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Multidimensional Fechnerian Scaling: Regular Variation Version

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The underlying assumptions of Fechnerian scaling are complemented by an assumption that ensures that any psychometric differential (the rise in the value of a discrimination probability function as one moves away from its minimum in a given direction) regularly varies at the origin with a positive exponent. This is equivalent to the following intuitively plausible property: any two psychometric differentials are comeasurable in the small (i.e., asymptotically proportional at the origin), without, however, being asymptotically equal to each other unless the corresponding values of the Fechner–Finsler metric function are equal. The regular variation version of Fechnerian scaling generalizes the previously proposed power function version while retaining its computational and conceptual simplicity.

1. INTRODUCTION

This paper introduces and justifies a special version of the theory of Fechnerian scaling proposed in its general form by Dzhafarov and Colonius (1999a, 1999b, 2001). This special version, in which the relationship between Fechnerian distances and discrimination probabilities is greatly simplified, is obtained by adding one intuitively plausible assumption to the three assumptions underlying the general theory.

To briefly outline the context, given an n-dimensional space of stimuli $\mathbf{x} = (x^1, \ldots, x^n)$ endowed with psychometric functions (Fig. 1)

$$\psi_{\mathbf{x}}(\mathbf{y}) = \text{Prob}[\mathbf{y} \text{ is discriminated from } \mathbf{x}],$$

the Fechnerian distances among stimuli are computed from *psychometric differentials*

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}),$$

where $s \geqslant 0$ is the *stimulus differential*, the magnitude of the transition from stimulus $\mathbf{x}$ to stimulus $\mathbf{x} + \mathbf{u}s$ in the direction $\mathbf{u} = (u^1, \ldots, u^n) \neq \mathbf{0}$. The underlying assumptions of Fechnerian scaling (briefly recapitulated in the next section) ensure

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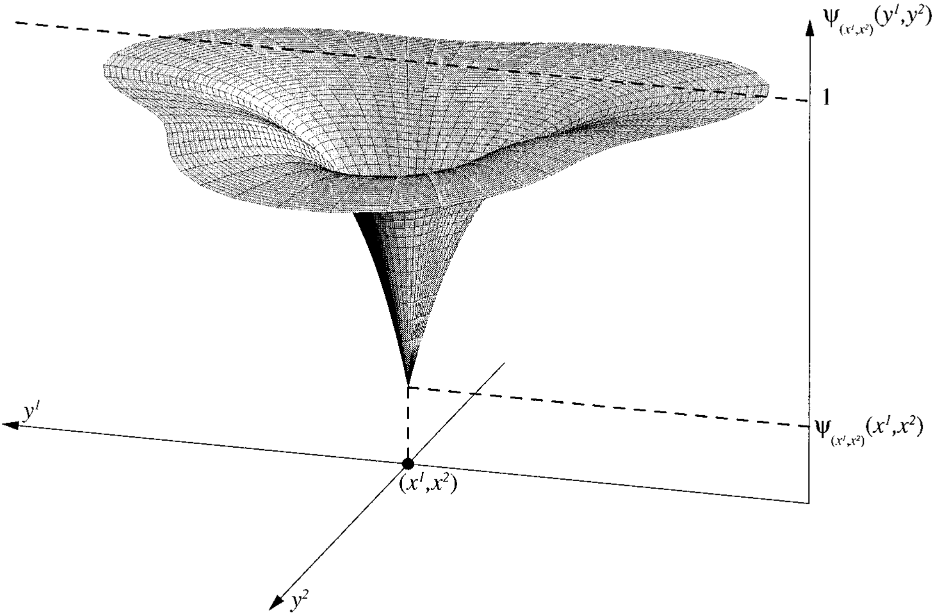

FIG. 1. Possible appearance of a psychometric function (in two-dimensional stimulus space).

that all psychometric differentials continuously increase with s in appropriately chosen (right-hand) vicinities of zero. Fechnerian scaling begins with computing from the psychometric differentials the *metric function* $F(\mathbf{x}, \mathbf{u})$, the function that determines the Fechnerian distances $G(\mathbf{x}, \mathbf{x}+\mathbf{u}s)$ between "infinitesimally close" stimuli (as $s \to 0+$). This in turn allows one to compute the *psychometric length* of any sufficiently smooth path connecting any two points **a** and **b** within the stimulus space, and, by finding the infimum of all such lengths, to compute the *Fechnerian distance* $G(\mathbf{a}, \mathbf{b})$ between these points. (See Dzhafarov and Colonius, 1999a, 2001, for details.)

The geometric aspects of Fechnerian scaling (i.e., the computation of Fechnerian distances from the metric function) are not discussed in this paper. It focuses instead on the initial step of Fechnerian scaling only, the computation of $F(\mathbf{x}, \mathbf{u})$ from psychometric differentials. The main assumption upon which this computation is based is that, for any two *line elements* $(\mathbf{x}_1, \mathbf{u}_1)$ and $(\mathbf{x}_0, \mathbf{u}_0)$ (i.e., two stimuli with attached to them directions of transition), the stimulus differentials s_1 and s_2 corresponding to equal psychometric differentials,

$$\psi_{\mathbf{x}_1}(\mathbf{x}_1+\mathbf{u}_1 s_1)-\psi_{\mathbf{x}_1}(\mathbf{x}_1)=\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s_0)-\psi_{\mathbf{x}_0}(\mathbf{x}_0)=h,$$

are comeasurable in the small (see Fig. 2). The precise meaning of the comeasurability in the small is that, as h in the expression above tends to zero, the two stimulus differentials, considered as functions of h, $s_0=s_0(h)$ and $s_1=s_1(h)$, tend to zero too, but their ratio converges to a finite positive quantity,

$$0<\lim_{h\to 0+}\frac{s_0(h)}{s_1(h)}<\infty.$$

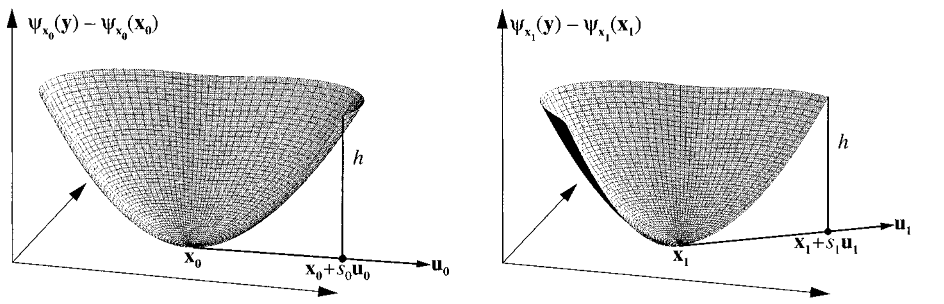

FIG. 2. Two psychometric differentials of equal magnitude and the corresponding stimulus differentials.

