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Preface: Mathematical Psychology in a Quest for Conceptual Clarity

In 1845 Edgar Allan Poe published a story titled "The Purloined Letter," in which a protagonist, Mr. C. Auguste Dupin, says the following:

> The mathematics are the science of form and quantity; mathematical reasoning is merely logic applied to observation upon form and quantity. The great error lies in supposing that even the truths of what is called pure algebra, are abstract or general truths. And this error is so egregious that I am confounded at the universality with which it has been received. Mathematical axioms are not axioms of general truth. What is true of relation — of form and quantity — is often grossly false in regard to morals, for example. In this latter science it is very usually untrue that the aggregated parts are equal to the whole. [...] two motives, each of a given value, have not, necessarily, a value when united, equal to the sum of their values apart. There are numerous other mathematical truths which are only truths within the limits of relation. But the mathematician argues, from his finite truths, through habit, as if they were of an absolutely general applicability — as the world indeed imagines them to be.

A safe reaction to this excerpt (especially in view of Mr. Dupin's subsequent remarks, omitted here) is that Mr. Dupin has a hopelessly approximate notion of mathematics. However, his appellation to morals and motives provides an opportunity for a more generous reaction, making Mr. Dupin's tirade relevant to a discussion of mathematical psychology. One could interpret this tirade as stating that

D1 given two motives or moral ideas A and B that are combined in some well-defined sense (e.g., co-occur chronologically),

D2 and assuming that each of them can be assigned a value represented by a real number, a and b, D3 and assuming that the combination of A and B can also be assigned a value c that is a real number,

D4 and assuming that the combination of A and B is represented by the sum of their individual values, $a + b$,

D5 we observe empirically that the value c is not generally equal to $a+b$;

D6 the contradiction between D4 and D5 shows that the laws of arithmetic do not apply to motives and moral ideas.

Of course, the assumptions D1-D4 are hidden, they are not explicated by Mr. Dupin. Nor would he stop to think about how he could know the truth of D5. Deny any of the assumptions D1-D5, and Mr. Dupin will lose any grounds to blame mathematics. For instance, if the assumption D4 is not made, then c does not have to be equal to $a + b$, it can instead be ab or $\max(a, b)$, or perhaps a and b alone do not determine c at all. Mathematics is perfectly fine with these possibilities. Mathematics is also fine with the possibility that the assumptions D2 and D3 are wrong, and the motives or moral ideas are not representable by anything that can be subjected to conventional addition. Perhaps a and b are dimensioned numbers, but their dimensionality is not the same (say, they are measured in "love units" and "revenge units," respectively).

Is there any useful lesson that can be derived from this admittedly too easy critique of Mr. Dupin's perorations? We think there is. The lesson is that mathematics in psychology (or chemistry, or wherever else it is applied to) is not about adding, multiplying, or, generally, computing. It is primarily about striving for conceptual clarity and avoiding conceptual confusions. Before we can compute, we need to explicate the hidden assumptions we make, and often when we do this we find out these assumptions are not all that compelling.

Take as an example the following piece of reasoning one can encounter in the modern literature. In logic, the conjunction of two statement is commutative, $A\&B$ is the same as $B\&A$. However, we have empirical evidence that the chronological order in which two statements are presented or evaluated does matter for one's judgement of the truth value (or probability) of their conjunction. Ergo, classical logic (probability theory) are not applicable to human judgments. Let us see what is involved in this reasoning.

L1 Assuming that if A is presented first and B is presented second, then their combination is represented by $A\&B$,

L2 whence, by symmetry, if A is presented second and B is presented first, their combination is represented by $B\&A$;

L3 and knowing from classical logic that $A\&B$ and $B\&A$ are equivalent,

L4 their truth value (or probability) M should be the same, $M(A\&B) = M(B\&A)$.

L5 But empirical observations tell us this is not generally the case.

L6 Ergo, classical logic (classical probability) here does not work.

The reasoning here is definitely "Dupinesque." Far from not being applicable, formal logic, if applied correctly, would lead one to reject, by reductio ad absurdum, the assumptions L1-L2. Indeed L3 and the implication L3→L4 are unassailable, and we assume L5 truthfully describes empirical facts. The ways to constructively deny L1-L2 readily suggest themselves. One way is to introduce a special, non-commutative operation $A\,then\,B$. Another way is to identify the statements not only by their content but also by their chronological position in the pair: a statement with content A, if presented first, is A_1, when presented second it is A_2. So the rejected representations $A\&B$ and $B\&A$ in L1 and L2 are in reality $A_1\&B_2$ and $A_2\&B_1$, respectively. The commutativity of the conjunction is perfectly preserved, e.g., $A_1\&B_2 \equiv B_2\&A_1$. But $A_1\&B_2$ and $A_2\&B_1$ are different propositions, and one should generally expect that

$$M\left(A_1\&B_2\right) \neq M\left(A_2\&B_1\right),$$

whatever M may be. One can further investigate which of the two solutions, the introduction of $A\,then\,B$ or the positional labeling, is preferable. Thus, if the truth values of the statements A and B themselves, and not just of their conjunction, depends on their chronological position, then the positional labeling clearly wins.

