Mathematical Theorizing versus Mathematical Metaphorizing: A Commentary on Rudolph

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Abstract

Science may benefit from aesthetically pleasing and intellectually stimulating mathematical metaphors. To make them into mathematical theories, however, one has to complement them by links to well-defined theoretical primitives in turn linked to well-defined empirical procedures and observable phenomena. Rudolph's mathematical metaphors for psychological time are fascinating, but the mathematical rigor with which they can be described does not compensate for the conspicuous lack of both theoretical and operational clarity in the notions these metaphors are supposed to pertain to, including the very notion of psychological time.

Keywords. Mathematical metaphor, mathematical modeling, mathematical psychology

Anyone who can appreciate aesthetic qualities of mathematics is likely to derive a sense of pleasure from the dazzling spectrum of constructs Lee Rudolph outlines in his paper as possible representations for psychological time. "Subtle and beautiful," Rudolph says of them, and one cannot but agree. In dealing with psychological time, his paper tells us, one is not confined to the dullness of the axis of reals, one can explore "bolder" representations, go through the "meshes of colored threads" and "gardens of forking paths" all the way to the globally unidimensional acyclic directed graphs "made full." There is not a hint of levity in this description. I share with Rudolph his sense of fascination, and I do think that the intellectual inspiration most certainly provided by the structures he describes is a valuable commodity. My comments therefore are not meant to be dismissive. Rather my goal is to characterize the genre in which Rudolph's paper is written, and the name I propose for this genre (again, with no disparaging connotations) is *mathematical metaphorizing*.

I would like to begin, however, with commenting on another aspect of Rudolph's work, which some may consider more important for psychology than a discussion of specific mathematical structures for psychological time. Rudolph presents his impressive display of such structures as an example of the remedy for what he views as a lamentable state of affairs in psychology. "It is a shame," his article begins, "that so much modern mathematics—surely (at least in mathematicians' estimation) among the greatest inventions, or discoveries, of the human mind—has so far not found much application to the study of the human mind." And near the end of the article: "Modern mathematicians have discovered many subtle and beautiful structures, particularly of a qualitative nature, that have not yet been used by researchers in other disciplines to construct models of phenomena that they find interesting. Psychologists, particularly, seem mostly to have either limited the mathematics they use for modeling to a very narrow range, or to have rejected mathematical modeling altogether."

As a mathematical psychologist I can testify to the fact that no topic is as prominent at coffee-table conversations among my colleagues as the under-utilization of mathematics in psychology and the under-appreciation of the mathematics being utilized. I am sympathetic to Rudolph's characterization. It may, however, be useful to put this characterization in a proper context. The pertinent fact is that in psychological literature one can find an impressive variety of mathematical constructs. Some of them are even introduced on a more abstract level and more rigorously than it is customary in physics, the unattainable ideal of true science for most behavioral and social scientists. I will provide a few illustrations, with no attempt to be systematic, and focusing only on topics that lie within my own sphere of research interests. The reader, however, could easily complement and amplify these example by having leafed through a few issues of the *Journal of Mathematical Psychology*, *Mathematical Social Sciences*, or one of other periodicals in the area.

Scientific psychology began as a mathematical theory, with Fechner's (1860, 1877) celebrated quantification of the notion of sensation magnitude. While Fechner's math was not especially sophisticated, it led to elaborate computations of the Riemannian metric relations in the space of aperture colors by Helholtz (1891) and Schrödinger (1920, the same Schrödinger who is better known as a founder of quantum mechanics), resulting in the present-day remarkable geometric edifice of color science (Wyszecki & Stiles, 1982). Fechner-Helholtz-Schrödinger's work was also the obvious precursor of the Generalized Fechnerian Scaling, a theory of subjective dissimilarities among stimuli in whose creation and development I participated myself, first by showing how one could impose on stimulus spaces a generalization of Riemannian geometry known as Finsler geometry (Dzhafarov & Colonius, 1999), then moving gradually to progressively more abstract intrinsic metrics, axiomatically presented (Dzhafarov, 2002; Dzhafarov & Colonius, 2001, 2005a-b). In other contexts relations among perceived stimuli were described by means of non-metric spaces with affine connections (Levin, 2000) and infinite-dimensional Riemannian metrics of a special form (Townsend, Solomon, & Spencer-Smith, 2001). Somewhat closer to the topic of Rudolph's paper, in early 1990's I attempted to systematically reconstruct the kinematics of visual space-time, which turned out to be described by spatiotemporal transformations in motion similar to but more general than the Lorentzian transformations of Special Relativity (Dzhafarov, 1992).

