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On the law of Regular Minimality: Reply to Ennis

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Abstract

Ennis's critique touches on issues important for psychophysics, but the points he makes against the hypothesis that Regular Minimality is a basic property of sensory discrimination are not tenable.

(1) Stimulus variability means that one and the same apparent stimulus value (as measured by experimenter) is a probabilistic mixture of true stimulus values. The notion of a true stimulus value is a logical necessity: variability and distribution presuppose the values that vary and are distributed (even if these values are represented by processes or sets rather than real numbers). Regular Minimality is formulated for true stimulus values. That a mixture of probabilities satisfying Regular Minimality does not satisfy this principle (unless it also satisfies Constant Self-Similarity) is an immediate consequence of my 2003 analysis. Stimulus variability can be controlled or estimated: the cases when observed violations of Regular Minimality can be accounted for by stimulus variability corroborate rather than falsify this principle. In this respect stimulus variability is no different from fatigue, perceptual learning, and other factors creating mixtures of discrimination probabilities in an experiment.

(2) Could it be that well-behaved Thurstonian-type models are true models of discrimination but their parameters are so adjusted that the violations of Regular Minimality they lead to (due to my 2003 theorems) are too small to be detected experimentally? This is possible, but this amounts to admitting that Regular Minimality is a law after all, albeit only approximate: nothing in the logic of the Thurstonian-type representations per se prevents them from violating Regular Minimality grossly rather than slightly. Moreover, even very small violations predicted by a given class of Thurstonian-type models can be tested in specially designed experiments (perhaps under additional, independently testable assumptions). The results of one such experiment, in which observers were asked to alternately adjust to each other the values of stimuli in two observation areas, indicate that violations of Regular Minimality, if any, are far below limits of plausible interpretability.

Keywords: Psychophysics; Perceptual discrimination; Same-different judgments; Regular Minimality; Self Similarity; Thurstonain modeling

1. Introduction

This paper pursues three goals:

1. To explain, more clearly than it is done in Ennis (2006), the law of Regular Minimality and its conceptual environment (observation areas, points of subjective equality, constant errors, Nonconstant Self-Similarity);
2. To explain what I mean by Thurstonian-type modeling, what my analysis of this type of modeling shows, and what it implies for the nature of the putative randomness in the process of comparing stimuli; and
3. To demonstrate the untenability of Ennis's criticisms.

The issues raised in or derived from Ennis's criticisms are treated differently from the arguments he uses to justify these criticisms. The issues are analyzed in the main text, primarily in Sections 4 and 5. Ennis's arguments I find to be confused, and not to interrupt the presentation flow I place my rejoinders to most of them in the Appendix, in the form of numbered comments. The reader specifically interested in these rejoinders may want first to look in the concluding section of this paper, and then to pay special attention to the comments referenced there.

I also relegate to the Appendix certain technical clarifications, whether or not related to Ennis's critique. For greater transparency, the amount of detail and the technical level in the main text are kept relatively low, lower than required by the logic of the theoretical

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constructs used. In particular, although the law of Regular Minimality pertains to arbitrary sets of stimuli, and my analysis of Thurstonian-type models deals with arbitrary continuous stimulus spaces (e.g., regions of $\mathbb{R}^n$), in this paper the discussion is primarily confined to stimulus representations comprising intervals of reals (and, in toy examples, finite sets).

The paper includes many figures, and their captions often contain essential information not necessarily replicated or even mentioned outside them. A reference to a figure, therefore, should be taken as that to additional information (sometimes the main information in a given paragraph) rather than to a mere illustration for a point being made.

2. Regular Minimality and related concepts

2.1. *What is Regular Minimality?*

Regular Minimality is formulated in terms of pairwise discrimination probabilities,

$$\psi(x, y) = \Pr[x \in \mathfrak{S}_1 \text{ and } y \in \mathfrak{S}_2 \text{ are judged to be different}]. \qquad (1)$$

To explain the symbols, consider the setup shown in Fig. 1. The sets of stimuli $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in this setup are identical sets of real numbers (segment lengths) but they have different operational meanings: $x \in \mathfrak{S}_1$ means that a segment of length x is presented on the left, and $y \in \mathfrak{S}_2$ means that a segment of length y is presented on the right. Fig. 1B illustrates the fact that $\mathfrak{S}_1$ and $\mathfrak{S}_2$, two *observation areas* as they were called in Dzhafarov (2002b), may also differ in respects other than stimulus spatial location or temporal order (see Comment 1).[1]

The law of Regular Minimality is a tripartite statement:

($\mathscr{P}_1$) for every $x \in \mathfrak{S}_1$, function $y \to \psi(x, y)$[2] achieves its global minimum at a single point $y \in \mathfrak{S}_2$, called the *point of subjective equality* (PSE) for x;

($\mathscr{P}_2$) for every $y \in \mathfrak{S}_2$, function $x \to \psi(x, y)$ achieves its global minimum at a single point $x \in \mathfrak{S}_1$, called the PSE for y;

($\mathscr{P}_3$) $y \in \mathfrak{S}_2$ is the PSE for $x \in \mathfrak{S}_1$ if and only if $x \in \mathfrak{S}_1$ is the PSE for $y \in \mathfrak{S}_2$. (See Comment 2.)

Fig. 2 illustrates the first two parts of the statement, with the emphasis on the fact that a stimulus and its PSE generally have different values. The non-coincidence of a stimulus value and its PSE value is called a *constant error*, by analogy with the use of this term in the context of greater–less discriminations.

[1]Here and thereafter, all capitalized numbered comments refer to the Appendix.

[2]The notation $y \to \psi(x, y)$ means that function $\psi(x, y)$ is viewed as a function of y alone, with x being fixed; analogously for $x \to \psi(x, y)$.

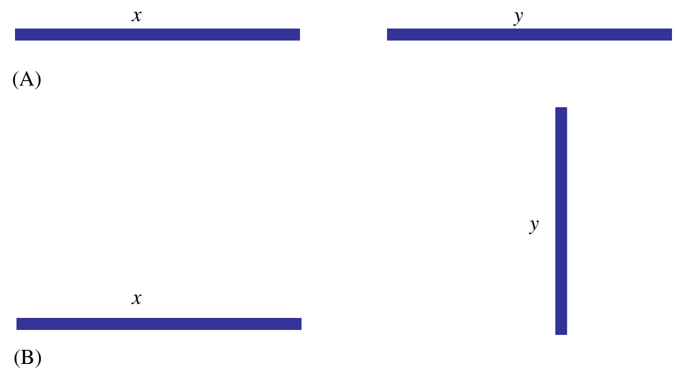

Fig. 1. Two possible operational meanings for the notion of observation area. Stimuli are line segments uniquely identified by their length. Observation area 1 ($x \in \mathfrak{S}_1$) is defined as "horizontal, presented on the left"; observation area 2 ($y \in \mathfrak{S}_2$) is defined as "horizontal, presented on the right" (in panel A) or "vertical, presented on the right" (in panel B). In these two examples the meaning of x and y being judged "the same" is that x and y appear identical in all respects except for being in different locations and (in B) differently oriented.

According to $\mathscr{P}_1$ the PSE for $x \in \mathfrak{S}_1$ is a well-defined function of x, say, $h(x) \in \mathfrak{S}_2$. Analogously, due to $\mathscr{P}_2$ the PSE for $y \in \mathfrak{S}_2$ is a well-defined function $g(y) \in \mathfrak{S}_1$. The third part of the Regular Minimality law, $\mathscr{P}_3$, then can be written as

$$g \equiv h^{-1}. \qquad (2)$$

The simplest special case is $h \equiv identity$ (equivalently, $g \equiv identity$): in this case no stimulus has a constant error associated with it, in either observation area (see Comment 3).

In its entirety the law of Regular Minimality is illustrated in Figs. 3 and 4. These figures also illustrate the notion of *Nonconstant Self-Similarity*[3]:

$$\psi(x, h(x)) \not\equiv const \quad \text{or, equivalently,} \quad \psi(g(y), y) \not\equiv const, \qquad (3)$$

where $\not\equiv$ can be read "is not always." In retrospect, the prefix "self" was a poor choice on my part, because one should always keep in mind that it does not indicate one and the same physical stimulus. Rather it refers to stimuli which are mutual PSEs. Thus, if one establishes that $\psi(a, a) \not\equiv const$ this does not constitute Nonconstant Self-Similarity, unless one knows additionally that $h \equiv identity$, in which case physically identical stimuli in the two observation areas are each other's PSEs. If $h \not\equiv identity$, properties of $\psi(a, a)$ have no special significance, it is the

[3]In recent work (e.g., Dzhafarov and Colonius, 2005a, 2005b) we prefer to speak of Self-Dissimilarity, since we usually deal with function (1) rather than its complement, Pr[x and y are judged to be the same]. I use the earlier version here (Dzhafarov, 2002b) because it is used in Ennis's paper.

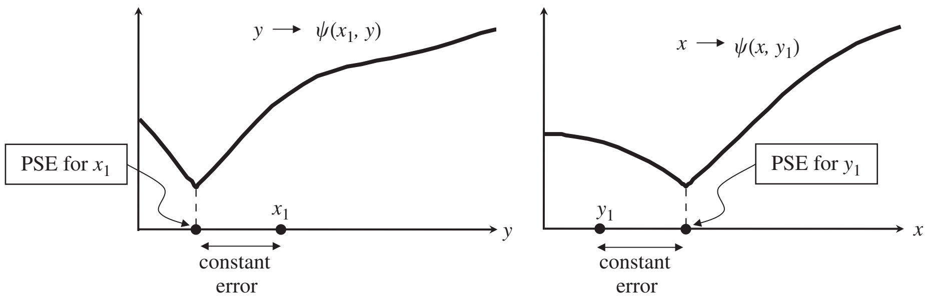

Fig. 2. Probability $\psi(x, y)$ with which stimuli $x \in \mathfrak{S}_1$ and $y \in \mathfrak{S}_2$ are judged to be different, shown for one value of x (left) and one value of y (right). For a given stimulus in either of the two observation areas, PSE is the least discriminable from it stimulus in the other observation area. According to the Law of Regular Minimality, this stimulus is defined uniquely.

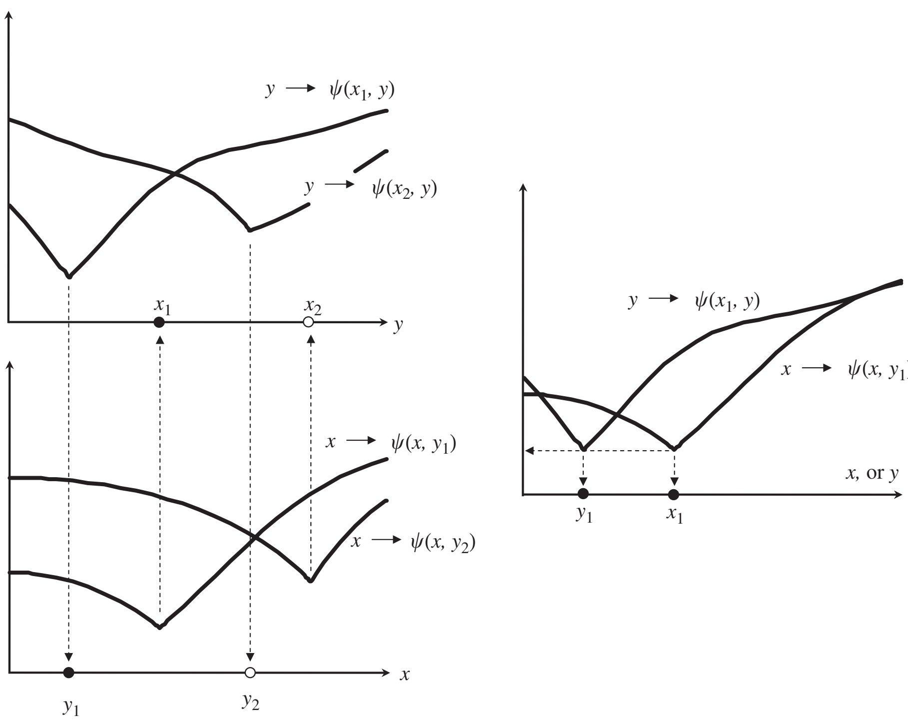

Fig. 3. An illustration for the Law of Regular Minimality. Every x in observation area $\mathfrak{S}_1$ has a PSE y in $\mathfrak{S}_2$, and vice versa; and x is the PSE for y if and only if y is the PSE for x. This is shown in the left panels for two PSE pairs, (x_1, y_1) and (x_2, y_2). Note that the minima $\psi(x_1, y_1)$ and $\psi(x_2, y_2)$ are different (Nonconstant Self-Similarity). The right-hand inset demonstrates the fact that the minima of the corresponding functions $x \to \psi(x, y_1)$ and $y \to \psi(x_1, y)$ are the same, because they both equal $\psi(x_1, y_1)$ (and analogously for x_2, y_2 and all other pairs of PSEs).

properties of $\psi(x, h(x))$, the minimum level of function $\psi(x, y)$, that matter.[4] An experimental illustration for Regular Minimality, Nonconstant Self-Similarity, and constant error is given in Fig. 5 (see Comment 4).

[4]Switching to a canonical form (see Comment 3) conceals this difference by virtue of a "notational trick," which is one reason I avoid using canonical transformations in this paper, where conceptual transparency is critical.

2.2. How can Regular Minimality be violated?

The statement of the law of Regular Minimality consists of three parts, and if a certain function $\psi(x, y)$ violates this law it can do this in two ways:

(a) either the statements $\mathscr{P}_1$ and/or $\mathscr{P}_2$ are violated, in which case the PSE functions h and/or g are not well-defined and $\mathscr{P}_3$ therefore cannot be formulated;
(b) or $\mathscr{P}_1$ and $\mathscr{P}_2$ are satisfied (i.e., h and g are well-defined functions) but $g \not\equiv h^{-1}$.