Based on this assumption, the *Fundamental Theorem* of Fechnerian scaling (Dzhafarov & Colonius, 2001) says that there exists a *global psychometric transformation* Φ, such that when this transformation is applied to psychometric differentials corresponding to one and the same stimulus differential s, it makes them all comeasurable in the small with s. The metric function $F(\mathbf{x}, \mathbf{u})$ is then the coefficient of the asymptotic proportionality between the Φ-transformed psychometric differentials and the stimulus differential s,

$$\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})] \sim F(\mathbf{x}, \mathbf{u})\, s \quad (\text{as } s \to 0+). \tag{1}$$

(Symbol $\sim$ indicates that the two expressions are *asymptotically equal*, i.e., their ratio tends to 1.) The transformation Φ in the general theory can be any function, provided it vanishes at zero and increases in some (right-hand) vicinity of zero.

The motivation for the present work comes from the *power function version* of Fechnerian scaling (Dzhafarov & Colonius, 1999a, 2001), in which all psychometric differentials are asymptotically representable as

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x}) \sim F(\mathbf{x}, \mathbf{u})^{\mu}\, s^{\mu} \quad (\text{as } s \to 0+). \tag{2}$$

The exponent $\mu > 0$ is referred to as the *psychometric order* of the stimulus space, and is determined uniquely. The representation (2) follows from the assumption that the global psychometric transformation Φ in (1) is, asymptotically, a power function (whose exponent, in reference to (2), is $1/\mu$):

$$\Phi(h) \sim \sqrt[\mu]{h} \quad (\text{as } h \to 0+).$$

Equation (2) makes the relationship between psychometric differentials and the corresponding values of the metric function especially transparent. To give just one example of the application of (2), consider the simple probability summation model in which $\psi_{\mathbf{x}}(\mathbf{x}) = 0$ and

$$1-\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = \prod_{i=1}^{n} [1-\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s)]$$

(where $\mathbf{u}_i$ is the vector obtained by projecting $\mathbf{u}$ on the ith coordinate). It can be shown (Dzhafarov, in press) that in view of (2) this model translates into the (local) Minkowski power function metric

$$F(\mathbf{x}, \mathbf{u})^{\mu} = \sum_{i=1}^{n} F(\mathbf{x}, \mathbf{u}_i)^{\mu}.$$

By contrast, with the general representation (1) the probability summation model does not lead to any discernible regularities in the metric function.

One argument in favor of the power function version of Fechnerian scaling is that the class of functions asymptotically equal to a power function is fairly broad. If, for example, the psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$ is differentiable at $\mathbf{y} = \mathbf{x}$ a sufficient number of times, and if μ is the order of the first of these derivatives that does not vanish (μ then is necessarily an even integer, as the derivative is taken at the minimum), then the Taylor expansion of the psychometric differential assumes the form

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = s^{\mu}\left[\frac{1}{\mu!}\sum_{i_1=1}^{n}\cdots\sum_{i_\mu=1}^{n}\frac{\partial^{\mu}\psi_{\mathbf{x}}(\mathbf{x})}{\partial x^{i_1}\cdots\partial x^{i_\mu}}u^{i_1}\cdots u^{i_\mu}\right] + o\{s^{\mu}\},$$

which is a special case of (2). The power function version, therefore, with μ being an arbitrary positive real, can be considered a generalization of this "sufficient smoothness at the minimum" assumption (that many would, erroneously, consider rather innocuous for applied purposes).

There is, however, a deeper argument in favor of the power function version. Consider the ratio of two psychometric differentials, taken at two line elements $(\mathbf{x}_1, \mathbf{u}_1)$ and $(\mathbf{x}_0, \mathbf{u}_0)$. In the power function version, as follows from (2),

$$\frac{\psi_{\mathbf{x}_1}(\mathbf{x}_1+\mathbf{u}_1 s) - \psi_{\mathbf{x}_1}(\mathbf{x}_1)}{\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)} \sim \frac{F(\mathbf{x}_1, \mathbf{u}_1)^{\mu}}{F(\mathbf{x}_0, \mathbf{u}_0)^{\mu}} \quad (\text{as } s \to 0+). \tag{3}$$

Any two psychometric differentials, therefore, are comeasurable in the small, and, moreover, they are not asymptotically equal to each other unless $F(\mathbf{x}_1, \mathbf{u}_1) = F(\mathbf{x}_0, \mathbf{u}_0)$. In view of (1), one might expect this intuitively plausible property to hold in general, but this is not the case. Consider, for example the following three models (formulated as equalities that hold in small right-hand vicinities of zero):

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = -\frac{1}{\log[F(\mathbf{x}, \mathbf{u})\, s]}, \tag{4}$$

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = \exp\left[-\frac{1}{F(\mathbf{x}, \mathbf{u})\, s}\right], \tag{5}$$

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = s\left\{1 + \frac{1}{2\pi+1}\sin[2\pi\log(F(\mathbf{x}, \mathbf{u})\, s)]\right\}. \tag{6}$$

One can verify that in all three models the psychometric differentials continuously increase with small values of $s>0$, because of which they satisfy (1), with Φ appropriately chosen for each of the models. At the same time, in the first model

$$\lim_{s\to 0+}\frac{\psi_{\mathbf{x}_1}(\mathbf{x}_1+\mathbf{u}_1 s)-\psi_{\mathbf{x}_1}(\mathbf{x}_1)}{\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s)-\psi_{\mathbf{x}_0}(\mathbf{x}_0)}=1,\ \text{ for any } (\mathbf{x}_1,\mathbf{u}_1), (\mathbf{x}_0,\mathbf{u}_0);$$

in the second model

$$\lim_{s\to 0+}\frac{\psi_{\mathbf{x}_1}(\mathbf{x}_1+\mathbf{u}_1 s)-\psi_{\mathbf{x}_1}(\mathbf{x}_1)}{\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s)-\psi_{\mathbf{x}_0}(\mathbf{x}_0)}=\begin{cases}0 & \text{if} \quad F(\mathbf{x}_1,\mathbf{u}_1)<F(\mathbf{x}_0,\mathbf{u}_0)\\ 1 & \text{if} \quad F(\mathbf{x}_1,\mathbf{u}_1)=F(\mathbf{x}_0,\mathbf{u}_0)\\ \infty & \text{if} \quad F(\mathbf{x}_1,\mathbf{u}_1)<F(\mathbf{x}_0,\mathbf{u}_0);\end{cases}$$

while in the third model the ratio of the two psychometric differentials tends to no definite limit.