The reason psychologists may need mathematical structures more abstract and even more diverse than those used in physics is that the empirical procedures upon which such theories are based often do not justify use of group symmetries, conservation laws, smoothness constraints, and similar considerations restricting the class of mathematical structures among which a physicist looks for the right ones. Another prominent reason is that in psychology we often know very little about the nature of our unobservables, such as internal representations of stimuli being compared with each other. Should such internal representations (perceptual images) be described by vectors of real numbers, by vectorial functions of such vectors, by functions on infinite-dimensional manifolds, etc.? The lack of a justification for any specific choice leads one to considering perceptual images on a very abstract level, say, as random entities defined on unspecified probability spaces (Dzhafarov, 2003b-c); and the necessity to understand what it might mean that each random entity is an image of a particular stimulus while these random entities are not necessarily stochastically independent leads to an abstract theory of selective influence under stochastic interdependence (Dzhafarov, 2003a).

I also mentioned that psychologists often introduce their mathematical constructs more rigorously than it is customary in physics. There, axiomatization as a rule only codifies an existing working theory, whereas a psychologist often presents a theory axiomatically from outset. Good recent illustrations can be found in

Luce (2002, 2004). Axiomatization as a methodology for psychological research (formulate several simple behavioral properties, usually in the language of abstract algebra, derive consequences, test them) lies in the heart of what is known as Representational Theory of Measurement (see, e.g., Krantz, Luce, Suppes, & Tversky, 1971, 1989, 1990, the three volumes cited in Rudolph's paper, mostly scornfully). A textbook example is the introduction of semiorders (Luce, 1956; Krantz et al., 1989), a structure designed to describe perceptual ordering with respect to a semantically unidimensional property (e.g., "later than").[1] Luce postulates (I use the relation "later than" for concreteness) that when presented a pair of stimuli (say, brief flashes x, y) an observer can sometimes judge one of them occurring later than the other ($x \succ y$ or $y \succ x$), that this judgment is irreflexive ($x \not\succ x$), and that it satisfies two other properties: if $x \succ y$ and $x' \succ y'$, then either $x \succ y'$ or $x' \succ y$; and if $x \succ y$ and $y \succ z$ then, for any u, either $x \succ u$ or $u \succ z$. A theorem says that if these properties are satisfied, one can map all our flashes into the set of reals (time line) in such a way that an observer judges x as occurring later than y if and only if the numerical value of x is greater than the numerical value of y by a constant quantity (say, 1). The axiomatic constructions proposed within the framework of Representational Theory of Measurement often lead to nontrivial functional equations, and the area of mathematics known by this name is primarily driven by behavioral and social applications (see, e.g., Aczel, 1987).

This is only a glimpse of the minuscule fraction of mathematics used in psychology (my list of cited literature alone would have easily exceeded the limits allocated to my commentary if I attempted to mention even most common mathematical themes within, say, just the area of psychophysics). This glimpse should suffice, however, to make the following point: although I presented my examples very much in the same manner Rudolph presents his (namely, by mentioning a mathematical structure being utilized), there is an important difference between the two, and this difference is depicted by the opposition of "mathematical theorizing" to "mathematical metaphorizing." The fact that Fechner used mathematics in his pioneering

[1] Rudoph could have used semiorders to complement his list, as well as many other structures described in Krantz et al.'s three volumes, especially the second. Rudolph's apparent negativism may be a result of not having recognized that virtually all "measurement-theoretic" constructions designed to deal with the classical issues of detection, discrimination, and scaling also apply to temporal judgments, pertaining thereby to (a large variety of different operationalizations of) the notion of perceptual time.