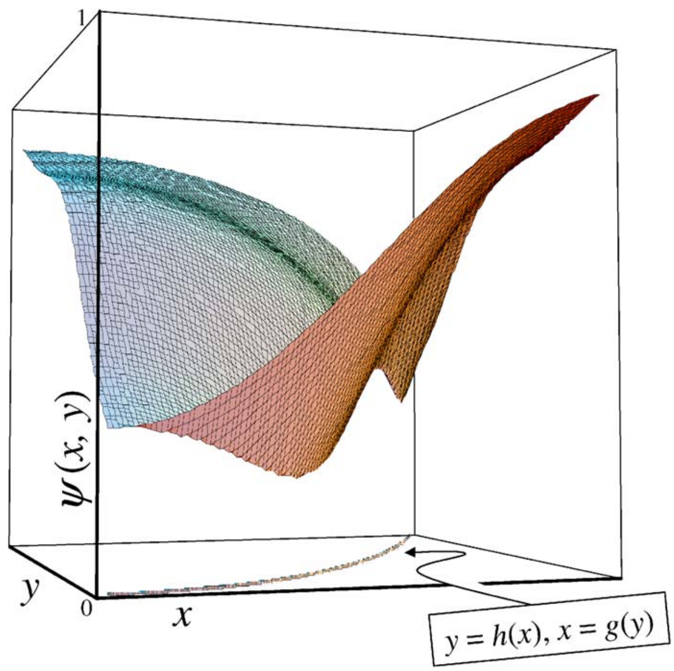

Fig. 4. A function $\psi(x, y)$ satisfying Regular Minimality and exhibiting Nonconstant Self-Similarity. (The curves shown in Fig. 3 are cross-sections of this function taken at two fixed values of x and two fixed values of y.) The line shown in the xy-plane is the PSE line, it can be described by either of the equations $y = h(x)$ (y is PSE for x) and $x = g(y)$ (x is PSE for y), because $g \equiv h^{-1}$. Along the PSE line the function is at its minimum: moving a point (x, y) away from this line along the x-axis or the y-axis increases the value of $\psi(x, y)$. Note the Nonconstant Self-Similarity property: the minimum level of $\psi(x, y)$ along the PSE curve is not constant.

	7.00	9.00	11.00	13.00	15.00	17.00	19.00	21.00	23.00
7.00			0.74						
9.00			0.50	0.77					
11.00	0.98	0.89	0.57	0.46	0.75				
13.00		1.00	0.87	0.60	0.42	0.67			
15.00			0.98	0.87	0.57	0.37	0.67		
17.00				0.98	0.86	0.53	0.37	0.63	
19.00					0.95	0.82	0.50	0.32	0.58
21.00						0.95	0.80		
23.00							0.92		

Fig. 5. Experimental estimates for $\psi(x, y)$, with x and y being as in Fig. 1A (data for one participant; segment lengths are given in pixels, with 1 px $\approx$ 0.86 min arc; within the central rectangle the estimates are based on $\approx$ 600 replications per point, outside the rectangle on $\approx$ 300 replications; for other details see Dzhafarov and Colonius, 2005a). The law of Regular Minimality is corroborated because the minimal value in every row (encircled) is also the minimal value in its column. The encircled values are clearly different from each other (Nonconstant Self-Similarity). Note the constant error at each stimulus value: all minima lie outside the main diagonal (indicated by the dashed line).

Fig. 6 provides an illustration on three toy examples. In matrix B, $\psi(x, y)$ violates property $\mathscr{P}_1$: $h(x)$ is not well-defined. In matrix C, $\psi(x, y)$ satisfies properties $\mathscr{P}_1$ and $\mathscr{P}_2$ (each row and each column has a single minimum) but it violates property $\mathscr{P}_3$.

Thurstonian-type models, discussed in the next section, may violate $\mathscr{P}_1$–$\mathscr{P}_2$, like matrix B, or only $\mathscr{P}_3$, like matrix C. The latter is usually the case with simpler Thurstonian-type models, of the kind proposed by Luce and Galanter (1963) and used for illustration purposes in Dzhafarov (2003a). With non-essential modifications Ennis's illustrations too are confined to such models. An example is provided in Figs. 7 and 8, depicting a discrimination probability function $\psi(x, y)$ generated by a model in which stimuli x and y are mapped into random variables $\mathbf{X}$ and $\mathbf{Y}$ independently normally distributed on the axis of reals, with their means and variances linearly related to stimulus values; the decision "x and y are different" in this model is made whenever $|\mathbf{X} - \mathbf{Y}|$ exceeds some preset critical value.

The difference of main interest between Figs. 7 and 4 is in the PSE lines: Regular Minimality means strictly one such line, while the model depicted in Fig. 7 has two. Note that my use of the notion of a PSE is not predicated on the law of Regular Minimality in its entirety, it only requires properties $\mathscr{P}_1$ and $\mathscr{P}_2$. This is convenient, because otherwise one would have to use different names for functions h, g depending on whether they are or are not each other's inverses. The same consideration applies to the notion of a constant error: not to multiply terms unnecessarily, one should speak of a constant error at a given value of x whenever $h(x) \neq x$ (and analogously for y, if $g(y) \neq y$), whether or not $g \equiv h^{-1}$ (see Comment 5).

2.3. *Regular Minimality and matching procedure*

The law of Regular Minimality is formulated in terms of the discrimination probability function $\psi(x, y)$, which means that the law presupposes an experimental procedure that provides an estimate of this function. Ideally, one deals with a representative sample of stimuli repeatedly presented in all possible pairwise combinations. In view of the experiment presented later in this paper, however, it is useful to consider another operationalization of the law, which some may find even more plausible intuitively than the formulation $\mathscr{P}_1$–$\mathscr{P}_3$. This is in fact how the notion of Regular Minimality was first introduced, in Dzhafarov (2002b).

Consider the procedure of *sensory-physical matching*, in which a participant is required to adjust the value of a stimulus in one observation area until it matches a fixed value of the stimulus in the other. In reference to Fig. 1, for example, the segment length $x \in \mathfrak{S}_1$ (or $y \in \mathfrak{S}_2$) in a trial may be kept fixed, while the participant changes the length $y \in \mathfrak{S}_2$ (respectively, $x \in \mathfrak{S}_1$). Let us idealize the situation by assuming that the participant makes no "adjustment errors," that is, the value of y that she judges to match a given value of x is the same in all trials involving this value of x; and analogously for trials involving a given value of y. The following tripartite statement then can be considered the *matching-based version* of the law of Regular Minimality:

($\mathscr{P}_1^*$) for every $x \in \mathfrak{S}_1$ there is a single matching value $y \in \mathfrak{S}_2$;

(A)	y_a	y_b	y_c	y_d
x_a	0.6	0.6	0.1	0.8
x_b	0.9	0.9	0.8	0.1
x_c	1	0.5	1	0.6
x_d	0.5	0.7	1	1

(B)	y_a	y_b	y_c	y_d
x_a	0.1	0.6	0.1	0.8
x_b	0.9	0.9	0.8	0.1
x_c	1	0.5	1	0.6
x_d	0.5	0.7	1	1

(C)	y_a	y_b	y_c	y_d
x_a	0.7	0.4	0.2	0.8
x_b	0.9	0.9	0.8	0.4
x_c	1	0.6	0.7	0.8
x_d	0.4	0.7	1	1

Fig. 6. Three toy examples of a stimulus space consisting of four stimuli. Each cell shows the probability with which the cell's row stimulus ($\in \mathfrak{S}_1$) and column stimulus ($\in \mathfrak{S}_2$) are judged to be different. Matrix A satisfies Regular Minimality because each row and each column in it has precisely one minimum, and a cell is minimal in its row if and only if it is minimal in its column. The pairs of PSEs in this matrix are (x_a, y_c), (x_b, y_d), (x_c, y_b), and (x_d, y_a). Matrix B violates Regular Minimality because the first row contains two identical minima. Matrix C violates Regular Minimality because while y_b is the PSE for x_c, the PSE for y_b is not x_c but x_a.

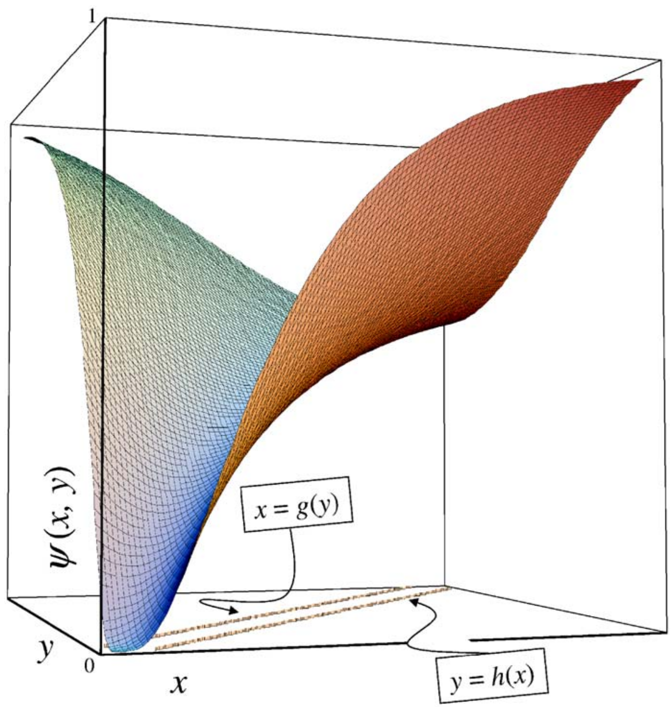

Fig. 7. A function $\psi(x, y)$ that does not satisfy Regular Minimality: although in this example the PSE lines $y = h(x)$ (y is PSE for x) and $x = g(y)$ (x is PSE for y) are well-defined, they do not coincide (i.e., $g \not\equiv h^{-1}$). This function is derived from a simple Thurstonian-type model used as an example in Dzhafarov (2003a).

($\mathscr{P}_2^*$) for every $y \in \mathfrak{S}_2$ there is a single matching value $x \in \mathfrak{S}_1$;

($\mathscr{P}_3^*$) $y \in \mathfrak{S}_2$ matches $x \in \mathfrak{S}_1$ if and only if $x \in \mathfrak{S}_1$ matches $y \in \mathfrak{S}_2$.

Let us further assume that this idealized matching procedure is *consistent* with the same–different procedure, in the following sense: y is judged to match a fixed value of x if and only if $\psi(x, y) < \psi(x, y')$ for all $y' \neq y$, and x is judged to match a fixed value of y if and only if $\psi(x, y) < \psi(x', y)$ for all $x' \neq x$. Clearly, this assumption implies that the functions h and g are well-defined, y is judged to match a fixed value of x if and only if $y = h(x)$, and x is judged to match a fixed value of y if and only if $x = g(y)$. The notion of a PSE now has the same meaning in the two procedures, and properties $\mathscr{P}_1^*$–$\mathscr{P}_3^*$ are equivalent to properties $\mathscr{P}_1$–$\mathscr{P}_3$ componentwise. Property $\mathscr{P}_3^*$, however, has a new (idealized) operational meaning, depicted in Fig. 9. Consider a "ping-pong" variant of the matching procedure, in which (1) y is adjusted until it matches a fixed value of x; (2) in the next trial the matching value of y achieved in the previous trial is fixed, while x is set at a random initial level and asked to be adjusted until it matches y; (3) in the next trial this matching value of x is fixed and y, being set at some random value, is adjusted to match x; and so on. Clearly, if Regular Minimality holds, we will have one and the same pair (x, y) at the end of each trial.

The situation is dramatically different if Regular Minimality does not hold (specifically, if $\mathscr{P}_3^*$ does not hold while $\mathscr{P}_1^*$ and $\mathscr{P}_2^*$ do). This is shown in Fig. 10, a free-hand generalization of the two non-coinciding PSE curves of Fig. 7 (without the linearity constraints imposed on the means and variances of the random images of stimuli). Consider the "ping-pong" adjustment scheme in this situation: in all odd-numbered trials y is adjusted until it matches a fixed value of x, in all even-numbered trials the procedure is reversed, and in each trial the fixed value is the matching value achieved in the previous trial. It is easy to see that now the matching values in either of the two observations areas will not stay unchanged, they will form a series of stimuli wandering away from the initial pair (x, y): starting with a fixed x, and denoting the statement "y is the match for x" by $x \curvearrowleft y$, we have

$$x \curvearrowleft y \curvearrowleft x' \curvearrowleft y' \curvearrowleft x'' \curvearrowleft y'' \curvearrowleft x''' \curvearrowleft y''' \curvearrowleft \ldots$$

Another way of looking at the difference between Figs. 9 and 10 is to say that Regular Minimality makes it possible to speak of stimuli x and y as matching or mismatching *each other*, whereas without Regular Minimality (but with $\mathscr{P}_1^*$ and $\mathscr{P}_2^*$ satisfied) one can only speak of y matching x or x matching y. If Regular Minimality does not hold, it should in fact be expected that at least in some cases (when the difference between the two PSE curves exceeds the precision of matching adjustments) an observer who has just adjusted y to match x to his satisfaction, will have to readjust the value of x as soon as his attention is drawn to the question of whether x matches y. This simple observation shows that the law of Regular Minimality is