These asymptotic properties correspond to the shapes of psychometric functions that one can safely consider empirically implausible. Consider the application of the models (4), (5), and (6) to the situation when $\mathbf{x}_1=\mathbf{x}_0=\mathbf{x}, \mathbf{u}_1=k\mathbf{u}_0=k\mathbf{u}$ $(k>0)$. Denoting

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})=\Psi_{\mathbf{x},\mathbf{u}}(s),$$

we have

$$\frac{\psi_x[\mathbf{x}+(k\mathbf{u})\,s]-\psi_{\mathbf{x}}(\mathbf{x})}{\psi_x[\mathbf{x}+\mathbf{u}s]-\psi_{\mathbf{x}}(\mathbf{x})}=\frac{\psi_x[\mathbf{x}+\mathbf{u}(ks)]-\psi_{\mathbf{x}}(\mathbf{x})}{\psi_x[\mathbf{x}+\mathbf{u}s]-\psi_{\mathbf{x}}(\mathbf{x})}=\frac{\Psi_{\mathbf{x},\mathbf{u}}(ks)}{\Psi_{\mathbf{x},\mathbf{u}}(s)}.$$

The ratio $\Psi_{\mathbf{x},\mathbf{u}}(ks)/\Psi_{\mathbf{x},\mathbf{u}}(s)$, taken at different values of $k>0$, characterizes the manner in which the psychometric differential $\Psi_{\mathbf{x},\mathbf{u}}(s)$ decreases to zero as $s\to 0+$. If, as in model (4), $\Psi_{\mathbf{x},\mathbf{u}}(ks)/\Psi_{\mathbf{x},\mathbf{u}}(s)$ always tends to 1, irrespective of k, then $\Psi_{\mathbf{x},\mathbf{u}}(s)$ is "infinitely sharp" in the vicinity of $s=0+$ (Fig. 3a). If, as in model (5), $\Psi_{\mathbf{x},\mathbf{u}}(ks)/\Psi_{\mathbf{x},\mathbf{u}}(s)$ tends to zero or infinity, for any $k\neq 1$, then in the vicinity of $s=0+, \Psi_{\mathbf{x},\mathbf{u}}(s)$, is "infinitely flat" (Fig. 3b). Finally, if, as in model (6), $\Psi_{\mathbf{x},\mathbf{u}}(ks)/\Psi_{\mathbf{x},\mathbf{u}}(s)$ oscillates between different limits, the $\Psi_{\mathbf{x},\mathbf{u}}(s)$ is "infinitely wavy" (Fig. 3c). Intuitively, these forms of asymptotic behavior can be thought of as "ungraphable": whatever the plot scale and however fine the plotting line, the "needle part" of Fig. 3a cannot be drawn as gradually diverging from the vertical axis, the "flat base" of Fig. 3b cannot be drawn as gradually diverging from the horizontal axis, and an infinite number of "waves" in Fig. 3c will always merge with the point of origin. Clearly, if $\Psi_{\mathbf{x},\mathbf{u}}(s)$ is a power function, $\Psi_{\mathbf{x},\mathbf{u}}(ks)/\Psi_{\mathbf{x},\mathbf{u}}(s)=k^{\mu}$, and the asymptotic behavior of $\Psi_{\mathbf{x},\mathbf{u}}(s)$ shows none of these irregularities. (See Comment 1 in the Appendix.)

A function $f(s)$, positive on some interval $0<s<a$, is referred to in this paper as a function *regularly varying at* $s=0+$ if

$$\lim_{s\to 0+}\frac{f(ks)}{f(s)}=\gamma(k)\not\equiv 1 \quad (\text{for any } k>0).$$

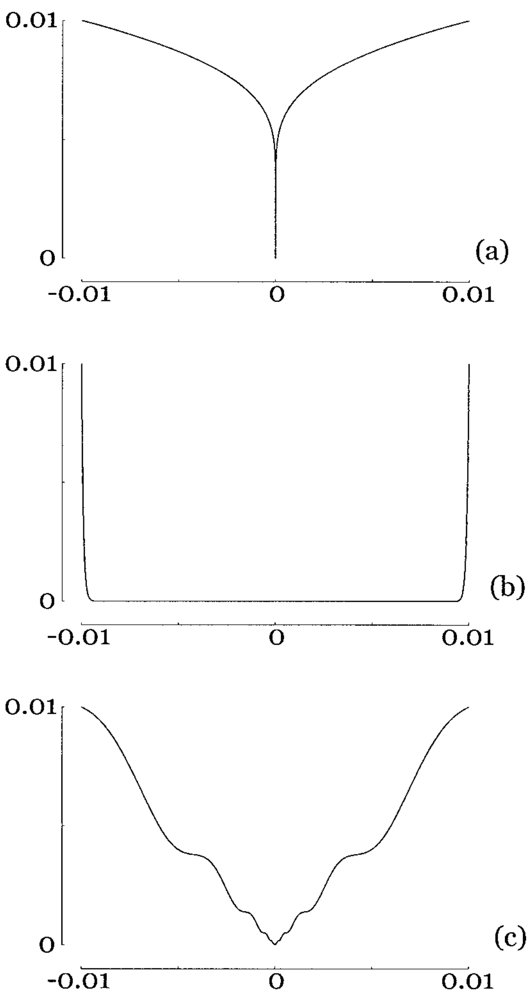

FIG. 3. Three psychometric differentials (shown with their symmetrically opposite pairs) corresponding to three models with irregular variation: (a) corresponds to Equation (4), (b) to (5), (c) to (6). The curves are scaled to reach the level 0.01 at $s = 0.01$.

(See Comment 2; here and in the remainder all references to numbered Comments refer the reader to the Appendix.) The *regular variation version* of Fechnerian scaling is obtained by adding to the underlying assumptions of the general theory the requirement that *at least one psychometric differential,* $\psi_{\mathbf{x}_0}(\mathbf{x}_0 + \mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)$, *be regularly varying at* $s = 0+$. I show in this paper that if this requirement is satisfied, then

(a) *all* psychometric differentials $\psi_x(\mathbf{x} + \mathbf{u}s) - \psi_x(\mathbf{x})$ are regularly varying;

(b) the global psychometric transformation Φ is regularly varying;

(c) all psychometric differentials are comeasurable in the small, being asymptotically equal to each other only if the corresponding values of the metric function $F(\mathbf{x}, \mathbf{u})$ are equal.

These three properties are satisfied in the power function version of Fechnerian scaling. The regular variation version is, in a sense, the broadest possible generalization of the power function version that retains these properties, retaining thereby the same transparency in the relationship between psychometric functions and metric functions.

2. REGULAR VARIATION VERSION

2.0. Underlying Assumptions

Fechnerian scaling is based on three assumptions (Dzhafarov & Colonius, 2001), that I describe here briefly.