work is not a sufficient reason for why we date the beginnings of scientific psychology from his work. Fechner did not merely declare that sensation magnitudes can be mapped on the set of reals, he did not merely propose (ingeniously) that the difference between the sensations caused by stimuli a and b can be measured as $\int_a^b f(x)\,dx$, by integrating a local discriminability measure $f(x)$ from a to b. Several decades prior to Fechner another German thinker, Herbart (1824/1890), proposed another, ostensibly more general mathematical theory, in which he described how various "mental ideas," each characterized by a real-valued magnitude changing in time, interact, compete, sink below the level of consciousness and resurface again, fuse with each other, etc. (not too dissimilar to how we describe lateral inhibition and summation phenomena in modern psychophysics of vision and hearing). Herbart's mathematics, however, was purely metaphorical: he did not provide any operational means by which his "mental ideas" could be identified as separate entities, computed the intensity of, and tracked the evolution of in time. Fechner, by contrast, made his mathematics part of a scientific theory, by identifying a sensation through a physical stimulus which caused it, and by specifying the operational meaning of $f(x)$ (the discriminability of stimulus x from its "immediate neighbors") through a well-defined experimental procedure, the one in which an observer had to judge which of two stimuli presented was greater with respect to a designated semantically unidimensional property, such as brightness. Put in modern language (see Dzhafarov, 2001), $f(x)$ is defined entirely in terms of observable probabilities $\gamma(x,y) = \Pr[y \text{ is greater than } x]$. In Generalized Fechnerian Scaling which does not impose unidimensionality on stimuli or their images the probability-of-greater $\gamma(x,y)$ is replaced with the equally observable and equally well-defined probability-of-different, $\psi(x,y) = \Pr[y \text{ is not the same as } x]$, and the difference of two sensations caused by a and b is replaced with subjective dissimilarity between a and b, defined in terms of the function $\psi(x,y)$ computed along various pathways connecting a and b. All topological, metric, and analytic properties of perceived stimulus spaces therefore are expressible through one well-delimited procedure (same-different comparisons) and one well-defined observable entity (the probability-of-different function). Luce (2002, 2004), on the other hand, retains Fechner's unidimensionality of both stimuli and sensory attributes, but he replaces the greater-less discrimination procedure as the operational basis for sensation magnitudes with direct numerical judgments of the type "the distance of b from a is β times the distance of c from a." He also assumes (and verifies experimentally) that these judgments can be made when

a, b, and c are pairs of stimuli (a_1, a_2), (b_1, b_2), (c_1, c_2) rather than single stimuli. He then imposes a few algebraic constraints on the relations between (a_1, a_2), (b_1, b_2), (c_1, c_2), and β, each of the constraints being testable separately or having testable consequences when combined with other constraints. The focus of this example, however, is not on the empirical falsifiability of axioms (which is untenable as a general criterion) but on the fact that Luce's theoretical concepts, in addition to being defined by their relations to each other (in axioms, definitions, and theorems derived therefrom), are also defined in their relations to well-delimited empirical procedures and observable phenomena. The issue here is conceptual clarity, and it can only be achieved by this combination of mathematical and operational definiteness.

When Rudolph presents his spectrum of possible representations for psychological time, he is squarely in Herbart's camp, not in Fechner's. It is a spectrum of mathematical metaphors. Some of them may very well be engendering future psychological theories, but from the descriptions given I cannot guess what theories (in particular, theories of what) and how one could get there. Rudolph suggests, for instance, that profinite integers may be used to describe the "Balinese time." But it becomes clear from the quotes and the discussion that what Rudolph is trying to encode by profinite numbers is the Balinese calendar, rather than the way a Balinese "feels" about or "thinks" of time. The difference is obvious. All measurements of physical time are based on cyclic processes occurring in time. A physicist can measure time by a potential infinity of co-occuring cycles with arbitrarily different periods without ever doubting that time is representable by reals with conventional order and conventional topology (as it is thought to be in all conventional physics, including theory of relativity).[2] A calendar (or clock) is nothing but an artificially created cyclic process to measure "linear" time. In my own tribe of people measurements of time are based on a large number of such cyclic or near-cyclic processes (12 irregular individually named months, 4 individually names seasons, 7 individually named days, breakfast-lunch-dinner cycle, etc.). Why then the division of a 210-day year

[2]Rudolph's description of Special Relativity as imposing a partial order on time is somewhat misleading. Every flash-point with respect to any observer (frame of reference) has a definite temporal coordinate, a real number. The difference between space-like, time-like, and null intervals pertains to such issues as invariance of the temporal order of the two flash-points separated by an interval with respect to multiple observers, causality relations, recordability of a flash-point by an observer, etc., but not to the existence of a definite chronological precedence relation for all definable events with respect to any given observer.