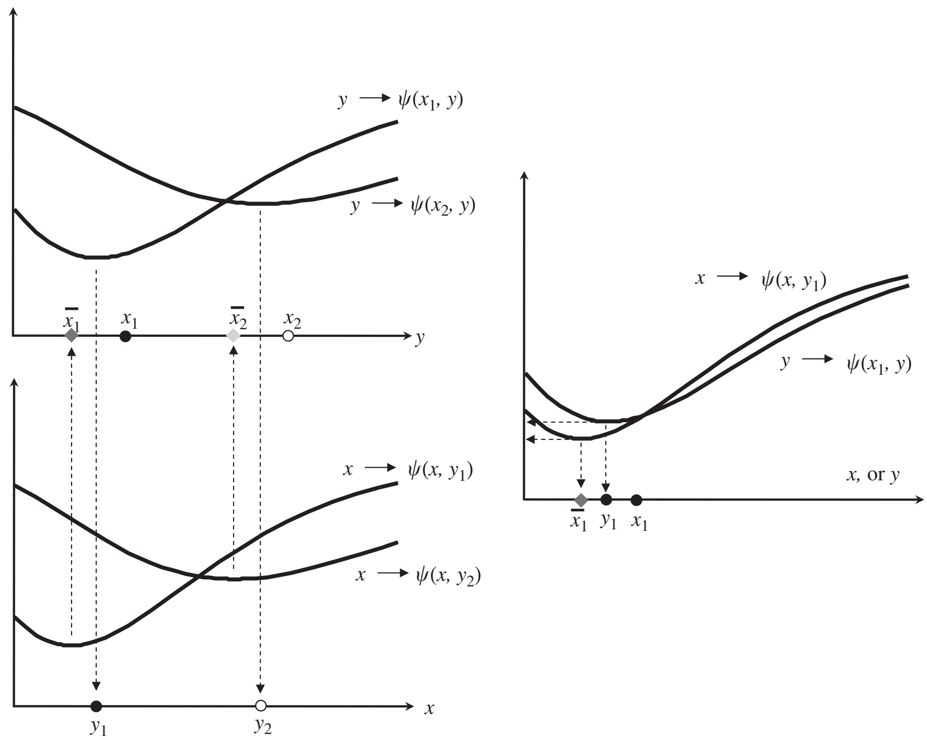

Fig. 8. Cross-sections of the function shown in Fig. 7 made at two fixed values of x (top left) and two fixed values of y (bottom left); y_1 and y_2 are chosen so that they are PSEs for x_1 and x_2, respectively. Note that the PSEs for y_1 and y_2 are not x_1 and x_2 but some stimuli $\bar{x}_1$ and $\bar{x}_2$, respectively (another way of stating the fact that the two PSE lines in Fig. 7 do not coincide). The right-hand inset demonstrates that the minimum levels of functions $y \to \psi(x_1, y)$ and $x \to \psi(x, y_1)$ are different even though y_1 is the PSE for x_1 (but not vice versa), because for the former function the minimum is $\psi(x_1, y_1)$ while for the latter function it is $\psi(\bar{x}_1, y_1)$.

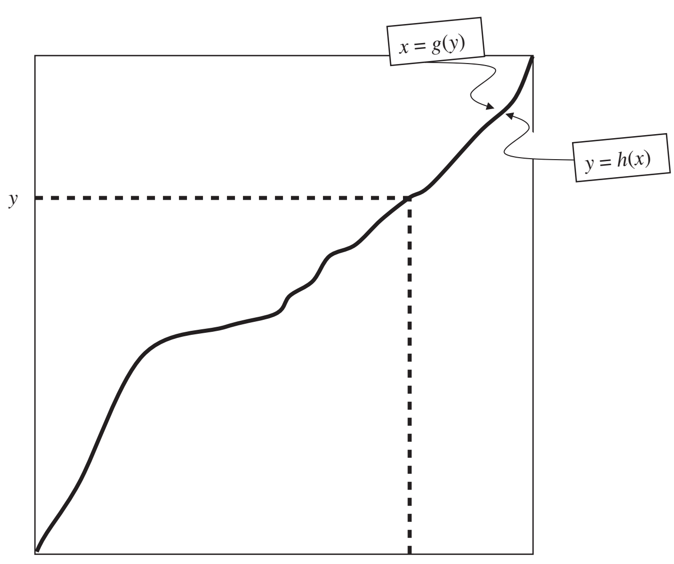

Fig. 9. A single PSE curve for $\psi(x, y)$ satisfying Regular Minimality. x and y shown are mutual PSEs. In the idealized matching procedure described in the text, y will be matched to x if and only if x is matched to y. Consequently any x and y can be said to be matched or mismatched, without specifying which of them is compared to which.

in fact an implicit foundation for a vast body of psychophysical research: to the extent the existing theories and summaries of empirical results involving matching do not have to habitually mention which stimulus was matched to which when they appear equal, Regular Minimality must be true at least to a very good degree of approximation.

A word of caution is due here: one should not forget that the consistency of the (idealized) matching procedure with same–different judgments is an additional assumption, however plausible. Without this assumption one may very well accept the matching-based version of the law of Regular Minimality while challenging its main, same–different-based version (or vice versa).

3. Thurstonian-type models

In Dzhafarov (2003a, 2003b) I have shown the following.

Theorem 1 (*Well-behaved Thurstonian-type models*). *Suppose a discrimination probability function $\psi(x, y)$ both satisfies Regular Minimality and exhibits Nonconstant Self-Similarity, at least in an arbitrarily small neighborhood of at least one pair of mutual PSEs (x_0, y_0). Then $\psi(x, y)$ cannot be generated (accounted for) by any well-behaved Thurstonian-type model (see Comment* 6).

This result is the cause of Ennis's critique of the law of Regular Minimality. To understand this result we have first to explain the terms ‘‘Thurstonian-type'' and ‘‘well-behaved,'' as they are defined in Dzhafarov (2003a, 2003b).

3.1. What does "Thurstonian-type" mean?

Not every model in which stimuli are mapped into random variables constitutes a Thurstonian-type model. Consider, for example, the construction presented in Fig. 11, in which x and y, the stimuli being compared, jointly evoke a random variable $\mathbf{S}$. By an appropriate choice of how its distribution's parameters depend on x and y one can generate *any* given discrimination probability function $\psi(x, y)$, including those with Regular Minimality, Nonconstant Self-Similarity, or any other property. This is not what I call a Thurstonian-type representation, although, as pointed out below, such a construction may very well turn out to be a better description of the stimulus comparison process than any Thurstonian-type model. The property which this construction lacks (and is, as a result, applicable to any function ψ) is *selective attribution* of random effects to individual stimuli. In a Thurstonian-type model, as I defined the term in Dzhafarov (2003a, 2003b), each of the two stimuli being compared is mapped into a random variable that can be interpreted as the image of *this* stimulus, and not of the other.

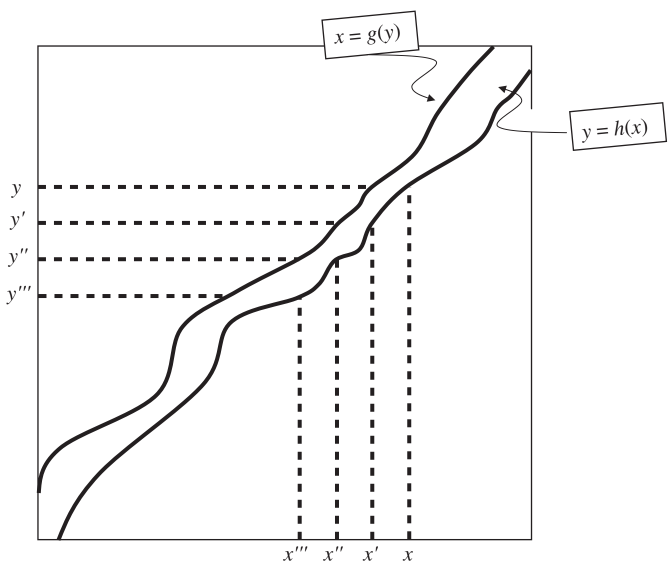

Fig. 10. Two distinct PSE curves like the ones shown in Fig. 7. The idealized "ping-pong" matching procedure described in the text forms a series of distinct stimuli in each of the two observations areas. Here, one cannot speak of x and y matching each other without specifying first which of them is matched to which: if y is judged to match x, then x will not be judged to match y, and vice versa.

This notion, that $\mathbf{X}$ is an image of x but not of y while $\mathbf{Y}$ is an image of y but not of x, is illustrated and explained in Fig. 12. The selective attribution of images to stimuli is not a trivial issue when the images are stochastically interdependent, and the definition given in Fig. 12 is derived from the theory presented in Dzhafarov (2003c). In the special case when $\mathbf{X}$ and $\mathbf{Y}$ are stochastically independent, however, their selective attribution to x and y, respectively, simply means that their distributions depend on x and y, respectively (see Comment 7). Note that if x and y are physically equal, or even if they are each other's PSEs, their respective images need not be identically distributed: the dependence of $\mathbf{X}$ on x may be completely different from that of $\mathbf{Y}$ on y.

3.2. What does "well-behavedness" mean?

Figs. 13 and 14 illustrate the notion of well-behavedness in the dependence of a random image on stimulus. A random entity $\mathbf{X}$ distributed on some probability space is entirely characterized by probabilities $\Pr[\mathbf{X} \in A]$ evaluated for all possible measurable sets (events) A. If $\mathbf{X}$ depends on x, then these probabilities generally change with x. For real-valued x the well-behavedness of $\mathbf{X}$ means that as x changes around some value, these probabilities change at well-defined rates (i.e., the directional derivatives

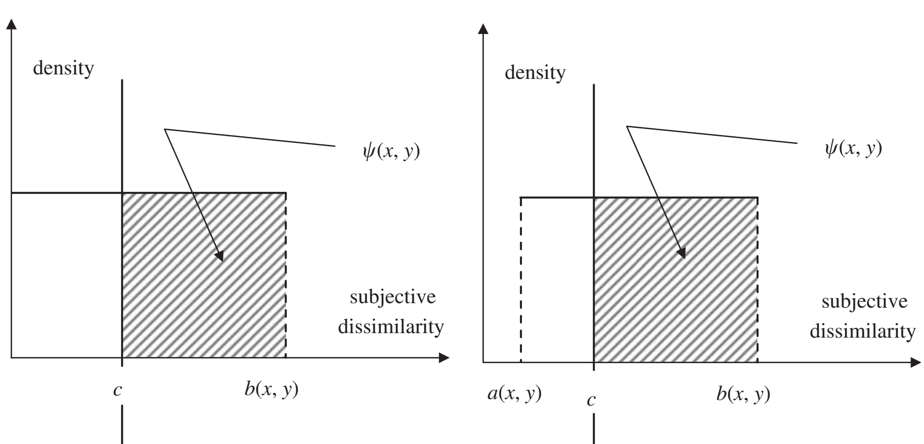

Fig. 11. A trivially universal random representation scheme, to illustrate why the selective attribution constraint is essential in the definition of Thurstonian-type models. Any discrimination probability function $\psi(x, y)$ can be modeled as shown in the right-hand panel: stimuli x, y evoke a random variable $\mathbf{S}$ (interpretable as "subjective dissimilarity") uniformly distributed between endpoints that depend on x and y, with the decision rule 'the stimuli are different iff $\mathbf{S} > c$' (c being an arbitrary fixed number). If $\psi(x, y)$ is never precisely 1 one can even use the "universal" model in the left-hand panel, with the left endpoint fixed at zero.

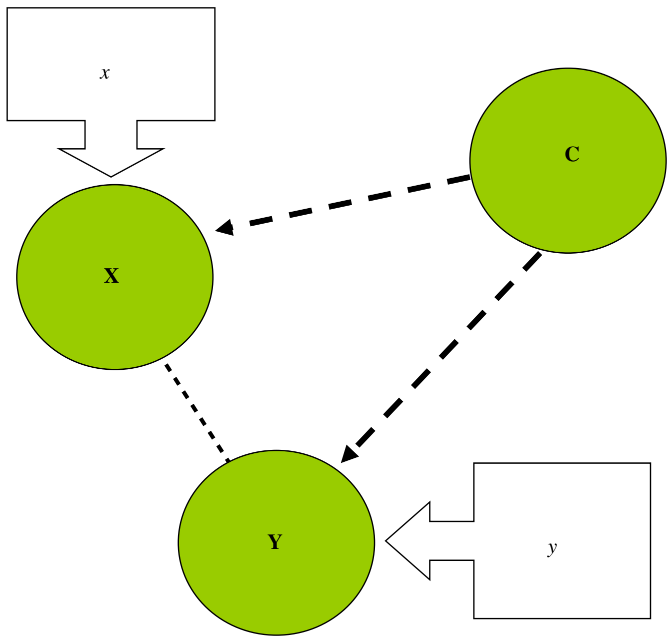

Fig. 12. A schematic illustration for the notion of selective attribution (in the sense of Dzhafarov, 2003c). Random images **X** and **Y** (generally stochastically interdependent) are said to be selectively attributable to (be images of) stimuli x and y, respectively, if one can find a random entity **C** whose distribution does not depend on either x or y and such that **X** and **Y** conditioned upon its value are stochastically independent, with the conditional distribution of **X** depending on x alone and the conditional distribution of **Y** depending on y alone. In a Thurstonian-type model the images evoked by two stimuli being compared are assumed to be selectively attributable to these stimuli. In a well-behaved Thurstonian-type model, in addition, it is assumed that **C** can be chosen so that for each of its values the conditional distributions of **X** and **Y** are well-behaved (as explained in Section 3.2).