The *First Assumption* is that the psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$ is continuous in $(\mathbf{x}, \mathbf{y})$ and, for any given $\mathbf{x}$, attains its single minimum at some point related to $\mathbf{x}$ by a smooth one-to-one function; and that within some neighborhood of this minimum the psychometric function increases in all directions. By a certain "recalibration" procedure (Dzhafarov & Colonius, 1999a, 2001) one can always ensure that the minimum of $\psi_{\mathbf{x}}(\mathbf{y})$ is attained at $\mathbf{y}=\mathbf{x}$, which makes all the psychometric differentials

$$\Psi_{\mathbf{x},\mathbf{u}}(s)=\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})$$

continuously decreasing to zero with $s\to 0+$.

(The term "decreasing to zero with $s\to 0+$" is used hereafter to designate "vanishing at $s=0$ and increasing on some interval $0<s<a$".)

The stimulus differential s in

$$h=\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})$$

can be presented as a function of h,

$$s=\Phi_{\mathbf{x},\mathbf{u}}(h) \quad (\text{as } h\to 0+).$$

The *Second Assumption* (already discussed in the Introduction) is that, for some fixed $(\mathbf{x}_0, \mathbf{u}_0)$ and for any $(\mathbf{x}, \mathbf{u})$, the limit ratio

$$\lim_{h\to 0+}\frac{\Phi_{\mathbf{x}_0,\mathbf{u}_0}(h)}{\Phi_{\mathbf{x},\mathbf{u}}(h)}=F(\mathbf{x},\mathbf{u})$$

is finite, positive, and continuous in $(\mathbf{x}, \mathbf{u})$. It follows (Dzhafarov & Colonius, 2001) that $F(\mathbf{x}, \mathbf{u})$ is positively Euler homogeneous:

$$F(\mathbf{x}, k\mathbf{u})=kF(\mathbf{x},\mathbf{u}) \quad (\text{for any } k>0).$$

The *Third Assumption* (that plays, however, no role in the present development) is that

$$\Phi_{\mathbf{x},\mathbf{u}}(h)\sim\Phi_{\mathbf{x},-\mathbf{u}}(h) \quad (\text{as } h\to 0+),$$

which is equivalent to

$$F(\mathbf{x}, -\mathbf{u}) = F(\mathbf{x}, \mathbf{u}).$$

I now add to this list the *Fourth Assumption of Fechnerian scaling*: *for some fixed* $(\mathbf{x}_0, \mathbf{u}_0)$, *the psychometric differential*

$$\Psi_0(s) = \psi_{\mathbf{x}_0}(\mathbf{x}_0 + \mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)$$

regularly varies at $s = 0+$. *This means that, for any* $k > 0$,

$$\gamma(k) = \lim_{s \to 0+} \frac{\Psi_0(ks)}{\Psi_0(s)}$$

is finite, positive, and varies with k. (Recall Comment 2.)

Note that the psychometric differential $\psi_{\mathbf{x}_0}(\mathbf{x}_0 + \mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)$ in the formulation of the Fourth Assumption is denoted by $\Psi_0(s)$ instead of the explicit $\Psi_{\mathbf{x}_0, \mathbf{u}_0}(s)$. This is done to simplify mathematical expressions in the subsequent development, where this particular psychometric differential is used repeatedly.

The Fourth Assumption links Fechnerian scaling with a well-known mathematical apparatus, the Karamata theory of slow and regular variation (Bingham, Goldie, & Teugels, 1987; Seneta, 1976). In the psychological literature this theory was first utilized by Colonius (1995), in the context of extreme-value distributions, where it plays a prominent role (Feller, 1971, pp. 275–284; Resnick, 1987). I show that the application of the Karamata theory to the First, Second, and the Fourth Assumptions of Fechnerian scaling leads one to the conclusion that psychometric differentials are representable as (compare with (2))

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) \sim F(\mathbf{x}, \mathbf{u})^{\mu}\, R^{\mu}(s) \quad (\text{as } s \to 0+),$$

where $\mu > 0$ is determined uniquely (one may continue to refer to it as the *psychometric order* of the stimulus space), and $R(s)$ is a regularly varying function of a special structure, determined asymptotically uniquely. The global psychometric transformation Φ is then also a regularly varying function, an asymptotic inverse of $R^{\mu}(s)$.

In the mathematics of this paper I rely on the systematic treatises by Bingham, Goldie, and Teugels (1987, primarily Chapter 1) and Seneta (1976), but with definitions and results modified to better suit our purposes (see Comment 3).

2.1. Immediate Consequences of the Fourth Assumption

The characterization of the psychometric differential $\Psi_0(s)$ satisfying the Fourth Assumption of Fechnerian scaling is based on the notion of *slow variation*. A function

$\ell(s)$ is said to be *slowly varying* (at $s \to 0+$) if it is positive on some interval $(0, a)$ and satisfies the equation

$$\lim_{s \to 0+} \frac{\ell(ks)}{\ell(s)} = 1 \quad \text{(for any } k > 0). \tag{7}$$

In this paper we are primarily concerned with continuous slowly varying functions, of which the following are simple examples (see Comment 4):

$$\ell(s) \equiv c > 0,\ \ell(s) = \log(1/s),\ \ell(s) = \log^{-1}(1/s),\ \ell(s) = e^s.$$

As $s \to 0+$, these functions tend, respectively, to c, ∞, 0, and 1. A slowly varying function need not, however, tend to any limit, finite or infinite.

Recall that by the First Assumption of Fechnerian scaling, psychometric differentials are positive and continuous at $s > 0$, and they decrease to zero with $s \to 0+$. The following lemma, therefore, applies to psychometric differentials.

LEMMA 2.1.1. *A positive function $f(s)$ continuously decreasing to zero with $s \to 0+$ satisfies the regular variation equation,*

$$\lim_{s \to 0+} \frac{f(ks)}{f(s)} = \gamma(k) \not\equiv 1 \quad (\textit{for any } k > 0), \tag{8}$$

only if, for some $\mu > 0$,

$$\gamma(k) = k^\mu, \tag{9}$$

in which case

$$f(s) = [s\ell(s)]^\mu, \tag{10}$$

where $\ell(s)$ is a slowly varying continuous function. Conversely, if $f(s)$ is of the form (10)*, then $\mu > 0$, and $f(s)$ satisfies* (8) *with $\gamma(k)$ given by* (9).

(See Comment 5.)