into several regular periods ("weeks") with individual names variably assigned to the weeks and their days[3] warrants a radically different number system to encode it? Rudolph writes: "In this infinite scheme, all 'moments' in the unordered plenum of time are mutually co-present, each distinguished from all the others by (and only by) its own inventory, unique to it, of exactly where it lies with respect to all of the infinitely many commensurable cycles." But where in the Balinese system does one see the infinity of different cycles? And how can a human being (or any other animal for that matter) function and survive in a system of "mutually co-present," unordered moments? One cannot seriously entertain the possibility that a Balinese woman would not realize, say, that the day she designates as Kajeng Pon Anggara (names of the coinciding days of the 3, 5, and 7-day weeks, see footnote 3) is happening now, that the last Beteng Paing Coma day has already passed (so she can only remember things she has done during that day), and that the next Beteng Paing Coma is yet to come (so she has to plan for what she would do on that day). This is probably not what Rudolph means, but without a clear definition of the empirical referents of mutually co-present moments and infinite numbers of cycles all we have here is a poetic metaphor added to a mathematical one. Rudolph, of course, is not alone in confounding cycles of events with cycles of time, this seems to be a venerable tradition in anthropological thinking. Thus, the creation-existence-destruction cycles of the Hindu mythology are often said to mean that the Hindus think of time as "circular," whereas in fact the very notion of "everything starting all over again" semantically presupposes "ambient time" in which the repetitions occurs.

The same lack of operational (hence also conceptual) clarity we find in the central and final construction presented in Rudolph's paper, the "full time" structure. This is how he characterizes it at the end of the paper: "I have sketched the beginnings of a new model, full time, that (I think) is at once bolder mathematically than the social scientists', philosophers', and computer scientists' proposed models, and better adapted to psychology than those models because it is developed axiomatically from psychological

[3]The website http://www.edvos.demon.nl/bali/calendar1.htm (pages 1-3) gives a clear account of how the names are assigned: to each day of each week (there are 1, 2, ..., 9, 10-day weeks) and to each of the 7-day weeks. Particular combinations of the days of the 3, 5, and 7-day weeks are then designated for particular religious rites. The system in fact seems to be quite transparent: the difficulties only arise in relating it to the Gregorian calendar. If this description is correct, it entirely obviates the need for any nontrivial mathematical encoding.

primitives: the notions of attention and ambivalence." I find it impossible to understand what precisely the state of ambivalence is, and it is definitely very naive to speak of attention as a psychological primitive. (An analogy would be to speak of a "number" as a mathematical primitive, lumping together the reals, quaternions, and profinite rationals indiscriminately.) It is even stranger that Rudolph considers what he describes an axiomatic development *from* the two "primitive" notions. Clearly, as a purely mathematical construction the "full time" structure can be described axiomatically, but would these axioms include the terms "attention" and "ambivalence"? It is hard to believe they would, given the extreme degree of vagueness with which these concepts are described. In any case, no such axioms are given in the present paper. Rudolph speaks of his "full time" structure as based on "one common hypothesis about psychological time (that it reckons 'acts of attention')," but all one gets in the way of an explanation as to what this hypothesis might be are two equally vague quotations from Whitrow (1980), containing such egregiously wrong assertions as "strictly speaking, we can consciously attend to only one thing at a time." I very much doubt that the hypothesis of psychological time "depending on the fact that our minds operate by successive acts of attention" can be called a hypothesis, let alone common. Hypotheses can be wrong but they should state something definite. It is not clear whether Rudolph implies that there are no chronological precedence relations among perceptual images falling within a single "act of attention," or that this applies to physical periods between two such successive "acts." He does not tell us from where one could in principle learn any of these things, whatever they are (cannot be from reading Whitrow alone). It is not even clear whether when Rudolph speaks of psychological time he is thinking of temporal content of one's perceptual images (e.g., if one perceives a motion of a dot from one position to another, the percept involves time), one's ability to judge intervals of time and temporal precedence relations, or even one's mental functioning in physical time (to which the meaning of successive acts of attention probably pertains). I suspect my speculating on these possible meanings and on their possible relations to the "full time" construction would be much less interesting than the intricacies of the Balinese calendar.

I would like to repeat what I said earlier, that my critique is not aimed at diminishing the value of Rudolph's metaphors as metaphors: it may be potentially high. Similarly, my examples of mathematical theories in psychology do not speak to their value as theories: it may very well be proved low. My only issue here is that the two genres should be distinguished rather than intermixed. This too is an issue of conceptual clarity.

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References

Aczèl, J. (1987). *A Short Course of Functional Equations. Based Upon Recent Applications to the Social and Behavioral Sciences.* Dordrecht, Holland: Reidel.

Dzhafarov, E.N. (1992). Visual kinematics III: Transformation of spatiotemporal coordinates in motion. *Journal of Mathematical Psychology*, 36, 524-546.