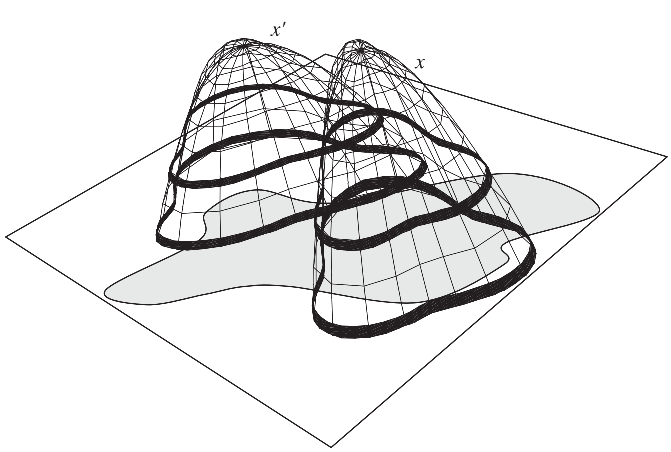

Fig. 13. An illustration for the notion of well-behavedness. The plane represents a set of images, the shaded area a measurable subset, the two figures represent distributions of images evoked by two values of stimulus, x and x' (in the same observation area). The well-behavedness means that if $x' \to x$ the probability with which a random image falls within a measurable subset changes at a rate bounded across all measurable subsets. For example, the random image will be well-behaved if it has a finite density in $\mathbb{R}^n$ smoothly depending on some parameters which in turn smoothly depend on x. A well-behaved distribution, however, need not be in $\mathbb{R}^n$, need not possess a density, moments, etc.

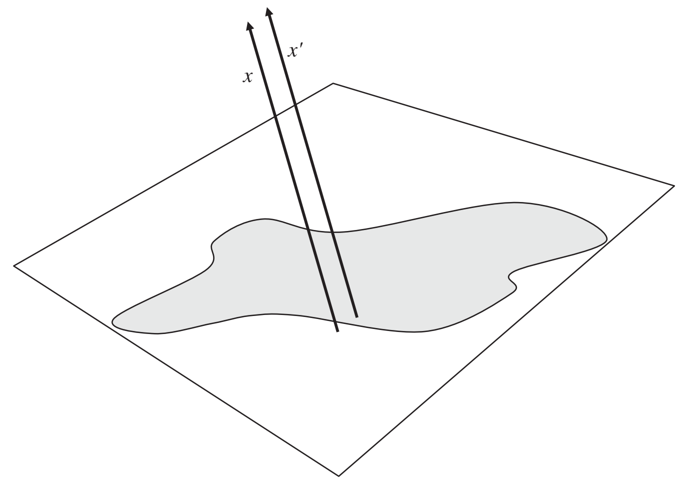

Fig. 14. Random images concentrated at a point in $\mathbb{R}^n$ (singular, or deterministic images) cannot be well-behaved: one can always find a measurable subset such that as $x' \to x$ the probability of the image falling within this subset jumps from 1 to 0 or vice versa.

$(\partial/\partial x\pm)\Pr[\mathbf{X} \in A]$ exist), and across all events A these rates are bounded. Intuitively, for all events A, in response to very small changes in x the probabilities of these events do not jump in value, and do not come arbitrarily close to jumping.

A Thurstonian-type model is well-behaved if **X** and **Y**, the respective images of x and y, are well-behaved for any given value of **C** (as explained in Fig. 12). The mentioning of **C** can be dropped if **X** and **Y** are assumed to be independent. Note that **X** and **Y** are distributed on some unspecified probability space: this means that values of **X** and **Y** may be entities of arbitrary complexity, such as functions or sets rather than real numbers or real-component vectors. Note also that the notion of a well-behaved Thurstonian-type model imposes no restrictions on decision rules: the decision to say that the two stimuli are the same or different in a given trial is an arbitrary deterministic or probabilistic function of the values of **X** and **Y** in this trial. Thus, x and y in a model may evoke two random processes $\mathbf{X} = \mathbf{x}(t)$, $\mathbf{Y} = \mathbf{y}(t)$ resulting in the response 'the stimuli are different' with some probability $p(\mathbf{X}, \mathbf{Y})$. The notion of well-behavedness seems to encompass all Thurstonian-type models that have been described in the literature (see Comment 8).

A simple example of a Thurstonian-type model would be one in which a set of perceptual images is partitioned into a finite number of areas (categories) $A_1, \ldots, A_n$, and stochastically independent images **X** and **Y** of stimuli x and y fall within these areas with probabilities $p_1(x), \ldots, p_n(x)$ and $q_1(y), \ldots, q_n(y)$, respectively. This model will be well-behaved if and only if the derivatives $(\partial/\partial x\pm)p_i(x)$ and $(\partial/\partial y\pm)q_i(x)$ exist. If one posits that the judgment 'different' is given if and only if **X** and **Y** fall in two different areas, then $\psi(x, y) = 1 - \sum p_i(x)q_i(y)$. Even without the general Theorem 1 it can be shown that this model cannot account for both Regular Minimality and Non-constant Self-Similarity in any area of (x, y)-values, however small. Another simple example we find in the models of Luce and Galanter's (1963) variety, with independently univariate-normally distributed $\mathbf{X}$ and $\mathbf{Y}$. Fig. 7 is generated by a model of this kind, with the usual, Luce–Galanter decision rule (respond 'different' iff $|\mathbf{X}-\mathbf{Y}|>\varepsilon$). Such a model is well-behaved if the means and variances of $\mathbf{X}$ and $\mathbf{Y}$ depend on, respectively, x and y sufficiently smoothly. Unfortunately, Ennis seems to be uncertain about how these models relate to Regular Minimality (see Comment 9).

3.3. What does Theorem 1 imply for stochasticity in the process of discrimination?

Although the well-behavedness constraint is only sufficient but not necessary for Theorem 1 to hold, some version of this restriction is indispensable, in view of the following result, also presented in Dzhafarov (2003a).

Theorem 2 (*Unrestricted Thurstonian-type models with independent images*). *Any discrimination probability function $\psi(x,y)$ can be generated by a Thurstonian-type model with stimuli mapping into stochastically independent random images.*

This means that if one accepts Regular Minimality and Nonconstant Self-Similarity but is set on using Thurstonian-type models in dealing with sensory discrimination, one might use non-well-behaved Thurstonian-type models, the ones in which $\Pr[\mathbf{X}\in A]$ in response to very small changes in x may change arbitrarily fast. In particular, a model with stimuli mapped into deterministic images which in turn map into responses probabilistically is, formally, a non-well-behaved Thurstonian-type model (see Fig. 14). In Dzhafarov (2003b) I proposed one such model ("uncertainty blobs") which generates Regular Minimality and Nonconstant Self-Similarity "automatically." In Dzhafarov and Colonius (2006) we reformulated and generalized this model as that of "quadrilateral dissimilarity." Using Ennis's classification, these are models of type II (but see Comment 10).

Deterministic images, however, are not the only alternative to well-behaved Thurstonian-type representations. Another alternative is to dispense with Thurstonian-type representations altogether, more specifically, with the idea that the choice between 'same' and 'different' is based on images $\mathbf{X}$ and $\mathbf{Y}$ selectively attributable to stimuli x and y, respectively. One can assume instead that the aspect of mental/neurophysiological processing which is responsible for same–different judgments is only representable by a random entity $\mathbf{S}$ (state, process, set of states or processes) whose distribution depends on both x and y, with no possibility of selective decomposition. The distribution presented in Fig. 11 provides an example: once the distribution of $\mathbf{S}$ in this figure is chosen to account for $\psi(x,y)$ with both Regular Minimality and Nonconstant Self-Similarity properties, it follows from Theorem 1 that this $\mathbf{S}$ cannot be decomposed into (computed from) two well-behaved images $\mathbf{X},\mathbf{Y}$ followed by any decision rule, deterministic or probabilistic. One can say that the property $\mathbf{S}$ here reflects in an irreducible way a *relationship* between x and y, say, their subjective dissimilarity in a given trial. A model of this variety has in fact been proposed in the literature (Takane and Sergent, 1983).

To prevent a misunderstanding: to say that same–different judgments depend on a random representation of a relationship between x and y rather than on their separate images does not mean that such separate images do not exist or are not random entities. This only means that the task of discrimination is not based on these separate random images. This could be one possible approach to the question Ennis poses in his paper, of how one could distinguish the processes in a perceiving system essentially continuing the presentation of stimuli to this system from the processes that begin the system's reaction to these stimuli.

4. Stimulus variability

Ennis's analysis of the relationship between the law of Regular Minimality and the issue of stimulus variability consists of the following. First he uses an example to demonstrate (by reasoning I find incorrect, see Comment 9) the fact that if stimuli x and y in $\psi(x,y)$ are names for random variables rather than deterministic quantities then $\psi(x,y)$ may not satisfy Regular Minimality. Then he states (correctly):

> Dzhafarov's analysis of well-behaved Thurstonian same–different models for precise-valued stimuli shows that under certain conditions these models will predict that h is not invertible. Since Dzhafarov (2003a, 2003b) defines RM [Regular Minimality] with respect to precise-valued stimuli, the lack of invertibility of h (i.e., the failure of $\mathscr{P}_3$ while $\mathscr{P}_1$–$\mathscr{P}_2$ are satisfied) for *noisy* stimuli is not relevant to his discussion of RM.

And he immediately contradicts this statement by saying:

> However, since all stimuli exhibit some degree of physicochemical noise, he exempts practical applications of Thurstonian models by requiring that stimuli are precise-valued.

I assume that by my "requiring that stimuli are precise-valued" Ennis means that I formulate the law of Regular Minimality in terms of precise-valued stimuli, and not that I deny the reality of measurement errors or physical fluctuations. If so, it is difficult to see how a law formulated for precise-valued stimuli can prevent one from considering situations in which these precise values vary. To give an analogy: Newton's second law of motion is not invalidated by and does not deny the fact that force, mass, and acceleration in any engineering application are measured with some errors and/or fluctuate in their values; if these errors and fluctuations are not negligible, their explicit

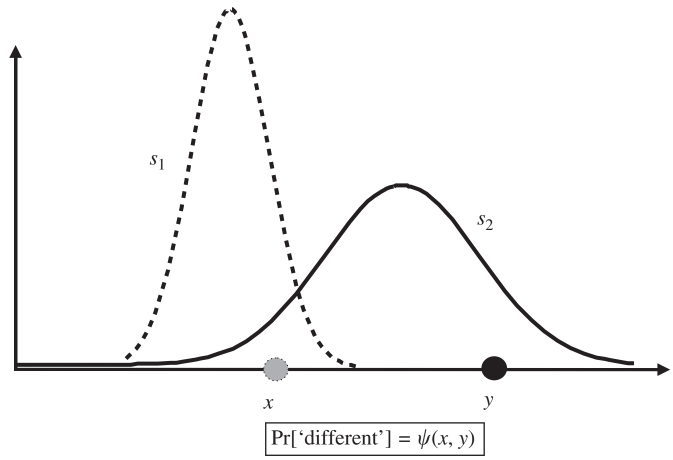

Fig. 15. A simple Thurstonian-type model with stochastic effects and a probabilistic decision rule: in a given trial stimuli s_1 and s_2 are independently mapped into randomly varying effects x and y, respectively, and the observer says 'different' with probability $\psi(x,y)$. The mathematical structure of the model remains the same irrespective of whether s_1, s_2 are true stimulus values mapped into perceptual images x, y, or s_1, s_2 are nominal (apparent) stimulus values mapped into true stimulus values x, y. If the distributions of x, y are well-behaved, Theorem 1 applies in either case.

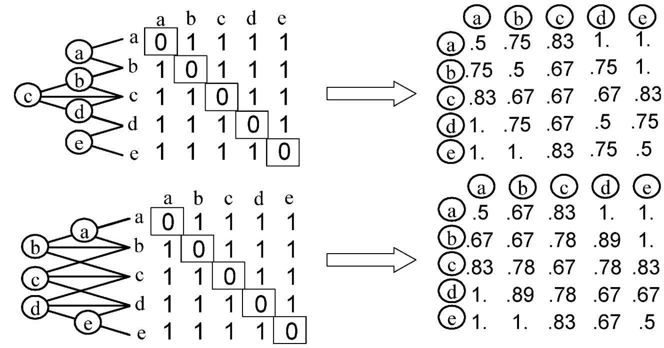

Fig. 16. A toy example illustrating the fact that stimulus uncertainty generally leads to violations of Regular Minimality with respect to apparent stimulus values. The stimulus space is assumed to consist of five stimuli with true values a, b, c, d, e. The apparent values (as known to the experimenter) are encircled. On the left: each fan of lines extending from an apparent value to several true values indicates that this apparent value can in reality designate one of these true values with equal probabilities. Thus, in both examples, the apparent value a may in reality designate a or b with equal probabilities, the apparent value c may in reality be b, c, or d with equal probabilities, etc. Even though the observer in these examples discriminates the stimuli perfectly (left-hand matrices), when formulated in terms of the apparent stimulus values (right-hand matrices) the probabilities violate Regular Minimality.

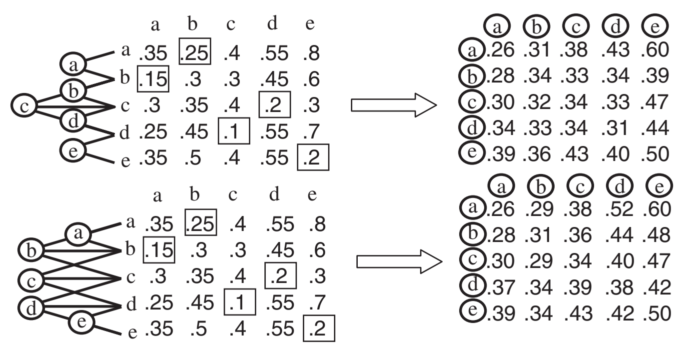

Fig. 17. The same two stimulus misidentification schemes as in Fig. 16, but applied to a matrix satisfying Regular Minimality in a more general form. The values placed in squares on the left are the row-column minima. The right-hand matrices violate Regular Minimality.

models should be combined with the second law of motion to produce correct predictions. This also applies to cases with irreducible stochasticity, such as the number of photons absorbed by a cone in retina: its Poisson distribution is compatible with and should be combined with any law relating the precise number of photons absorbed to the cone's electrochemical response.