A *unit-regularly varying* function is defined as

$$R(s) = s\ell(s),$$

where $\ell(s)$ is slowly varying. Using this notion and Lemma 2.1.1, it follows from the First and Fourth Assumptions of Fechnerian scaling states that, for a certain line element $(\mathbf{x}_0, \mathbf{u}_0)$, the psychometric differential

$$\Psi_0(s) = \psi_{\mathbf{x}_0}(\mathbf{x}_0 + \mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)$$

can be presented as

$$\Psi_0(s) = R^\mu(s), \tag{11}$$

where μ is some positive constant and $R(s) = s\ell(s)$ is a unit-regularly varying function (continuously decreasing to zero with $s \to 0+$).

A function $R^{\alpha}(s)$, with $\alpha \neq 0$, is said to be *regularly varying* (*at* $s = 0+$) *with exponent* α. Thus the psychometric differential satisfying (11) is regularly varying with exponent $\mu > 0$, while any unit-regularly varying function $R(s)$ is regularly varying with exponent 1 (see Comment 6).

The proof of the following two simple facts is given in Comment 7.

LEMMA 2.1.2. (i) *The representation* $f(s) = [s\ell(s)]^{\alpha}$ *for a regularly varying function is unique.* (ii) *Any function* $f_1(s)$ *asymptotically equal to* $f(s)$ *is regularly varying with the same exponent and with the slowly varying component asymptotically equal to* $\ell(s)$,

$$f_1(s) = [s\ell_1(s)]^{\alpha},\ \ell_1(s) \sim \ell(s) \quad (\textit{as } s \to 0+).$$

Observe in conclusion that a regularly varying function $f(s)$ in (10) may, as a special case, asymptotically equal a power function,

$$f(s) \sim cs^{\mu} \quad (\text{as } s \to 0+).$$

Clearly, this happens if and only if the slowly varying component $\ell(s)$ is a positive constant or converges to such a constant as $s \to 0+$ (e.g., $\ell(s) = c + \log^{-1}(1/s)$).

2.2. *Characterization of the Global Psychometric Transformation*

The identification of

$$\Psi_0(s) = \psi_{\mathbf{x}_0}(\mathbf{x}_0 + \mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)$$

as a regularly varying function with a positive exponent only applies to a single line element $(\mathbf{x}_0, \mathbf{u}_0)$. In the next subsection I show that this characterization can be extended to psychometric differentials in general. To achieve this extension, however, one first must establish the relationship between $\Psi_0(s)$ and the global psychometric transformation Φ.

We know from the Fundamental Theorem of Fechnerian scaling that $\Phi(h)$ decreases to zero with $h \to 0+$, and, when applied to the line element $(\mathbf{x}_0, \mathbf{u}_0)$ of the Fourth Assumption,

$$\Phi[\Psi_0(s)] = \Phi[\psi_{\mathbf{x}_0}(\mathbf{x}_0 + \mathbf{u}_0 s) - \psi_{\mathbf{x}_0}(\mathbf{x}_0)] \sim F(\mathbf{x}_0, \mathbf{u}_0)\, s \quad (\text{as } s \to 0+).$$

Since the uniqueness of the metric function $F(\mathbf{x}, \mathbf{u})$ is only up to multiplication by a positive constant, one can put $F(\mathbf{x}_0, \mathbf{u}_0) = 1$ with no loss of generality, and rewrite the expression above as

$$\Phi[\Psi_0(s)] \sim s \qquad (\text{as } s \to 0+). \tag{12}$$

By definition, this characterizes Φ as an *asymptotic inverse* of Ψ_0 (see Comment 8).

Clearly, $\Phi(h)$ is determined by $\Psi(s)$ asymptotically uniquely: since the latter is continuously increasing in some neighborhood of $s=0+$, the asymptotic equality

$$\Phi_1[\Psi_0(s)] \sim \Phi_2[\Psi_0(s)] \sim s \quad (\text{as } s \to 0+)$$

is equivalent to

$$\lim_{h\to 0+} \frac{\Phi_1(h)}{\Phi_2(h)} = \lim_{s\to 0+} \frac{\Phi_1[\Psi_0(s)]}{\Phi_2[\Psi_0(s)]} = \frac{\lim_{s\to 0+} \{\Phi_1[\Psi_0(s)]/s\}}{\lim_{s\to 0+} \{\Phi_2[\Psi_0(s)]/s\}} = 1.$$

One can therefore use (12) to compute any one from the set of *asymptotically equal variants* of Φ, and (as Φ is determined only asymptotically uniquely in the general theory) identify Φ as any function asymptotically equal to this variant. The obvious variant of Φ to consider is the *precise inverse* Ψ_0^{-1} of Ψ_0. This inverse exists on any vicinity of $s=0+$ that is sufficiently small for $\Psi_0(s)$ to be continuously increasing.

A characterization of Ψ_0^{-1} can be achieved with the help of the following lemma (also used in the next subsection; its proof is given in Comment 9).

LEMMA 2.2.1. *Let $f(t)$ be a regularly varying (at $t=0+$) function continuously decreasing to zero with $t\to 0+$. Let $g(\tau)$ be continuously decreasing to zero with $\tau\to 0+$. Then*

$$f[g(\tau)] \sim f[\tilde{g}(\tau)] \quad (as\ \tau \to 0+)$$

implies

$$g(\tau) \sim \tilde{g}(\tau) \quad (as\ \tau \to 0+).$$

Now, $\Psi_0(s)$ continuously decreases to zero with $s\to 0+$ and regularly varies with an exponent $\mu > 0$, while $\Psi_0^{-1}(h)$ continuously decreases to zero with $h\to 0+$. Since

$$\lim_{h\to 0+} \frac{\Psi_0[\sqrt[\mu]{k}\,\Psi_0^{-1}(h)]}{h} = \lim_{h\to 0+} \frac{\Psi_0[\sqrt[\mu]{k}\,\Psi_0^{-1}(h)]}{\Psi_0[\Psi_0^{-1}(h)]} = \lim_{s\to 0+} \frac{\Psi_0[\sqrt[\mu]{k}\,s)}{\Psi_0(s)} = k,$$

on using the identity

$$\frac{\Psi_0[\Psi_0^{-1}(kh)]}{h} = k$$

we conclude that

$$\Psi_0[\sqrt[\mu]{k}\,\Psi_0^{-1}(h)] \sim \Psi_0[\Psi_0^{-1}(kh)] \quad (\text{as } h \to 0+).$$

From this and the lemma above,

$$\sqrt[\mu]{k}\,\Psi_0^{-1}(h) \sim \Psi_0^{-1}(kh) \quad (\text{as } h \to 0+),$$

whence we conclude that $\Psi_0^{-1}(h)$ is regularly varying with exponent $1/\mu$. Replacing now $\Psi_0^{-1}(h)$ with an arbitrary variant of $\Phi(h)$ (continuously decreasing to zero with $h \to 0+$), we have, in accordance with Lemma 2.1.2,

$$\lim_{h \to 0+} \frac{\Phi(kh)}{\Phi(h)} = \sqrt[\mu]{k} \quad (\text{for any } k > 0).$$