The quest for conceptual clarity and explication of hidden assumptions often faces greater and subtler difficulties than in the examples above. The greater then are the rewards ensuing from resolving these difficulties. Take as an illustration the question of whether the ways we measure certain quantities constrain the way these quantities can be related to each other. The historical context for this question is the emergence in mathematical psychology in the second half of the 20th century of the line of research referred to as representational theory of measurement. It is an unusual theory, in the sense that while it is a firmly established branch or part of mathematical psychology, its aim is to formalize all empirical measurements, across sciences, and even provide necessary conditions for all possible scientific laws and regularities.

One of the tenets of this theory, widely accepted in modern psychology (and in textbooks of elementary statistics), is that all entities we deal with, physical or mental, are measured on specific scales, such as ordinal, interval, or ratio scales. We need not get here into the details of the qualitative, or pre-numerical, symmetries (automorphisms) postulated for the entities being measured. Suffice it to mention that the scale type assigned to these entities is characterized by the class (usually, a parametric group) of interchangeable mathematical representations, i.e., measurement functions, mapping the entities being measured into mathematical objects, usually real numbers. Thus, if entities $\mathbf{x} \in \mathcal{X}$, say, stimulus intensity or sensation magnitude values, are said to be measured on a ratio scale, it means that the measurement functions for $\mathcal{X}$ map this set into intervals of real numbers, and that if f and g are such measurement functions, then, for every $\mathbf{x} \in \mathcal{X}$,

$$f\left(\mathbf{x}\right) = kg\left(\mathbf{x}\right),$$

for some positive constant k. R. Duncan Luce, arguably the greatest mathematical psychologist of the 20th century, made use of the notion of a measurement scale to theoretically restrict the class of possible psychophysical functions, those relating the magnitude of stimulus to the magnitude of sensation it causes. Luce proposed this idea in a book titled "Individual choice behavior" and in a journal paper (Luce 1959). The idea is so attractive aesthetically that it deserves being reproduced here, mutatis mutandis.

Let $x = f(\mathbf{x})$ and $s = \varphi(\mathbf{s})$ represent measurement functions for the stimulus magnitude $\mathbf{x}$ and sensation magnitude $\mathbf{s}$, respectively, and let the psychophysical function relating $\mathbf{s}$ to $\mathbf{x}$, written in terms of these specific measurement functions, be

$$s = \psi(x).$$

Assume that both $\mathbf{x}$ and $\mathbf{s}$ are of the ratio scale type. Consider another admissible measurement function for $\mathbf{x}$,

$$x' = kf(\mathbf{x}),$$

for some $k > 0$. Then, Luce hypothesized, if one switches from x to x', the psychophysical function should be presentable as

$$s' = \psi(x'),$$

where

$$s' = c\varphi(\mathbf{s}),$$

for some $c > 0$. That is, s' is another admissible measurement function for $\mathbf{s}$. Put differently, the function ψ is invariant with respect to admissible changes of the measurement function for $\mathbf{x}$, provided that the measurement function for the dependent variable $\mathbf{s}$ can also change to other measurement functions accordingly. The last word, "accordingly," means that the choice of the measurement function for $\mathbf{s}$ generally depends on the choice of the measurement function for $\mathbf{x}$, i.e.,

$$c = K(k),$$

for some function K.

The reasoning here is seductively plausible, and Luce thought that examples of the well-established laws of physics confirmed its validity. Thus, Newton's law of gravitation is conventionally written as

$$F = \gamma\frac{m_1 m_2}{r^2}.$$

If we assume that everything in the right-hand side is fixed except for the distance measurement function r, then augmenting this measurement function by the factor of $k = 10$ would result in the same expression, except that the measurement function F for force will have to be multiplied by $c = k^{-2} = 1/100$.

Having accepted Luce's hypothesis (Luce called it a "principle of theory construction"), we are led to a surprising conclusion: the psychophysical function cannot be anything but a power function. What is surprising here is that this conclusion is based on no empirical evidence, it is obtained deductively, by merely assuming that he magnitudes of stimulus and sensation are of the ratio-scale type. Indeed, the reasoning above translates into

$$\psi(kx) = \psi(x') = s' = cs = c\psi(x) = K(k)\psi(x),$$

whence, by eliminating all but the marginal terms, we get the functional equation

$$\psi(kx) = K(k)\psi(x).$$

Here, the values of x and k are positive, and the functions ψ and K are positive and increasing. Since the functional equation holds for all positive k and all x on some interval of positive reals, its only solution is known to be (Aczél, 1987)

$$\psi(x) = bx^{\beta}, K(k) = k^{\beta},$$

for some positive b and β.