The point of logic is that the notion of stimulus variability and of a distribution of stimulus values presupposes something that varies and values that are distributed. Without creating an infinite regress, these values are deterministic, and they can be called "true" or "precise" stimulus values. If an experimenter repeatedly presents two lines of apparent lengths s_1 and s_2 (measured, say, in min of arc), in reality she may be presenting values x, y that are randomly distributed around these apparent values, as shown in Fig. 15. Formally, we have a Thurstonian-type model here (of type III, in Ennis's classification), with

$$\psi_{\text{obs}}(s_1, s_2) = \iint \psi(x, y)\, \mathrm{d}F_{s_1}(x)\, \mathrm{d}F_{s_2}(y), \tag{4}$$

where $\psi_{\text{obs}}(s_1, s_2)$ is the observed probability with which s_1 and s_2 are judged to be different, $F_{s_1}(x), F_{s_2}(x)$ are distribution functions corresponding to s_1, s_2, and the integration is over all possible values of (x, y). As with any model of this kind, to make predictions about $\psi_{\text{obs}}(s_1, s_2)$ it is equally important to know the properties of the distribution functions $F_{s_1}(x), F_{s_2}(x)$ and of the decision-making probability function $\psi(x, y)$. In particular, it is useful to know whether $\psi(x, y)$ satisfies the law of Regular Minimality and whether it exhibits Nonconstant Self-Similarity. Whether $\psi_{\text{obs}}(s_1, s_2)$ has or does not have these properties then can be determined mathematically. In fact, if $F_{s_1}(x)$ and $F_{s_2}(x)$ are well-behaved (a natural assumption when dealing with measurement errors), Theorem 1 tells us that $\psi_{\text{obs}}(s_1, s_2)$ cannot have both.

One does not need Theorem 1, however, to demonstrate that $\psi_{\text{obs}}(s_1, s_2)$ will generally violate Regular Minimality when $\psi(x, y)$ satisfies it. Two such demonstrations are provided in Figs. 16 and 17. It would hardly be a tenable position to maintain that because the stimuli in these examples are probabilistically misidentified, it is of little value or interest to know that true stimulus values are discriminated perfectly (Fig. 16) or according to another pattern satisfying Regular Minimality (Fig. 17).

The stimulus variability schemes shown in Figs. 15–17 are very simple. In a real psychophysical experiment they may be more complex. True stimulus values x, y, for instance, may have a higher dimensionality than the apparent stimuli s_1, s_2 (e.g., line segments identified by their length may in fact vary not only in length but also in width, shape, or intensity). True stimulus values may be processes $x = x(t)$, $y = y(t)$, that is, they may change within a trial (recall that this makes no difference for the applicability of Theorem 1). Finally, as Ennis's example with chemical stimuli indicates, there may be situations when one cannot tell, at least at one's present state of knowledge, precisely what processes constitute a stimulus. In criticizing a potentially general law, however, one should not argue from especially complicated special cases. Newton's second law of motion would be very difficult to establish on water jets in a waterfall, but once it is established on more easily analyzable objects, it helps us to understand waterfalls as well. In a psychophysical experiment we usually identify a stimulus by its controllable source and by controllable aspects of the observation conditions. The stimuli we call "line segments," for example, are identified by geometric characteristics of their sources (light distributions in a frontoparallel plane), with such aspects as intensity, overall illumination, viewing distance, etc., being held constant. Stimulus variability can never be entirely eliminated, but one can always try additional measures to reduce it below a level already achieved (e.g., by fixing the observer's head in a chin-rest, monitoring eye movements, etc.). If these measures result in a reduction or elimination of the observed violations of Regular Minimality, it is reasonable to view the latter as empirically corroborated. Epistemologically, it would have been worse for the law of Regular Minimality if an apparent compliance with it could somehow be created rather than destroyed by uncontrollable stimulus variability.

5. Approximate law of Regular Minimality?

Ennis's second line of criticism consists in suggesting the following possibility: discrimination probability function $\psi(x, y)$ (now assuming that x and y are true stimulus values) may be generated by a well-behaved Thurstonian-type model, but the parameters of the latter could be so chosen that the violations of Regular Minimality, as predicted by Theorem 1, would be too small to be detectable in a realistic experiment. Once again, the substantiation Ennis gives to this possibility is not error-free (see Comments 11 and 12), but this does not invalidate the possibility itself. The latter, or at least a certain interpretation thereof can be arrived at by means of a simpler argument.

Consider Fig. 5. The set of stimuli used in this or any other experiment is necessarily finite, involving therefore an error of discretization. The conjunction of Regular Minimality and Nonconstant Self-Similarity in Fig. 5 is only corroborated within a two-pixel precision (≈ 1.8 min arc). One cannot exclude the possibility, for example, that the PSE for $x = 11$ px is $y = 13$ px, as the data table suggests, but the PSE for $y = 13$ px is, say, $x = 11.5$ px, rather than 11 px. Another illustration is given in Comment 12, using Ennis's own simulation study. On a more general level, suppose that $\psi(x, y)$ is generated by a well-behaved Thurstonian-type model with two disparate PSE curves $y = h(x)$ and $x = g(y)$, as shown in Figs. 7 and 10. Since the stimulus scale in these figures is not specified, it is possible that the difference between the two PSE lines is so small that the grid of stimulus pairs (a_i, a_j) used in a realistic experiment $(i, j = 1, \ldots, k)$ will be relatively too coarse. It is easy to imagine a combination of two very close PSE curves with a sparse grid structure in which every a_i will belong to a pair (a_i, a_j) which is so much closer to both $(a_i, h(a_i))$ and $(g(a_j), a_j)$ than the rest of the experimental pairs that $\psi(a_i, a_j)$ will be found to be the smallest value in both the ith row and the jth column, creating thereby an "illusion" of Regular Minimality. Thus, if $h(a) = g(a) < a$ for all a (as it may happen in the simple symmetric models of the Luce–Galanter variety), the disparate pairs $(a_i, h(a_i))$ and $(g(a_i), a_i)$ may very well be much closer to (a_i, a_i) than to any other pair from the experimental grid, creating the impression that Regular Minimality holds in its simplest form. Note that this argument is entirely non-statistical: it holds true even if one knows the probabilities $\psi(x, y)$ for all experimental points (x, y) precisely.

I will assume that this "close-PSE-curves" interpretation can be taken as the intended meaning of Ennis's "subtle and difficult to detect" violations of Regular Minimality. It is important to see the implications of this interpretation. Saying that $\psi(x, y)$ is generated by a well-behaved Thurstonian-type model but that this model's parameters are such that the two PSE curves $y = h(x)$ and $x = g(y)$ are very close to each other does not mean that the closeness of these PSE curves is "predictable" from the principles underlying well-behaved Thurstonian-type models. On the contrary, there is nothing in their internal logic that would compel the violations of Regular Minimality to be small rather than arbitrarily gross. To assume the closeness of the PSE curves within the framework of well-behaved Thurstonian-type models amounts to assuming that *Regular Minimality holds as an approximate law*, as an external constraint in constructing these models. A simple but rather apt analogy may help to understand this clearly. Consider two competing assumptions about the form of some relationship $u = f(v)$: one assumption is that $f(v) = \exp(kv)$, the other says that $f(v) = P_m(v)$, a polynomial of an unspecified order m. Obviously, if the exponential model holds, then no polynomial model can be true (an analogue of our Theorem 1). Suppose that the exponential model is corroborated by experimental data. A proponent of the polynomial model may claim, however, that it is the polynomial model which is true, but the polynomial order and coefficients have to be adjusted so that $P_m(v)$ closely approximates $\exp(kv)$. Clearly, this is a possibility, but it does not follow from the internal structure of polynomial functions. Rather for a proponent of the polynomial model it amounts to adopting an additional assumption that the exponential model holds too, albeit approximately.

The question that arises is how small the hypothetical small violations of Regular Minimality are in real data. An obvious way of addressing this question is to conduct experiments like the one depicted in Fig. 5 but with progressively increasing density of the experimental grid of stimuli. With a large number of replications per stimulus pair, however, this approach may quickly run into technical difficulties (for one thing, an observer's perception and judgments are unlikely to remain unchanged throughout an experiment that extends over many weeks). In the remainder of this section I present an alternative approach, using the matching-based version of the law of Regular Minimality with the consistency assumption, as described in Section 2.3. Suppose that the discrimination probability function ψ which is consistent with matching adjustments is generated by a well-behaved Thurstonian-type model with the two PSE curves $y = h(x)$ and $x = g(y)$ as shown in Fig. 10, with no cross-overs. Then the idealized "ping-pong" matching procedure illustrated in the same figure yields a series of matched values in each observation area which will deviate from the initial values by intervals monotonically increasing with the number of adjustment steps. Even if the two curves are very close to each other, the eventual separations after a sufficiently large number of steps can be expected to be large enough to be experimentally detectable.

To switch from the idealized "ping-pong" procedure to a realistic one I assume that when y is adjusted to match a fixed value of x the matching value (referred to as *balance point*) is achieved at

$$y = h(x) + \varepsilon^{(2)}, \tag{5}$$

where $\varepsilon^{(2)}$ is an *adjustment error* in the second observation area. Analogously, for x-balance points,

$$x = g(y) + \varepsilon^{(1)}. \tag{6}$$

I assume that the adjustment errors are symmetrically distributed around zero (but are not necessarily stochastically independent). The details of the experimental procedure are given in Figs. 18 and 19. Experiment C referred to in the latter figure is the control experiment whose main purpose was to help in determining time-series properties of the balance points. This experiment ("semi-ping-pong" procedure) is described in Comment 13, together with a summary of the time-series properties.

The results of Experiments A, B1, and B2 (for a single participant naive as to the aims of the experiments) are presented in Figs. 20–23. Two aspects of these results are relevant to the present discussion.

1. The distributions of the first-order differences in Figs. 20 and 22 being almost perfectly symmetrical, they provide no support even for very small violations of Regular Minimality.

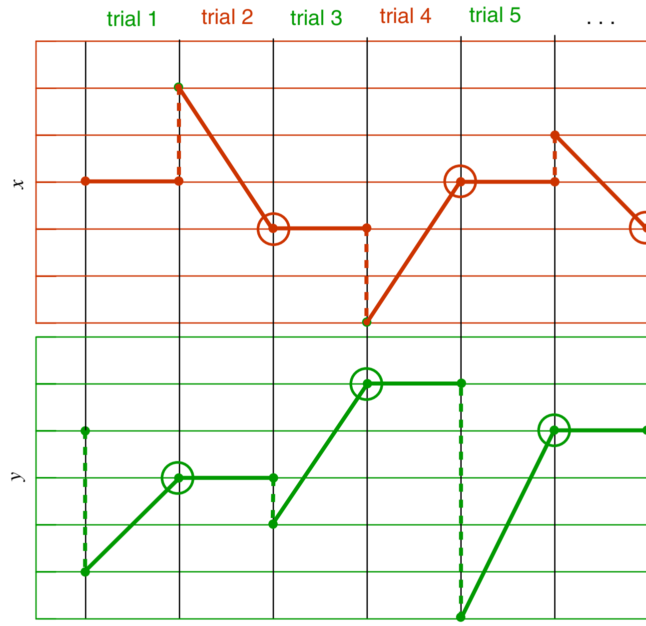

Fig. 18. A schematic representation of the "ping-pong" matching procedure. The vertical scale represents values of two stimuli (lengths of light segments) in two observation areas (x and y). The horizontal scale shows successive trials. At the end of each trial x and y appear equal. A new trial begins by a randomly chosen "disbalancing" change (indicated by vertical point lines) in the stimulus that was kept fixed in the previous trial. This stimulus is then being adjusted (as indicated by tilted solid lines) until it appears to match the value of the other stimulus which remains fixed throughout the trial (horizontal solid lines). This procedure continues for a series of 200 trials yielding a series of 100 x-values and a series of 100 y-values (referred to as x and y "balance" points, encircled). An experiment consists of 10–25 such 200-trial series, all starting with one and the same fixed x-value.

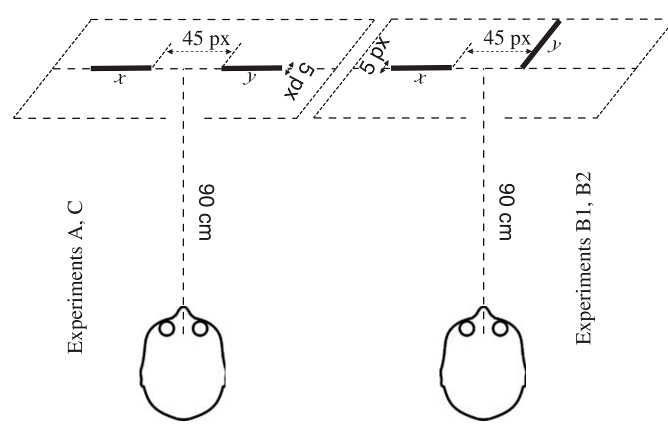

Fig. 19. Details of Experiments A, C (with stimulus displays corresponding to Fig. 1A) and Experiments B1, B2 (corresponding to Fig. 1B). 1 px $\approx$ 0.92 min arcs. In reference to the procedure depicted in Fig. 18, the initial value of x in each series was 45 px in Experiments A, C, and B1, and it was 90 px in Experiment B2; the randomly chosen disbalance changes were uniformly distributed on the set $[-15, -5] \cup [5, 15]$ (px) in Experiments A, C, B1, and on the set $[-30, -10] \cup [10, 30]$ (px) in Experiment B2.