Hence, by Lemma 2.1.1,

$$\Phi(h) = \sqrt[\mu]{R^*(h)} = \sqrt[\mu]{h\ell^*(h)}, \tag{13}$$

where $R^*(h)$ is some (continuously decreasing to zero with $h \to 0+$) unit-regularly varying function with some slowly varying component $\ell^*(s)$. This completes the proof of

THEOREM 2.2.1. *The global psychometric transformation $\Phi(h)$ is an asymptotic inverse of the psychometric differential $\Psi_0(s)$ of the Fourth Assumption of Fechnerian scaling, and $\Phi(h)$ is determined by $\Psi_0(s)$ asymptotically uniquely. As $\Psi_0(s)$ is regularly varying* (*at $s = 0+$*) *with exponent μ, $\Phi(h)$ is regularly varying* (*at $h = 0+$*) *with exponent $1/\mu$, thereby satisfying* (13).

To characterize the relationship between the slowly varying components of $\Psi_0(s) = [s\ell(s)]^\mu$ and $\Phi(h) = \sqrt[\mu]{h\ell^*(h)}$, it is convenient to rewrite these two functions as

$$\Psi_0(s) = \frac{s^\mu}{L(s)^\mu}, \; \Phi(h) = \frac{\sqrt[\mu]{h}}{L^*(\sqrt[\mu]{h})}, \tag{14}$$

where, as one can easily check, the functions

$$L(s) = \frac{1}{\ell(s)}, \; L^*(h) = \frac{1}{\sqrt[\mu]{\ell^*(h^\mu)}}$$

are both slowly varying at the origin. Substituting $\Psi_0(s)$ for h in the expression for $\Phi(h)$,

$$\Phi[\Psi_0(s)] = \frac{\dfrac{s}{L(s)}}{L^*\left[\dfrac{s}{L(s)}\right]} = s\frac{1}{L(s)\, L^*\left[\dfrac{s}{L(s)}\right]},$$

one concludes that (12) is satisfied if and only if

$$\lim_{s \to 0+} \left\{ L(s)\, L^*\left[\frac{s}{L(s)}\right]\right\} = 1. \tag{15}$$

Slowly varying function $L(s)$ and $L^*(s)$ related to each other in this way are called *de Bruijn conjugates* of each other. As an example, if $L(s) = \log(1/s)$, then one of its de Bruijn conjugates is $L^*(s) = \log^{-1}(1/s)$, because

$$\log(1/s) \frac{1}{\log\left[\frac{\log(1/s)}{s}\right]} \to 1 \quad (\text{as } s \to 0+).$$

We can now complement Theorem 2.2.1 by the following statement.

THEOREM 2.2.2. *In Theorem* 2.2.1, *$\Psi_0(s)$ and $\Phi(h)$ satisfy* (14), *with $L(s)$ and $L^*(s)$ being de Bruijn conjugates of each other.*

It is known (Bingham *et al.*, 1987, p. 29; Seneta, 1976, pp. 25–29), that a de Bruijn conjugate $L^*(s)$ exists and is asymptotically unique for any slowly varying function $L(s)$, and that $L^{**}(s) \sim L(s)$. The latter means that, in addition to (15),

$$\lim_{s \to 0+} \left\{ L^*(s)\, L\left[\frac{s}{L^*(s)}\right]\right\} = 1. \tag{16}$$

Techniques for computing de Bruijn conjugates for several classes of slowly varying functions are presented in Bingham *et al.* (1987, pp. 433–435; recall Comment 3).

2.3. *Asymptotic Representation of Psychometric Differentials*

Now we are ready to develop a characterization for arbitrary psychometric differentials

$$\Psi_{\mathbf{x},\mathbf{u}}(s) = \psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}).$$

Without mentioning this every time, *in all statements below $\Psi_{\mathbf{x},\mathbf{u}}(s)$ is taken to satisfy the First, Second, and the Fourth Assumptions of Fechnerian scaling.*

The Fundamental Theorem of Fechnerian scaling tells us that

$$\Phi[\Psi_{\mathbf{x},\mathbf{u}}(s)] \sim F(\mathbf{x}, \mathbf{u})\, s \quad (\text{as } s \to 0+),$$

whence

$$\Phi\{\Psi_{\mathbf{x},\mathbf{u}}[s/F(\mathbf{x}, \mathbf{u})]\} \sim s \quad (\text{as } s \to 0+).$$

At the same time, referring to the line element $(\mathbf{x}_0, \mathbf{u}_0)$ of the Fourth Assumption,

$$\Phi[\Psi_0(s)] \sim s \quad (\text{as } s \to 0+),$$

and, by Lemma 2.2.1, $\Psi_{\mathbf{x},\mathbf{u}}[s/F(\mathbf{x}, \mathbf{u})]$ and $\Psi_0(s)$ must be asymptotically equal. Equivalently,

$$\Psi_{\mathbf{x},\mathbf{u}}(s) \sim \Psi_0[F(\mathbf{x}, \mathbf{u})\, s] \quad (\text{as } s \to 0+),$$

whence, on recalling (11),

$$\Psi_{\mathbf{x},\mathbf{u}}(s) \sim R^{\mu}[F(\mathbf{x},\mathbf{u})\, s] \sim F(\mathbf{x},\mathbf{u})^{\mu}\, R^{\mu}(s) \quad (\text{as } s \to 0+).$$

This completes the proof of

THEOREM 2.3.4. *All psychometric differentials are asymptotically representable as*

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) \sim F(\mathbf{x},\mathbf{u})^{\mu}\, R^{\mu}(s) \quad (\text{as } s \to 0+). \tag{17}$$

with one and the same exponent $\mu > 0$, determined uniquely, and one and the same unit-regularly varying function $R(s) = s\ell(s)$, determined asymptotically uniquely.

Observe that being asymptotically equal to a regularly varying function, any psychometric differential, by Lemma 2.1.2, is itself a regularly varying function (with the same exponent),

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = F(\mathbf{x},\mathbf{u})^{\mu}\, [s\ell_{\mathbf{x},\mathbf{u}}(s)]^{\mu}.$$

This does not, however, allow one to replace (17) with precise equalities, because $\ell_{\mathbf{x},\mathbf{u}}(s)$ is not one and the same for all line elements $(\mathbf{x}, \mathbf{u})$.