Dzhafarov E.N. (2001). Fechnerian Psychophysics. In N.J. Smelser, P.B. Baltes (Eds.) *International Encyclopedia of the Social and Behavioral Sciences*, v. 8 (pp. 5437-5440). New York: Pergamon Press.

Dzhafarov, E.N. (2002). Multidimensional Fechnerian scaling: Pairwise comparisons, regular minimality, and nonconstant self-similarity. *Journal of Mathematical Psychology*, 46, 583-608.

Dzhafarov, E.N. (2003a). Selective influence through conditional independence. *Psychometrika*, 68, 7-26.

Dzhafarov, E.N. (2003b). Thurstonian-type representations for "same-different" discriminations: Deterministic decisions and independent images. *Journal of Mathematical Psychology*, 47, 208-228.

Dzhafarov, E.N. (2003c). Thurstonian-type representations for "same-different" discriminations: Probabilistic decisions and interdependent images. *Journal of Mathematical Psychology*, 47, 229-243.

Dzhafarov, E.N., & Colonius, H. (1999). Fechnerian metrics in unidimensional and multidimensional stimulus spaces. *Psychonomic Bulletin and Review*, 6, 239-268.

Dzhafarov, E.N., & Colonius, H. (2001). Multidimensional Fechnerian scaling: Basics. *Journal of Mathematical Psychology*, 45, 670-719.

Dzhafarov, E.N., & Colonius, H. (2005a). Psychophysics without physics: A purely psychological theory of Fechnerian Scaling in continuous stimulus spaces. *Journal of Mathematical Psychology*, 49, 1-50.

Dzhafarov, E.N., & Colonius, H. (2005b). Psychophysics without physics: Extension of Fechnerian Scaling from continuous to discrete and discrete-continuous stimulus spaces. *Journal of Mathematical Psychology*, 49, 125-141.

Fechner, G. T. (1860). *Elemente der Psychophysik* [Elements of psychophysics]. Leipzig: Breitkopf & Härtel.

Fechner G.T. (1877). *In Sachen der Psychophysik* [In the Matter of Psychophysics]. Breitkopf & Härtel, Leipzig.

Helmholtz, H. von (1891). Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem [An attempt at a generalized application of Fechner's Law to the color system]. *Zeitschrift für die Psychologie und die Physiologie der Sinnesorgane*, 2, 1-30.

Herbart, J. F. (1824). *Psychologie als Wissenschaft* [Psychology as science]. In K. Kehrbach & O. Flügel (Eds.), *Jon. Fr. Herbart's sämtliche Werke in chronologischer Reihenfolge* (Vol. 5, pp. 177-434). Langensalza, Germany: Hermann Beyer und Söhne, 1890.

Krantz, D. H, Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of Measurement,* vol. 1. San Diego, CA: Academic Press.

Krantz, D. H, Luce, R. D., Suppes, P., & Tversky, A. (1989). *Foundations of Measurement,* vol. 2. San Diego, CA: Academic Press.

Krantz, D. H, Luce, R. D., Suppes, P., & Tversky, A. (1990). *Foundations of Measurement,* vol. 3. San Diego, CA: Academic Press.

Levin, D. (2000). A differential geometric description of the relationships among perceptions. *Journal of Mathematical Psychology*, 44, 241-284 .

Luce, R.D. (1956). Semiorders and a theory of utility discrimination. *Econometrica*, 24, 178-191.

Luce, R. D., 2002. A psychophysical theory of intensity proportions, joint presentations, and matches. *Psychological Review,* 109, 520-532.

Luce, R. D., 2004. Symmetric and asymmmetric matching of joint presentations. *Psychological Review,* 111, 446-454.

Schrödinger, E. von (1920). Farbenmetrik [Color metrics]. *Zeitschrift für Physik*, 12, 459-466.

Townsend, J. T., Solomon, B., & Spencer-Smith, J. B. (2001). The perfect Gestalt: Infinite dimensional Riemannian face spaces and other aspects of face cognition. In M. J. Wenger and J. T. Townsend (Eds.), *Computational, geometric and process issues in facial cognition: Progress and challenges*. Mahwah, NJ: Erlbaum.

Whitrow, G. J. (1980). *The natural philosophy of time*. Oxford : Clarendon Press ; New York : Oxford University Press.

Wyszecki G., Stiles W.S. (1982). *Color Science: Concepts and Methods, Quantitative Data and Formulae*. Wiley, New York

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