2. The magnitude of a systematic trend in the mean balance points (Figs. 21 and 23) should reflect an average value of the hypothetical difference between two PSE curves in

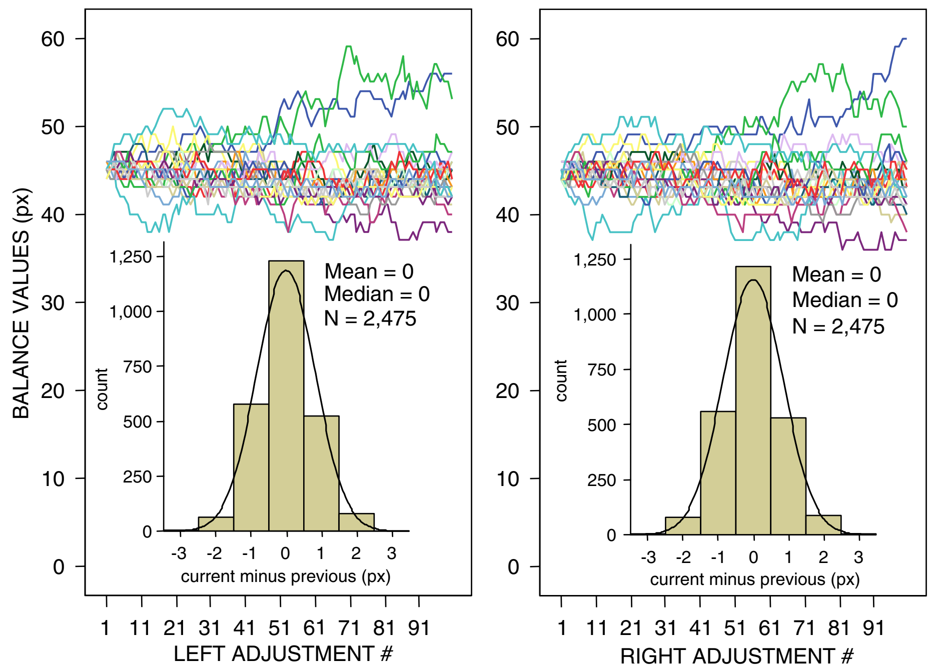

Fig. 20. The results of Experiment A, consisting of 25 "ping-pong" matching series of 200 trials. Each experimental curve represents a series of 100 successive balance points for each of the two observation areas. The fan-like appearance of the tangle indicates a Winer-like process (see Comment 13). The insets represent the distributions of changes (first-order differences) between successive balance points in a series. The distributions are almost perfectly symmetrical.

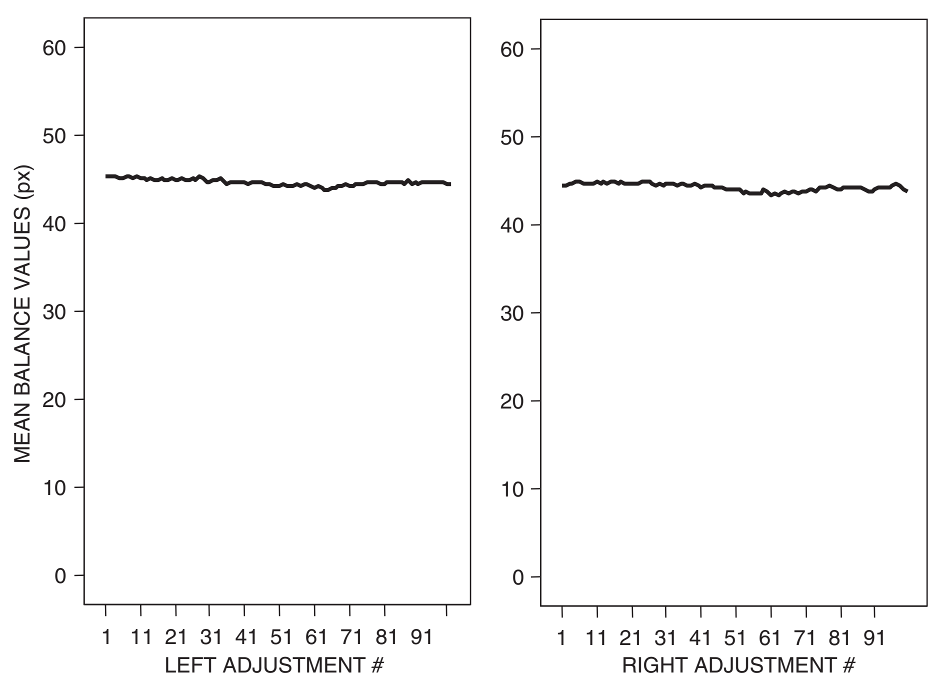

Fig. 21. The means of the balance point series shown in Fig. 20. The rate of the overall linear trend is about −0.008 px/trial for both x and y balance values. 0.008 px make up less than 0.5 sec arc, or 0.0003% of the initial value (45 px). The negative sign of the trend is not reliable (e.g., the trend is positive for the second half of the adjustments, trials 51–100).

Fig. 10 (horizontal difference in the case of x-balance points and vertical for y-balance points). The observed magnitudes of the overall trends are clearly below the level of physiological plausibility, at least by a factor of 100. Moreover, the trends are of different signs in different experiments and for different parts of the series. (That the trends are spurious is also evident from the fact that smaller but comparable trends are found in the control Experiment C, Fig. 27, where no trends are predicted irrespective of whether Regular Minimality holds.)

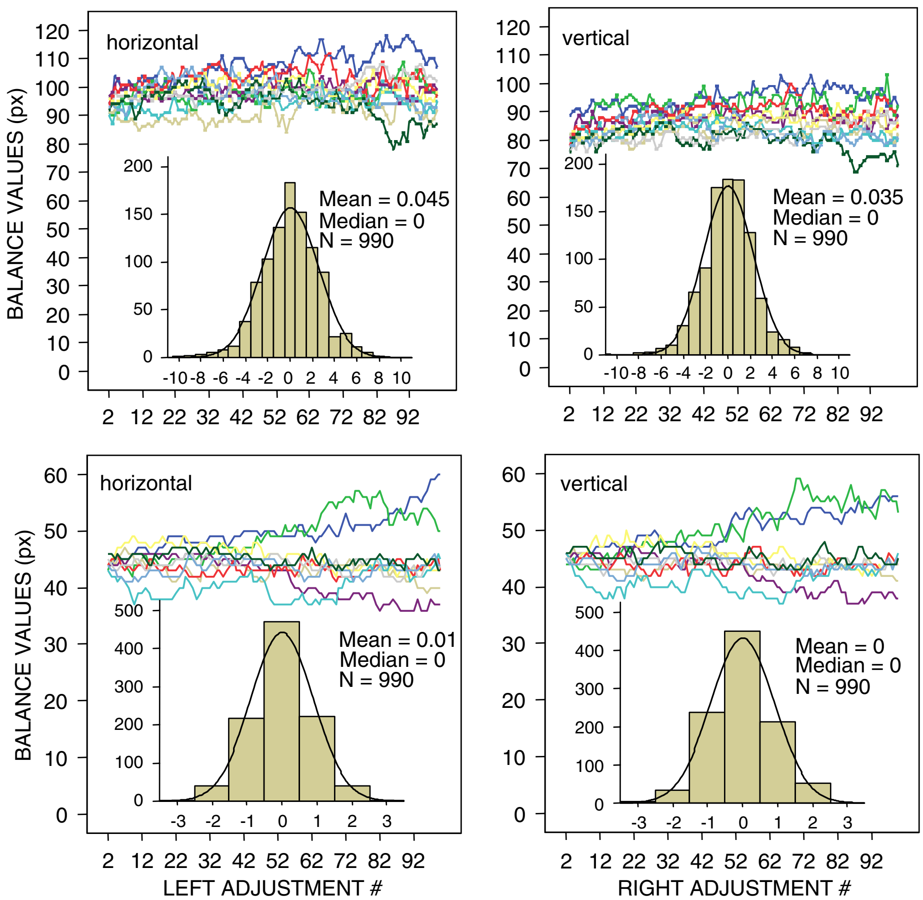

Fig. 22. The results of Experiments B1 (bottom panels) and B2 (top), each consisting of 10 "ping-pong" matching series of 200 trials. Recall from Fig. 19 that in B1 the initial length on the left (x-value) was 45 pixels, in B2 it was 90 pixels. The insets are as in Fig. 20, and they too exhibit almost perfect symmetry.

These conclusions, however, should be taken with caution, as their validity is based on two additional assumptions: the absence of cross-overs between the two hypothetical PSE curves, and a specific adjustment error model, (5)–(6). Moreover, the relevance of these results for discrimination probabilities hinges on the assumption of consistency (Section 2.3). By presenting these experiments therefore I do not claim a definitive corroboration of the law of Regular Minimality, either in its main version ($\mathscr{P}_1$–$\mathscr{P}_3$) or the matching-based one ($\mathscr{P}_1^*$–$\mathscr{P}_3^*$). Rather the goal is to demonstrate that the issue of "subtle violations" can be meaningfully addressed by experimental means.

6. Conclusion

The concluding section of Ennis's paper provides a convenient way of summarizing my analysis of his criticisms. He writes in this section:

> ... it is easy to show that the mapping between observation areas may not be invertible when stimuli possess stimulus noise. The imposition of invertibility of this mapping on models of discrimination probabilities would be highly restrictive, particularly when stimulus variance is a primary source of perceptual variance as occurs in practical applications of Thurstonian models.

The logical problems with this criticism and with the arguments by which it was arrived at are explained in Section 4 and in Comment 9. The summarizing statement itself seems to augment the confusion by implying that I *impose* the property $\mathscr{P}_3$ on situations where, according to my own analysis, this property cannot be satisfied.

Ennis also writes:

> ... violations of RM may be subtle and difficult to detect experimentally, often not exhibited in matrices of same–different discrimination probabilities. Under the assumptions made it seems that the conditions leading to more confidence in observing NCSS [Nonconstant Self-Similarity] are those that increase the likelihood of observing RM in an experiment.

The confusions in the arguments by which Ennis arrives at these conclusions are pointed out in Comments 11 and 12. In particular, Comment 12 demonstrates that the second sentence in the quotation is factually incorrect. The possibility of "subtle and difficult to detect" violations of Regular Minimality, however, becomes a valid

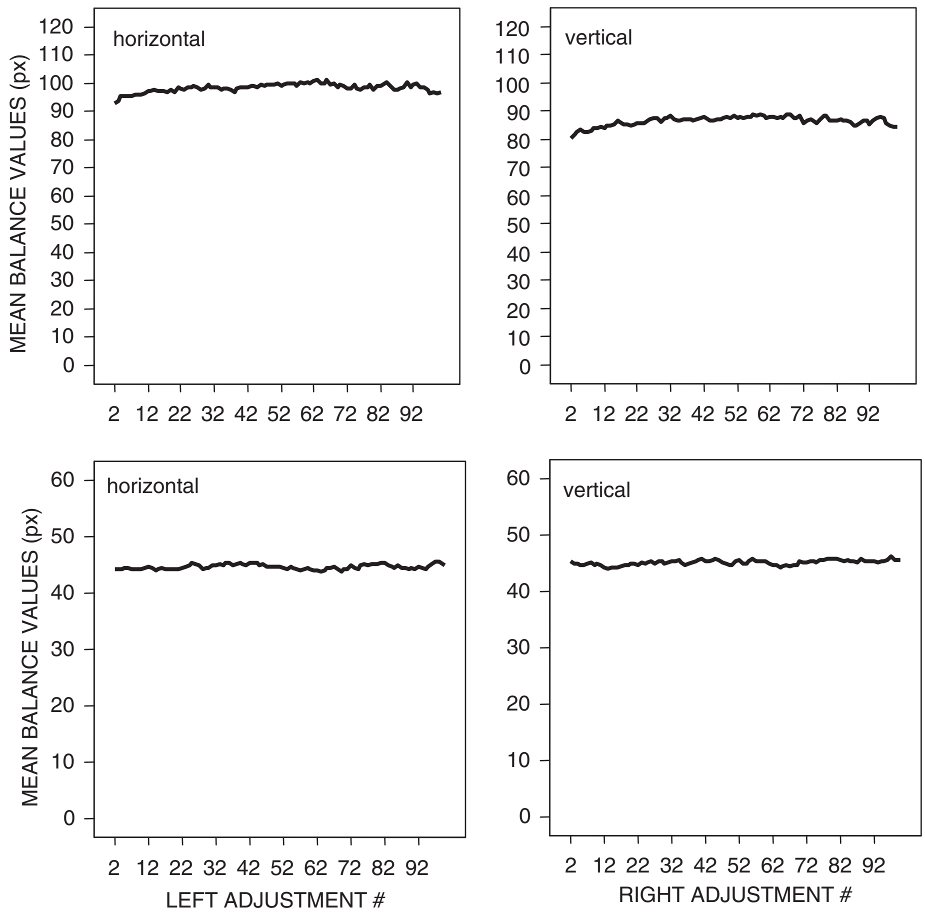

Fig. 23. The means of the balance point series shown in Fig. 22. In the bottom panels (45 px initial value) the rates of the overall linear trends are about +0.004 and +0.008 px/trial for, respectively, left and right adjustments. In the top panels (90 px initial value) the linear trend is +0.028 px/trial. Compare these figures (all making up less than 0.0005% of the initial values) with the negative overall trend in Fig. 21.

consideration under a reinterpretation of Ennis's position. This reinterpretation (involving sparsity of the experimental grid of stimulus pairs) and its theoretical implications ("approximate law of Regular Minimality") are presented in Section 5. The experiments with length discriminations described in that section do not exhibit even small violations of Regular Minimality, provided "small" is not meant to fall outside the range of plausible interpretability (say, below 0.1 min arc).