Observe also that as $R(s)$ is the same in both (17) and (11), it continuously decreases to zero with $s \to 0+$. At the same time, $R(s)$ in (17) may, obviously, be replaced with any of its asymptotically equal variants, $\bar{R}(s) \sim R(s)$, including those that converge to zero without being continuous or strictly increasing in any vicinity of zero (i.e., without being continuously decreasing to zero with $s \to 0+$). As all asymptotic variants of $R(s)$ would lead to the same $F(\mathbf{x}, \mathbf{u})$, such a replacement cannot affect Fechnerian computations. This is the reason why the monotonic and continuous decrease of $R(s)$ is not mentioned in the formulation of Theorem 2.3.1.

As a corollary to Theorem 2.3.1, we have

THEOREM 2.3.2. *There is a unique constant $\mu > 0$ (called the psychometric order of the stimulus space) and there is an asymptotically unique unit-regularly varying function $R(s)$, such that the metric function $F(\mathbf{x}, \mathbf{u})$ is representable as*

$$F(\mathbf{x},\mathbf{u}) = \lim_{s \to 0+} \frac{\sqrt[\mu]{\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \Psi_{\mathbf{x}}(\mathbf{x})}}{R(s)}. \tag{18}$$

This proposition provides an alternative to the use of the global psychometric transformation in the Fechnerian theory. In the regular variation version of Fechnerian scaling one can always use (18) instead of the equivalent representation

$$F(\mathbf{x},\mathbf{u}) = \lim_{s \to 0+} \frac{\Phi[\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \Psi_{\mathbf{x}}(\mathbf{x})]}{s},$$

where Φ, as we know, is regularly varying with exponent $1/\mu$.

Another immediate consequence of Theorem 2.3.1 is the following proposition.

THEOREM. *Any two psychometric differentials are comeasurable in the small, with*

$$\frac{\psi_{\mathbf{x}_1}(\mathbf{x}_1+\mathbf{u}_1 s)-\psi_{\mathbf{x}_1}(\mathbf{x}_1)}{\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s)-\psi_{\mathbf{x}_0}(\mathbf{x}_0)} \sim \frac{F(\mathbf{x}_1, \mathbf{u}_1)^\mu}{F(\mathbf{x}_0, \mathbf{u}_0)^\mu} \quad (as\ s \to 0+). \tag{19}$$

The importance of this fact is in that it is *precisely the same as in the power function version of Fechnerian scaling,* (3).

3. CONCLUSION

Considerations of simplicity, computational or conceptual, have always been powerful, if not philosophically noncontroversial, guides in constructing scientific theories. When, however, one chooses a particular version of a general theory on the grounds that this version affords a special degree of computational simplicity or conceptual transparency, the question arises whether this particular version of the theory is the only one or the most general one to have this property. When this question is applied to the power function version of Fechnerian scaling, the answer, as shown in this paper, turns out to be negative. The attractiveness of the power function version stems from the fact that the ratio

$$\frac{\psi_{\mathbf{x}_1}(\mathbf{x}_1+\mathbf{u}_1 s)-\psi_{\mathbf{x}_1}(\mathbf{x}_1)}{\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s)-\psi_{\mathbf{x}_0}(\mathbf{x}_0)}$$

in this version tends (as $s \to 0+$) to a finite nonzero limit whose value is not independent of the two line elements involved. This property is intuitively plausible and it greatly simplifies the Fechnerian analysis of psychometric differentials. This paper demonstrates, however, that this property of the power function version does not imply this version, being equivalent instead to a more general version, where psychometric differentials are asymptotically described by functions regularly varying at the origin with a positive exponent.

Remarkably, in virtually all conceivable computations the regularly varying functions can be treated as if they were power functions. For instance, in the example of the simple probability summation model given in the Introduction,

$$\begin{cases} \psi_{\mathbf{x}}(\mathbf{x}) = 0 \\ 1-\psi_x(\mathbf{x}+\mathbf{u}s) = \prod_{i=1}^{n} [1-\psi\mathbf{x}(\mathbf{x}+\mathbf{u}_i s)] \end{cases}$$

(where $\mathbf{u}_i$ is the projection of $\mathbf{u}$ on the ith coordinate), the conclusion that the ensuing Fechnerian metric is the (locally) Minkowski power function metric,

$$F(\mathbf{x}, \mathbf{u})^\mu = \sum_{i=1}^{n} F(\mathbf{x}, \mathbf{u}_i)^\mu,$$

is derived identically whether the psychometric differentials are described by (2) or by (17). This conclusion is reached by equating

$$\psi_x(\mathbf{x}+\mathbf{u}s) = F(\mathbf{x}, \mathbf{u})^{\mu}\, R^{\mu}(s) + o\{R^{\mu}(s)\}$$

to

$$\begin{aligned} 1-\prod_{i=1}^{n} [1-\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s)] &= 1-\prod_{i=1}^{n} [1-F(\mathbf{x}, \mathbf{u}_i)^{\mu}\, R^{\mu}(s)+o\{R^{\mu}(s)\}] \\ &= R^{\mu}(s) \sum_{i=1}^{n} F(\mathbf{x}, \mathbf{u}_i)^{\mu} + o\{R^{\mu}(s)\}. \end{aligned}$$

An elaboration of this relationship leads to a productive theory of perceptual separability of stimulus dimensions, presented elsewhere (Dzhafarov, in press). The point made here is that the simple algebra involved in this derivation, being the same as in the power function version of Fechnerian scaling, does not generalize beyond its regular variation version.

APPENDIX: TECHNICAL COMMENTS

1. Counterintuitively, within the class of functions converging to zero as $s \to 0+$, the "infinitely sharp" appearance belongs to functions traditionally referred to as *slowly varying* (at $s = 0+$), while the "infinitely flat" appearance belongs to functions called *rapidly varying* (at $s = 0+$). Slowly varying functions (though not necessarily converging to zero) play a central role in the subsequent development.

2. The definition of regular variation used in this paper is more narrow than the traditional definition. The latter does not impose the restriction $\gamma(k) \not\equiv 1$, which is, since $\gamma(1) = 1$, equivalent to the requirement that $\gamma(k)$ vary with k. The traditional definition incorporates thereby in the class of regularly varying functions all slowly varying functions. This includes the "infinitely sharp" functions like the function described by (4), as well as functions that do not converge to zero with $s \to 0+$ (e.g., $f(s) \equiv 1$, or $f(s) = -\log(s)$). Such functions are not suitable for describing psychometric differentials.

3. As in other mathematical texts known to me, the treatises by Seneta (1976) and Bingham *et al.* (1987) deal with variation of functions at the infinity rather than at the origin. To relate a statement made in this paper to a corresponding statement in this literature, the reader should replace: (a) every occurrence of s (as $s \to 0+$) with $1/x$ (as $x \to \infty$); (b) every occurrence of $\ell(\ldots)$, where ℓ slowly varies at the origin, with $l(1/\ldots)$, where l slowly varies at the infinity; (c) every mentioning of an exponent α of regular variation with that of the exponent $-\alpha$.