It looks like we have here an immaculate piece of deductive reasoning, with all concepts rigorously defined and all assumptions explicated. However, what shall we do with the fact that psychophysical laws of other forms have been proposed too? Most notably, every psychologist knows of the logarithmic law proposed by Gustav Theodor Fechner in 1861,

$$s = s_0 \log \frac{x}{x_0}.$$

Here, x_0 is the numerical representation of the absolute threshold magnitude $\mathbf{x_0}$, one at which the numerical representation of $\mathbf{s}$ is zero, for all measurement functions.

We can see that Fechner's law does not violate any of Luce's assumptions. Since $\mathbf{x}$ and $\mathbf{x_0}$ are measured by the same measurement function, the value of

$$\frac{f(\mathbf{x})}{f(\mathbf{x_0})} = \frac{kx}{kx_0}$$

is the same for all admissible f. The magnitude of the absolute lower threshold is defined irrespective of the measurement function chosen for $\mathbf{x}$, because so is defined $s = 0$. Even if one denies the existence of absolute threshold as a fixed constant, such operational definitions of $\mathbf{x_0}$ as "the value of $\mathbf{x}$ detected with probability p" are independent of the measurement function for $\mathbf{x}$. The measurement function for the dependent variable $\mathbf{s}$ is chosen independently, which formally translates into $K(k) = 1$. The value s_0 is the numerical representation of the value of $\mathbf{s}$ corresponding to the value of $\mathbf{x}$ at which $\log \frac{x}{x_0} = 1$.

Since the logarithm law is not the same as the power law, Luce must have made a hidden assumption that Fechner's derivation of his law violates. This hidden assumption is not difficult to detect. It is the assumption that the dependence of $\mathbf{s}$ on $\mathbf{x} \in \mathcal{X}$ contains no parameters (constants with respect to $\mathbf{x}$) that belong to the same set $\mathcal{X}$ and are therefore represented by the same measurement function. Such parameters are called measurement-dependent constants, or dimensional constants in the case of ratio scales. An expression

$$s = s_0 \psi \left(\frac{x}{x_0} \right),$$

with dimensional constants x_0 and s_0, can hold for any positive increasing function ψ. Using examples of physical laws, this was pointed out to Duncan Luce by William W. Rozeboom in a 1962 article. Being a true scientist, Luce accepted this criticism and withdrew his "principle of theory construction" (Luce, 1962). Interestingly, in the formulation of this principle, Luce did in fact mention dimensional constants: the form of the dependence ψ should be invariant, he wrote, "except for the numerical values of parameters that reflect the effect on the dependent variables of admissible transformations of the independent variables." This is precisely what dimensional constants are. Using Luce's own example of the universal gravitation law, in the formula

$$F = \gamma \frac{m_1 m_2}{r^2},$$

if one uses the distance-time-mass-force system of units, changing the dimensionality of mass or distance in no way leads to the change of the dimensionality of force. Rather the dimensional constant γ, whose dimensionality is

$$force \cdot distance^2 \cdot mass^{-2},$$

changes its numerical value. In essence, γ is a coalesced form (using the expression coined by Percy Williams Bridgman) of the "individual" dimensional constants in the formula

$$\frac{F}{F_0} = \frac{\frac{m_1}{m_0} \frac{m_2}{m_0}}{\left(\frac{r}{r_0} \right)^2}.$$

The lesson we learn from the story of Duncan Luce's "principle of theory construction" is that hidden assumptions and lack of conceptual clarity due to the failure to explicate them can be present even in very rigorous treatments. Moreover, explication of these hidden assumptions, while resolving the issue at hand, leads to new conceptual problems and opens new venues of conceptual research. In our example the new conceptual problems can be formulated thus:

P1 What is the nature of dimensional (more generally, measurement-dependent) constants in empirical laws? Where do they come from?

P2 How do we know the scale type (the group of admissible measurement functions) of a given entity? Is it imposed on the entity by the human mind, or is it objectively present in it, to be uncovered?

These questions are at the foundations of all empirical science, and it is an interesting historical fact that their development owes a great deal to mathematical psychology (see, e.g., Dzhafarov, 1995; Falmagne, Narens & Dobb, 2018; Narens, 2007). This preface, of course, is not a place to discuss these questions in any detail.

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