The notion of the range of plausible interpretability deserves a comment. If the law of Regular Minimality is valid, its sphere of applicability, as with any other law, is limited to a certain scale of consideration. In this trivial sense Regular Minimality is bound to be an approximation, and its relationship to Thurstonian-type modeling is only meaningful to discuss within the framework of appropriately chosen description of a stimulus space. Thus, since the notion of well-behavedness in Theorem 1 (Section 3) does not apply to discrete stimulus spaces, the applicability of Theorem 1 may break down on a "microscopic" scale of consideration, where the continuity in the description of stimuli gets into a conflict with the discreteness in the structure of receptors, molecular interactions, or quantal phenomena.

Ennis is doubtlessly right when he says that Thurstonian-type models are simple and intuitive. One should not forget, however, that they are still only models rather than facts, and that part of their appeal may be in their potentially unlimited fitting power, due to the unlimited freedom one has in choosing these models' unobservables: probability spaces, multiparametric distributions for random images, and decision rules. The correspondence between the hypothetical random images in these models and what is known as facts about mental or neurophysiological processing is far from being straightforward or uniquely determinable. Thus, the fact that any sensory input gives rise to a complex system of stochastic processes does not by itself justify the use of well-behaved Thurstonian-type models, for two reasons: (a) not all stochasticity has a Thurstonian-type structure (see Section 3.1), and (b) there is a potential infinity of aspects and properties of observable stochastic processes that may pertain to a given psychophysical task (such as same–different judgments). For instance, a large number of parallel realizations of a stochastic process invoked by a stimulus, if averaged across, may give rise to an essentially deterministic process representing the stimulus, excluding thereby all well-behaved Thurstonian models (see Section 3.2).

In spite of my disagreements with Ennis's reasoning I find his critical paper very useful for drawing one's attention to issues that can be easily glossed over in a mathematical theory which begins with an abstraction like "let $\mathfrak{S}$ be a set of stimuli and ψ a discrimination probability function." The first of these issues is the necessity of being specific and precise in describing one's set of stimuli. The second issue is the necessity of being aware that the observed proportions of responses in an experiment generally estimate mixtures of true probability functions rather than a fixed probability function. Stimulus variability is only one mechanism of creating such mixtures, the other ones being the numerous sources of non-stationariness in observer's behavior, depicted by such notions as fatigue, attention fluctuations, and perceptual learning (see Dzhafarov & Colonius, 2006). As explained in Section 4, mixtures do not inherit the law of Regular Minimality from the individual probability functions they contain. In those cases where the violations of Regular Minimality are suspected to be due to mixtures, one faces the necessity of modeling these mixtures explicitly, as in (4). If one is willing to use the adjective "Thurstonian-type" to designate such mixture models, then one can view this as the possibility, and even necessity, of combining Thurstonian-type modeling with the law of Regular Minimality in a cooperative rather than antagonistic way.

Acknowledgments

This research was supported by the NSF grant SES 0318010. I am grateful to Daniel Ennis for interesting discussions and for the opportunity his critique provides for clarifying and elaborating issues we both find important. I thank Daniel Ennis, Jun Zhang, and J.D. Balakrishnan for critically reading and commenting on this paper, and Philip Smith for moderating earlier discussions that led to Ennis's paper and my reply.

Appendix A. Additional comments

Comment 1: Stated more rigorously, a stimulus $x \in \mathfrak{S}_1$ is characterized by its variable properties (e.g., segment length) and fixed properties (e.g., being on the left); and analogously for $y \in \mathfrak{S}_2$. The fixed characteristics of a stimulus define its observation area and are therefore generally different in $\mathfrak{S}_1$ and $\mathfrak{S}_2$. The variable characteristics in $\mathfrak{S}_1$ and $\mathfrak{S}_2$ form identical sets and are referred to as *stimulus values*. By convenient abuse of language the term "stimulus" is often used in the sense of a stimulus value. Thus, two segments x and y can be said to be physically equal, $x = y$, even though it is only their lengths that are equal. Similarly, $x \in \mathfrak{S}_1$ and $y \in \mathfrak{S}_2$ can be said to belong to a single stimulus set, with their observation areas being indicated by their ordinal position within a stimulus pair, (x, y). For detailed discussions see Dzhafarov and Colonius (2005a, 2006).

Comment 2: For completeness, it should be noted that Regular Minimality is predicated on the assumption that neither $\mathfrak{S}_1$ nor $\mathfrak{S}_2$ may contain distinct elements that are psychologically indistinguishable: that is, if $\psi(a_1, y) = \psi(a_2, y)$ for all y, then $a_1 = a_2$, and if $\psi(x, b_1) = \psi(x, b_2)$ for all x, then $b_1 = b_2$. For example, before one posits Regular Minimality for aperture colors viewed in daylight, all photopic metamers of any given color should be labeled identically (treated as a single stimulus). Fig. 24 provides a schematic illustration on a toy example with a finite set of stimuli.

Comment 3: It often simplifies mathematical analysis to present stimuli in the two observation areas in what I called "a canonical form" (Dzhafarov, 2002b). Essentially, it means that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are both bijectively mapped onto a set $\mathfrak{S}$ so that whenever $x \in \mathfrak{S}_1$ and $y \in \mathfrak{S}_2$ are mutual PSEs, they are mapped into one and the same element ("stimulus label") $z \in \mathfrak{S}$. This is always possible to achieve (in an infinite number of ways) due to the law of Regular Minimality. In those cases (e.g., in the analysis of Thurstonian-type modeling) where it is important for the

		y_a		y_b		y_c	y_d	
	A**	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_a	x_1	0.6	0.6	0.6	0.6	0.1	0.8	0.8
	x_2	0.6	0.6	0.6	0.6	0.1	0.8	0.8
x_b	x_3	0.9	0.9	0.9	0.9	0.8	0.1	0.1
	x_4	0.9	0.9	0.9	0.9	0.8	0.1	0.1
x_c	x_5	1	1	0.5	0.5	1	0.6	0.6
x_d	x_6	0.5	0.5	0.7	0.7	1	1	1
	x_7	0.5	0.5	0.7	0.7	1	1	1

A	y_a	y_b	y_c	y_d
x_a	0.6	0.6	0.1	0.8
x_b	0.9	0.9	0.8	0.1
x_c	1	0.5	1	0.6
x_d	0.5	0.7	1	1

Fig. 24. Entries of the two matrices represent probabilities $\psi(x, y)$. Regular Minimality is assumed to hold on *reduced* stimulus sets, schematically shown by the rows ($\mathfrak{S}_1$) and columns ($\mathfrak{S}_2$) of matrix A. It is obtained from the two initial stimulus sets, the rows and columns of matrix A^{**}, by lumping together (identically labeling) any two stimuli x_i, x_j in the first observation area such that $\psi(x_i, y) = \psi(x_j, y)$ for all y in the second observation area (and analogously, lumping together any y_i, y_j with $\psi(x, y_i) \equiv \psi(x, y_j)$).

analysis that stimuli form an open connected region of $\mathbb{R}^n$, the relabeling $\mathfrak{S}_1 \to \mathfrak{S}$, $\mathfrak{S}_2 \to \mathfrak{S}$ must be confined to diffeomorphisms. In spite of their great convenience in formal computations, canonical forms should be treated with caution. In a canonical form, a pair (z, z) is always a pair of mutual PSEs, but the two z's may very well be physically different (i.e., the physical identity of a stimulus is encoded by both a label and its ordinal position within a pair). One should remember therefore that a canonical form does not eliminate constant errors (unless they were not there to begin with). To prevent confusions on this account, I do not use canonical representations in this paper.

Comment 4: When Ennis writes that he acknowledges "extensive empirical support" for Nonconstant Self-Similarity but challenges the validity of Regular Minimality, or that in the experiments that manifest Nonconstant Self-Similarity it may be difficult to detect violations of Regular Minimality, he is glossing over a terminological subtlety: Nonconstant Self-Similarity, in my meaning of the term, presupposes Regular Minimality. Clearly, if functions h and/or g are not defined, then one cannot ask whether the values of $\psi(x, h(x))$ and/or of $\psi(g(y), y)$ are constant. If properties $\mathscr{P}_1$ and $\mathscr{P}_2$ of the law of Regular Minimality hold but $\mathscr{P}_3$ does not, $\psi(x, y)$ does not have a single minimum level function. It has instead two distinct functions, $\psi(x, h(x))$ and $\psi(g(y), y)$, either of which can be constant or nonconstant independent of the other. To make Ennis's remarks consistent one should redefine the notion of Nonconstant Self-Similarity as applying to these two functions separately, under the assumptions $\mathscr{P}_1$ and $\mathscr{P}_2$. Such a redefinition may be warranted in the context of a debate over the validity of $\mathscr{P}_3$, but one should clearly mark the change of the meaning. In this respect the situation here is different from that with the notion of a constant error (which, ironically and contrary to my usage, Ennis does not wish to consider unless Regular Minimality holds in its entirety).

Comment 5: Ennis writes that he does not consider constant error a well-defined concept unless Regular Minimality holds. Irrespective of one's definitions, however, if one does not accept the law of Regular Minimality but assumes (as Ennis does) its properties $\mathscr{P}_1$ and $\mathscr{P}_2$, one must allow for the possibility that x and its PSE $h(x)$ are different, and that the same may be true for $g(y)$ and y (even if $g \not\equiv h^{-1}$). One cannot eliminate the non-coincidence of stimuli with their PSEs (which is what constant error is, in my usage) by means of refusing to call it constant error. This is relevant to one of Ennis's mistakes mentioned in Comment 11.

Comment 6: In Dzhafarov (2003a, 2003b) this theorem (in fact, a series of theorems) is proved for $x, y \in \mathbb{R}^n$, and it can be generalized to continuous spaces as defined in Dzhafarov and Colonius (2005a). The theorem does not apply, however, to discrete stimulus spaces, such as spaces of Morse codes or consumer products, where the notion of well-behavedness is not defined (for a general definition of discrete spaces see Dzhafarov and Colonius, 2005b).

Comment 7: In the context of greater–less discriminations Thurstone (1927a, 1927b) assumed that x and y are mapped into a bivariate normally distributed $(\mathbf{X}, \mathbf{Y})$ with the correlation coefficient generally determined by both x and y, $r = r(x, y)$. It is not known, however, whether it is always possible to find a variable $\mathbf{C}$ (independent of x and y) such that for any its value the variables $\mathbf{X}, \mathbf{Y}$ have independent conditional distributions whose parameters depend on x and y, respectively (see a discussion of this issue in Dzhafarov, 2003c). Thurstone's most general case therefore does not necessarily describe a Thurstonian-type model in my meaning of the term.

Comment 8: The definition given in the text is the most restrictive version of well-behavedness. In Dzhafarov (2003b) it is shown how this notion can be relaxed without affecting the validity of what in the present paper is Theorem 1. Moreover, even these relaxed definitions are only sufficient conditions for this theorem, it might be provable under still weaker constraints.

Comment 9: In the context of stimulus variability Ennis considers a model in which $\mathbf{X}$ and $\mathbf{Y}$ are independently normally distributed on $\mathbb{R}$ with means μ_x, μ_y and variances σ_x^2, σ_y^2. (I change Ennis's notation, because the decomposition of σ^2 into "stimulus variance" σ_{S}^2 and constant "neural variance" σ_{N}^2 is not relevant to this discussion.) He assumes the distributions of $\mathbf{X}$ and $\mathbf{Y}$ to be independent of the observation area: if $x = y$, then $\mu_x = \mu_y$ and $\sigma_x^2 = \sigma_y^2$. Then he considers the situation (again, changing his notation for greater clarity) when, for some stimulus values a and b (whether they are values of x or of y), $\mu_a = \mu_b$ but $\sigma_a^2 > \sigma_b^2$. He observes that in this situation $\psi(a, a) > \psi(a, b) = \psi(b, a) > \psi(b, b)$, and he claims that this is a violation of Regular Minimality. There is one logical error and one misleading implication in this reasoning.

The logical error is this: finding that $\psi(a, a) > \psi(a, b) = \psi(b, a) < \psi(b, b)$ for two specific values of a and b does not indicate a violation of Regular Minimality. To establish the latter at $x = a$ one has to do three things: (a) to find out if the function $y \to \psi(a, y)$ achieves a global minimum at some $y = h(a)$ (if this is not the case, $\mathscr{P}_1$ is violated); (b) to find out if the function $x \to \psi(x, h(a))$ achieves a global minimum at some $x = g(h(a))$ (if this is not the case, $\mathscr{P}_2$ is violated); and (c) to find out if $g(h(a)) = a$ (if this is not the case, $\mathscr{P}_3$ is violated). This analysis is easy to perform, with the result that if μ_x, σ_x^2 (hence also μ_y, σ_y^2) are sufficiently smooth function of x (respectively, y) and if σ_x^2 (hence also σ_y^2) is not a constant, then $\mathscr{P}_1$ and $\mathscr{P}_2$ hold but $\mathscr{P}_3$ is violated. Ennis does speak of the "non-invertibility of h" but makes no attempt to find $h(a)$. Fig. 25 illustrate this fallacy on a toy example.

The misleading implication in Ennis's reasoning is in its suggesting that the equality $\mu_a = \mu_b$ in combination with the inequality of variances is somehow responsible for the fact that this model violates Regular Minimality. In fact, the specific dependence of the means and variances on

	a	s	b
a	0.48	0.30	0.36
s	0.30	0.40	0.35
b	0.36	0.35	0.08

	a		b
a	0.48		0.36
b	0.36		0.08

Fig. 25. Illustration of an error in thinking of Regular Minimality in terms of two arbitrarily chosen stimulus values. The left matrix represents $\psi(x, y)$ for a three-element stimulus set. Regular Minimality is satisfied, with the PSE pairs $(x = a, y = s)$, $(x = s, y = a)$, and $(x = b, y = b)$. If one picks just a and b from the set $\{a, s, b\}$, however, Regular Minimality will "fail": the apparent PSE for $x = a$ is $y = b$, but the apparent PSE for $y = b$ is $x = b \neq a$. The pattern is $\psi(a, a) > \psi(a, b) = \psi(b, a) > \psi(b, b)$, the same as in Ennis's paper.

stimulus values is irrelevant, insofar as one can show that $\psi(x, h(x))$ is not constant around at least a single point $x = a$. This follows from Theorem 1, and even without it, it is clearly explained in the introduction to Dzhafarov (2003a) on an illustrating example which, except for superfluous details, is mathematically identical to Ennis's.