4. According to the Karamata Representation Theorem (Bingham *et al.*, 1987, pp. 12–13; Seneta, 1976, pp. 2–3), a function $\ell(s)$ considered on some interval $0 < s < a$ is slowly varying at the origin if and only if it is representable as

$$\ell(s) = \exp\left\{c + \eta(s) + \int_s^a \frac{\varepsilon(u)}{u}\,du\right\}, \qquad 0 < s < a,$$

where $\eta(s) \to 0$ and $\varepsilon(s) \to 0$ as $s \to 0+$, both functions being continuous for continuous $\ell(s)$. For example, choosing $\varepsilon(s) = \log^{-1}(1/s)$, $\eta(s) \equiv 0$, one gets $\ell(s) = const \cdot \log(1/s)$; changing $\varepsilon(s)$ to $\log^{-1}(s)$ yields $\ell(s) = const \cdot \log^{-1}(1/s)$.

5. The proof of Lemma 2.1.1 is an adaptation of Bingham *et al.*, 1987, pp. 16–18. I outline it here briefly, emphasizing only those aspects that are specific for positive functions continuously decreasing to zero as their positive argument tends to zero.

Since, for any positive k_1 and k_2,

$$\frac{f(k_1k_2s)}{f(s)} = \frac{f(k_1k_2s)}{f(k_1s)} \cdot \frac{f(k_1s)}{f(s)},$$

it follows from (8) that $\gamma(k_1k_2) = \gamma(k_1)\,\gamma(k_2)$. By putting $g(x) = \log\gamma(e^x)$, this equation is transformed into the Cauchy functional equation, with $\gamma(k) = k^\mu$ as the only possible solution. As $k^\mu \not\equiv 1$, $\mu \neq 0$. Presenting $f(s)$ as $s^\mu \ell^\mu(s)$ one deduces that $\ell(s)$ should be continuous and satisfy (7). Finally, with $k > 1$ one observes that, since $f(ks) > f(s)$ for sufficiently small s,

$$k^\mu = \lim_{s \to 0+} \frac{f(ks)}{f(s)} \geqslant 1,$$

whence it follows that $\mu > 0$. As this follows from (10) alone, to prove the converse statement of the lemma it remains to check that

$$\lim_{s \to 0+} \frac{f(ks)}{f(s)} = \lim_{s \to 0+} \left[\frac{(ks)\,\ell(ks)}{s\ell(s)}\right]^\mu = k^\mu \left\{\lim_{s \to 0+} \left[\frac{\ell(ks)}{\ell(s)}\right]\right\}^\mu = k^\mu.$$

6. A traditional presentation of a function regularly varying at $s = 0+$ with exponent μ (following the correspondence rules of Comment 3) is $s^\mu \mathscr{L}(s)$, where $\mathscr{L}(s)$ slowly varies at $s = 0+$. Clearly, $\mathscr{L}(s)$ corresponds to $\ell^\mu(s)$ in (11). The difference between the two forms, however, becomes apparent as one makes μ decrease to zero. The traditional form then tends to a slowly varying function, $\mathscr{L}(s)$, while (11) tends to 1. While the traditional form is superior for the general theory of regular and slow variation, (11) is more appropriate in the present context (see Comment 2).

7. If $f(s) = [s\ell(s)]^\alpha = [s\ell_1(s)]^\beta$, then, for any $k > 0$,

$$\frac{[s\ell(s)]^\alpha}{[s\ell_1(s)]^\beta} = \frac{[ks\ell(ks)]^\alpha}{[ks\ell_1(ks)]^\beta} = 1,$$

and

$$\frac{[ks\ell(ks)]^{\alpha}}{[s\ell(s)]^{\alpha}} = \frac{[ks\ell_1(ks)]^{\beta}}{[s\ell_1(s)]^{\beta}},$$

whence $\alpha = \beta$ because the two ratios tend to, respectively, k^{α} and k^{β} (as $s \to 0+$). The equality $\ell(s) = \ell_1(s)$ then follows.

For the second part of the lemma, if $f_1(s) \sim f(s)$, then $f_1(ks) \sim f(ks)$ $(k > 0,$ $s \to 0+)$, and

$$1 = \left\{ \lim_{s \to 0+} \frac{f(ks)}{f_1(ks)} \right\} \left\{ \lim_{s \to 0+} \frac{f_1(s)}{f(s)} \right\} = \lim_{s \to 0+} \left\{ \frac{f(ks)}{f(s)} \div \frac{f_1(ks)}{f_1(s)} \right\} = \frac{k^{\alpha}}{\displaystyle \lim_{s \to 0+} \frac{f_1(ks)}{f_1(s)}}$$

from which the rightmost limit equals k^{α}. By Lemma 2.1.1 then $f_1(s) = [s\ell_1(s)]^{\alpha}$, with $\ell_1(s) \sim \ell(s)$ following trivially.

8. In Bingham *et al.* (1987, pp. 28–29) and Seneta (1976, pp. 27–29) the characterization of asymptotic inverses for regularly varying functions is formulated only for, in our notation, $[s\ell(s)]^{\alpha}$ with $\alpha < 0$, which is not our case. Therefore I present the formal argument in extenso.

9. To prove Lemma 2.2.1, assume, to the contrary, that $f[g(\tau)] \sim f[\tilde{g}(\tau)]$ (as $\tau \to 0+$) but $\tilde{g}(\tau)/g(\tau)$ does not tend to 1. Then there is a sequence $\tau_i \to 0+$, such that, for some $\varepsilon > 0$, either $\tilde{g}(\tau_i)/g(\tau_i) > 1 + \varepsilon$ for all its elements, or $\tilde{g}(\tau_i)/g(\tau_i) < 1 - \varepsilon$ for all its elements. Assuming the first of these possibilities (the second is treated analogously), and using the fact that f is increasing, we have

$$f[\tilde{g}(\tau_i)] = f\left[g(\tau_i) \frac{\tilde{g}(\tau_i)}{g(\tau_i)} \right] > f[g(\tau_i)(1+\varepsilon)] \quad \text{(for all } i = 1, 2, \ldots).$$

But f is regularly varying with some exponent $\alpha > 0$, because of which

$$\lim_{i \to \infty} \frac{f[\tilde{g}(\tau_i)]}{f[g(\tau_i)]} \geqslant \lim_{i \to \infty} \frac{f[g(\tau_i)(1+\varepsilon)]}{f[g(\tau_i)]} = (1+\varepsilon)^{\alpha} > 1,$$

contrary to the assumption that $f[g(\tau)] \sim f[\tilde{g}(\tau)]$.

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