natural question to ask, therefore, is whether $\xi_a(b)$ (provided it is not zero or one) is uniquely determined by $G(a,b)$ for all stimulus pairs a, b. More generally: can one find a continuous *internal* metric $D(a,b)$ and a continuous function f such that

$$\xi_a(b) = f\big[D(a,b)\big], \tag{5}$$

and if so, what is the relationship between $D(a,b)$ and the Fechnerian metric $G(a,b)$? Metric $D(a,b)$ is called *internal* (also, *inner*, or *intrinsic*) to the given unidimensional continuum if $D(a,b) + D(b,c) = D(a,c)$ for any $a < b < c$ on this

continuum. Thus the Fechnerian metric G is internal. The assumption that Eqn. 5 holds true can be called the *probability-distance hypothesis*, and the problem of finding D and f satisfying this hypothesis is traditionally referred to as "*Fechner's problem*" (Luce and Galanter 1963).

Primarily due to the criticism mentioned in Sect. 3.1, Fechnerian psychophysics since the 1960's was essentially reduced to the probability-distance hypothesis. One should note, however, that the classical theory presented in Sect. 1 does not imply the truth of this hypothesis. At the same time, the implication holds in the reverse direction: Pfanzagl (1962) proved that (subject to some regularity conditions) Eqn. 5 may hold for an internal metric D only if $D(a,b) = G(a,b)$. That is, discrimination probabilities may not depend on any internal metric other than the Fechnerian one.

Falmagne (1985) proved that Eqn. 5 is equivalent to either of the following two conditions: (*weak bicancellation*) $\xi_a(c)$ is uniquely determined by $\xi_a(b)$ and $\xi_b(c)$, for any a,b,c; (*weak quadruple condition*) $\xi_a(b) = \xi_a(b\,)$ if and only if $\xi_a(a\,) = \xi_b(b\,)$.

5. Multidimensional Fechnerian Psychophysics

A generalization of Fechnerian psychophysics to continuous stimulus spaces of arbitrary dimensionality (such as the CIE color space, or space of tones varying in both frequency and amplitude) was proposed by Dzhafarov and Colonius (1999). A special form of this generalization, however, can be traced back to the tradition,

originated by Helmholtz and Schrödinger, of computing colors metrics from color-discrimination data (Wyszecki and Stiles 1982). The discrimination of multidimensional stimuli $\mathbf{x} = (x_1,..., x_n)$ cannot be defined in terms of "greater-less", and the function $\xi_x(y)$ of the classical theory should be replaced with $\psi_x(\mathbf{y})$, the probability of $\mathbf{y}$ being perceived *different* from $\mathbf{x}$. Let $\psi_x(\mathbf{y})$ attains its minimum at $\mathbf{y} = \mathbf{x}$ (if necessary, after an appropriate recalibration of reference stimuli $\mathbf{x}$). The local discriminability measure $F(\mathbf{x}, \mathbf{u})$ *at stimulus* $\mathbf{x} = (x_1,..., x_n)$ *in direction* $\mathbf{u} = (u_1,..., u_n)$ is defined as

$$F(\mathbf{x}, \mathbf{u}) = C \lim_{s \to 0+} \frac{[\psi_x(\mathbf{x} + \mathbf{u}s) - \psi_x(\mathbf{x})]}{s}, \tag{6}$$

where the transformation is one and the same for all $\mathbf{x}$ and $\mathbf{u}$. The assumptions about $\psi_x(\mathbf{y})$ underlying the theory guarantee that $F(\mathbf{x}, \mathbf{u})$ is determined by $\psi_x(\mathbf{y})$ uniquely, and is continuous and positive, while is determined asymptotically uniquely (put roughly, uniquely in a small vicinity of zero). The local discriminability $F(\mathbf{x}, \mathbf{u})$ can be used to compute the *psychometric length* L of any path $\mathbf{x}(t)|_{\mathbf{a}}^{\mathbf{b}}$, $a \leq t \leq b$, lying within the stimulus space and connecting stimuli $\mathbf{a} = \mathbf{x}(a)$ and $\mathbf{b} = \mathbf{x}(b)$:

$$L[\mathbf{x}(t)|_{\mathbf{a}}^{\mathbf{b}}] = \int_a^b F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]dt. \tag{7}$$

The Fechnerian distance between $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$G(\mathbf{a},\mathbf{b}) = \inf L\left[\mathbf{x}(t)\Big|_{\mathbf{a}}^{\mathbf{b}}\right], \tag{8}$$

where the infimum is taken over all paths connecting $\mathbf{a}$ with $\mathbf{b}$. Under certain conditions, this infimum may equal the psychometric length of a certain ("shortest") curve connecting $\mathbf{a}$ with $\mathbf{b}$, a *Fechnerian geodesic*. Fechnerian distances are determined uniquely and are invariant with respect to all smooth one-to-one physical reparametrizations of the stimulus space. If $\psi_x(\mathbf{x}) = \text{const}$ (*constant self-similarity condition*), then Fechnerian distances are also invariant with respect to smooth monotone transformations $T\left[\psi_x(\mathbf{y})\right]$ (e.g., due to *response bias* changes).

The construction just described identifies the Fechnerian metric as *internal to the stimulus space*. (In general, a distance function is internal to a space if the distance between any two points in the space is the infimum of the lengths of all paths connecting these points and lying entirely within the space.) A multidimensional generalization of Pfanzagl's theorem on the probability-distance hypothesis holds: $\psi_a(\mathbf{b})$ (different from zero and one) may be uniquely determined by an internal metric $D(\mathbf{a},\mathbf{b})$ only if $D(\mathbf{a},\mathbf{b}) = G(\mathbf{a},\mathbf{b})$.

The multidimensional Fechnerian theory, when specialized to unidimensional continua, sheds additional light on some aspects of the classical theory. While the concept of a unidimensional Fechnerian distance readily generalizes to spaces of arbitrary dimensionality, no natural generalizations exist for the notions of a

Fechnerian ("sensation") magnitude and psychophysical law, such as Fechner's or Stevens's. This suggests that the importance traditionally attached to these notions may be unwarranted. By the same criterion (multidimensional generalizability), it is desirable to reformulate the classical theory in terms of the "same-different" discrimination probabilities $\psi_x(y)$, instead of the traditional "greater-less" probabilities $\xi_x(y)$, and to accordingly revise the empirical procedures for computing the local discriminability measure $F(x)$.

6. Concluding Remark

With discrimination among stimuli being arguably the most basic cognitive function, Fechnerian psychophysics is motivated by an expectation that distances computed from discrimination probabilities should have a fundamental status among behavioral measurements, and that in a final analysis various kinds of perceptual judgments could be shown to prominently depend on Fechnerian distances among stimuli involved. The eventual status of Fechnerian psychophysics will depend on the extent to which this expectation will be confirmed by future experimental and theoretical developments.

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