Comment 10: The specific model listed in Ennis's table under type II and attributed to Shepard (1987) is most obviously incompatible with Regular Minimality and Nonconstant Self-Similarity, at least if one does not consider (as Shepard did not) the possibility that $x = y$ but x and y (because they belong to different observation areas) are represented by different vectors of perceptual attributes (the z's). The model yields $\psi(x, y) = 0$ whenever $x = y$, which makes it the simplest and most easily dismissible version of the Probability–Distance hypothesis (Dzhafarov, 2002a). De facto, however, Shepard used $\phi(x, y)\phi(y, x)/\phi(x, x)\phi(y, y)$ (where $\phi \equiv 1 - \psi$) rather than $\psi(x, y)$ (see Shepard, 1957, for details). With this understanding Shepard's theory is consistent with a version of the quadrilateral dissimilarity model discussed in Dzhafarov and Colonius (2006).

Comment 11: Ennis's argument can be summarized as follows. He generates $\psi(x, y)$ by means of a well-behaved Thurstonian-type model which satisfies properties $\mathscr{P}_1$ and $\mathscr{P}_2$: that is, for every x the function $y \to \psi(x, y)$ achieves its minimum at some $y = h(x)$, and for every y the function $x \to \psi(x, y)$ achieves its minimum at some $x = g(y)$. The values of $\psi(x, h(x))$ and $\psi(g(y), y)$ vary with x and y, respectively, and it follows from Theorem 1 that $\mathscr{P}_3$ cannot be satisfied: in fact in Ennis's examples $g(h(x)) \neq x$ for all x. Moreover, $h(x) \neq x$ and $g(y) \neq y$ at all points (constant errors, in my terminology). Then Ennis considers an imaginary data-analyst who is supposed to check the compliance of $\psi(x, y)$ with the law of Regular Minimality. With one exception (discussed separately, in Comment 12), the data-analyst does this by testing the null-hypothesis that $h(a) = a$, for a specific value $x = a$. Ennis thinks that he makes his point by showing that, when tested against the correct value of $h(a)$, the probability of Type 2 error in this case is not sufficiently low, even if estimates of ψ are computed from a large number of replications. There are at least two logically independent problems with this reasoning.

One is in Ennis's assertion that in testing whether Regular Minimality holds for his function $\psi(x, y)$, one would only have to consider whether it holds with no constant error, i.e., with $h(x) \equiv x$. He gives two reasons for this assertion.

(a) "The mapping is not invertible so constant error is not a well-defined concept." This amounts to saying that $h(x)$ can be made coincide with x by means of not calling their non-coincidence a constant error (see Comment 5).

(b) Since $x = a$ and $y = a$ in his model are mapped into identically distributed random variables, Ennis says that there cannot be a constant error. This is a double-confusion: first, in his model de facto $h(x) \not\equiv x$ and $g(y) \not\equiv y$, and second, the imaginary data-analyst, in testing ψ for compliance with the law of Regular Minimality, is not supposed to know the model by which ψ was generated (especially if the model is known to violate Regular Minimality).

The second problem is in Ennis's overlooking the fact that the experimental corroborations of the law of Regular Minimality reported in Dzhafarov (2002b, 2003a) and Dzhafarov and Colonius (2005a) are never based on his imaginary procedure of a priori choosing a specific pattern of Regular Minimality (canonical or not) and then failing to reject it as a null-hypothesis. This would not be good science. What we have factually done should be apparent from inspecting Fig. 5. If the true discrimination probabilities in neighboring cells of such a matrix were very close to each other, the probability of observing a pattern of minima satisfying Regular Minimality would be negligibly small.

A legitimate issue that can be taken with our experimental data is different from Ennis's power analysis, and in fact is entirely non-statistical: it has to do with the possibility that the discrete experimental grid of stimulus pairs may be too coarse. This issue is taken on in the main text and in the next Comment.

Comment 12: With reference to a personal communication from me (in which I pointed out some of the problems mentioned in Comment 11), Ennis cites a simulation study in which a matrix of statistical estimates of discrimination probabilities $\psi(x, y)$ was found to be in compliance with Regular Minimality in 99.3% of cases, in spite of being generated by a well-behaved Thurstonian-type model known to violate this law. This may sound as a valid version of Ennis's low-statistical-power claim, but it is not. The issue has nothing to do with statistics, as one can see by calculating the *true*, population-level probabilities $\psi(x, y)$ for Ennis's simulation study (from

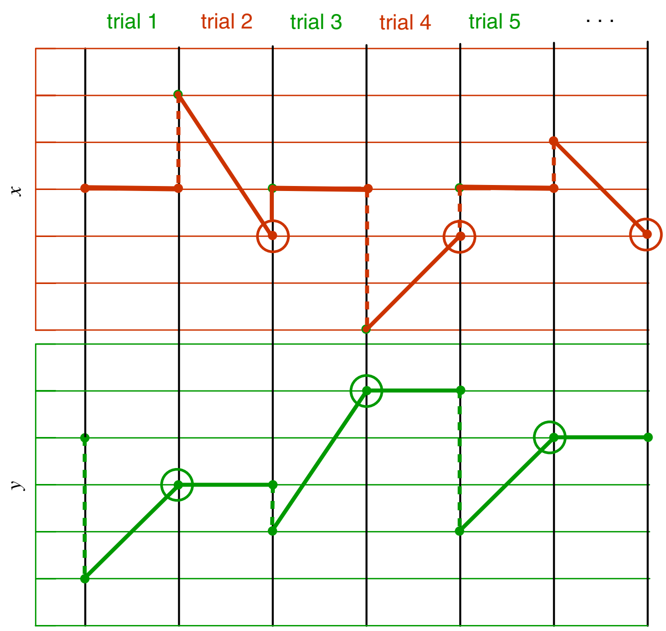

Fig. 26. A schematic representation of the "semi-ping-pong" control (Experiment C) for the main Experiment A. The procedure is identical to the "ping-pong" one (Fig. 18), except that the fixed x-value (of the left segment) during every y-adjustment (of the right segment) was always set equal to the initial x-value of 45 pixels.

his Eq. (2)):

Stimuli	2.0	2.5	3.0	3.5	4.0
2.0	0.444 |	0.538	0.697	0.837	0.926
2.5	0.538	0.500 |	0.577	0.711	0.837
3.0	0.697	0.577	0.545 |	0.610	0.725
3.5	0.837	0.711	0.610	0.583 |	0.638
4.0	0.926	0.837	0.725	0.638	0.615 |

We see that the true, population-level row-column minima in this matrix are all located on the main diagonal. It is hardly surprising therefore that with 600 replications per cell the statistical estimates of the minima in this matrix should be located *correctly* with probability close to 1. The reason why this matrix does not exhibit violations of Regular Minimality on the population level is that Ennis chose a grid of stimulus pairs which is too sparse to include the true PSE values $h(x)$ and $g(y)$ predicted by his model, and the pairs closest to them happen to be the diagonal values.

A direct minimization shows that in Ennis's simulation $h(x) \approx x - 0.1$, i.e., the true PSEs for the row stimuli $x = 2, 2.5, 3, 3.5, 4$ are $y = 1.9, 2.4, 2.9, 3.4, 3.9$, respectively (and $g \equiv h$ due to the symmetry of his model). To reveal

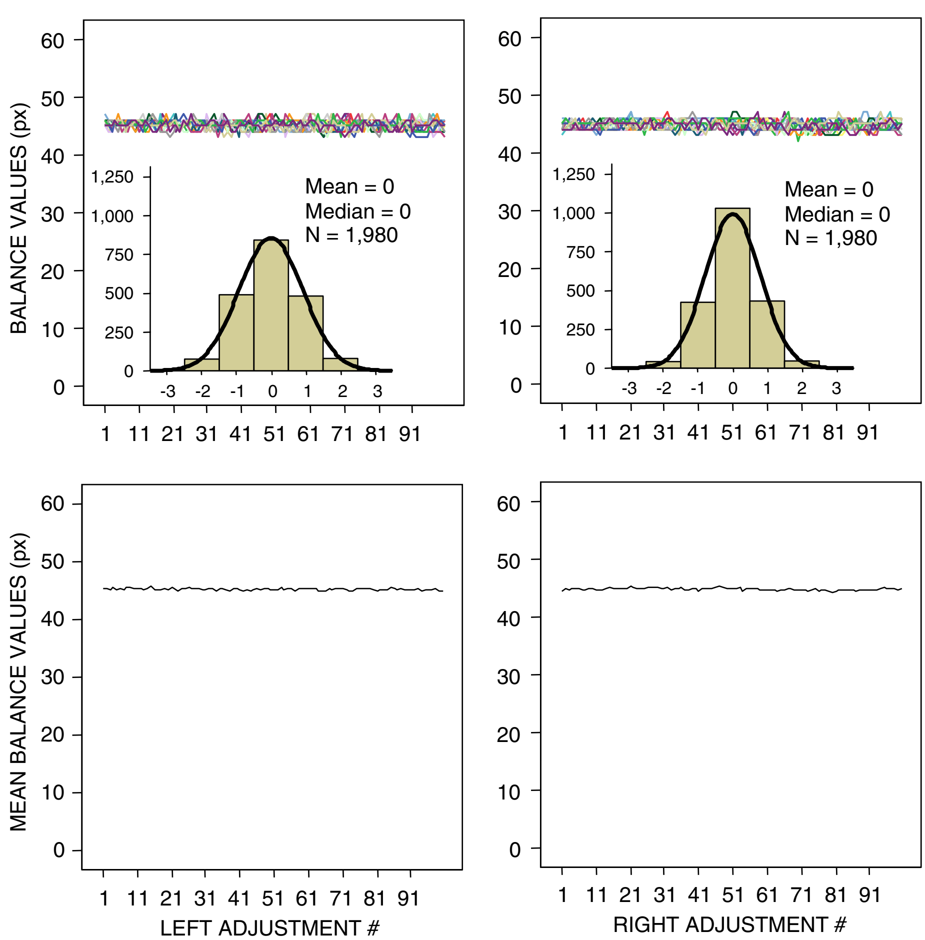

Fig. 27. The results of Experiment C (control) consisting of 20 series of 200 trials obtained by means of the "semi-ping-pong" procedure depicted in Fig. 26. The rest is the same as in Figs. 20 and 21. The overall linear trend for the two mean curves in the bottom panels is -0.0002 px/trial.

violations of Regular Minimality by Ennis's model one has to choose stimuli with 0.1 rather than 0.5 steps, e.g., as shown below

Stimuli	1.9	2.0	2.1	2.2	2.3
1.9	0.431|	<u>0.441</u>	0.456	0.477	0.501
2.0	0.441|	0.444	<u>0.453</u>	0.468	0.487
2.1	0.456	0.453|	0.456	<u>0.465</u>	0.479
2.2	0.477	0.468	0.465|	0.468	<u>0.476</u>
2.3	0.501	0.487	0.479	0.476|	0.479

Horizontal and vertical lines indicate column and row minima, respectively: clearly, the two sets of minima are different. One does not have to resort to a simulation to calculate that with a large number of replications per cell the probability with which all estimated minima would fall on the main diagonal in this matrix is close to zero.

Note that the issue here is in the *density* of experimental points rather than in their *range*: nothing except for typographic considerations prevents me from extending the matrix above to include the entire range of stimuli used in the first matrix (say, from 1.9 to 4.1). Ennis is wrong therefore when he says "The conditions required to see NCSS [Nonconstant Self-Similarity] are incompatible with those required to detect a violation of RM [Regular Minimality] in the cases discussed."

Comment 13: The "semi-ping-pong" procedure of the control Experiment C is presented in Fig. 26, the results are presented in Fig. 27. If Regular Minimality holds, it can be shown that the nth x-balance point and the nth y-balance point in this experiment, in a first-order approximation, are given by

$$x_n \approx x_0 + \frac{1}{h'(x_0)}\varepsilon_n^{(2)} + \varepsilon_n^{(1)},$$

$$y_n \approx h(x_0) + \varepsilon_n^{(2)},$$

where x_0 is the initial value and $\varepsilon_n^{(1)}, \varepsilon_n^{(2)}$ are the adjustment errors.

In the main Experiment A the corresponding first-order approximations are

$$x_n \approx x_0 + \frac{1}{h'(x_0)}\sum_{k=1}^{n}\varepsilon_k^{(2)} + \sum_{k=1}^{n}\varepsilon_k^{(1)},$$

$$y_n \approx h(x_0) + h'(x_0)\sum_{k=1}^{n-1}\varepsilon_k^{(1)} + \sum_{k=1}^{n}\varepsilon_k^{(2)}.$$

Assuming that $h(x)$ can be closely approximated by $x + const$ one can reconstruct the variance–covariance structure of the adjustment errors. Without getting into details, the observed time series in Experiments A and C seem to be reasonably well accounted for by $Var[\varepsilon_k^{(1)}] \approx Var[\varepsilon_k^{(2)}] \approx 0.4$ and $Corr[\varepsilon_k^{(1)}, \varepsilon_k^{(2)}] \approx -0.5$, with all other correlations close to zero. If the adjustment errors are normally distributed these parameters characterize the time series of balance points completely.

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