Journal of Mathematical Psychology **45**, 670–719 (2001)
doi:10.1006/jmps.2000.1341, available online at http://www.idealibrary.com on IDEAL®

Multidimensional Fechnerian Scaling: Basics

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Fechnerian scaling is a theory of how a certain (Fechnerian) metric can be computed in a continuous stimulus space of arbitrary dimensionality from the shapes of psychometric (discrimination probability) functions taken in small vicinities of stimuli at which these functions reach their minima. This theory is rigorously derived in this paper from three assumptions about psychometric functions: (1) that they are continuous and have single minima around which they increase in all directions; (2) that any two stimulus differences from these minimum points that correspond to equal rises in discrimination probabilities are comeasurable in the small (i.e., asymptotically proportional), with a continuous coefficient of proportionality; and (3) that oppositely directed stimulus differences from a minimum point that correspond to equal rises in discrimination probabilities are equal in the small. A Fechnerian metric derived from these assumptions is an internal (or generalized Finsler) metric whose indicatrices are asymptotically similar to the horizontal cross-sections of the psychometric functions made just above their minima.

1. INTRODUCTION

1.0. *Outlines.* Intuitively, Fechnerian scaling is a method for computing distances among stimuli from the probabilities with which each of these stimuli can be discriminated from its very close neighbors. Dzhafarov and Colonius (1999a) proposed a comprehensive theory that applies this *metric-from-discriminability* idea

This article was handled by Guest Editor Thomas S. Wallsten.

This research was supported by NSF Grant SES-0001925 to Purdue University. The authors are indebted to Thomas Wallsten and two anonymous reviewers for valuable suggestions and to Damir Dzhafarov and Radomil Dzhafarov for their help in preparing the manuscript.

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0022-2496/01 $35.00

to continuous stimulus spaces of arbitrary dimensionality (such as the CIE space of colors, the amplitude–frequency space of tones, and a space of parametrized geometric shapes). What motivates this theory is the vague belief that, the discrimination among stimuli being arguably the most basic cognitive function and the probability of discrimination being a universal measure of discriminability, distances computed from discrimination probabilities should have a fundamental status among behavioral measurements. In other words, the expectation is that, although the theory of Fechnerian scaling in its present form makes no *predictions* of the kind, many different behavioral measures, such as response times, direct estimates of stimulus dissimilarities, and the discrimination probabilities themselves, in a final analysis could be expressed as functions of Fechnerian distances among the stimuli involved.

In the present work we further develop the theory of Fechnerian scaling by elaborating its mathematical foundations and establishing operational meanings for its principal concepts and assumptions (by which we mean their linkage to observables and empirical procedures). More specifically, the development presented in this paper is as follows.

We place the notion of a Fechnerian metric in the context of the general geometry of *internal metrics*, which means that the Fechnerian distance between two stimuli in a stimulus space is defined as the infimum of the *psychometric lengths* of all well-behaved curves connecting the two stimuli within the space. The psychometric length, in turn, is defined through the notion of an *indicatrix* attached to a stimulus, a geometric device that allows one to measure the magnitude of any vector of change that originates at the stimulus. We establish the empirical meaning of Fechnerian indicatrices in terms of the shapes of the discrimination probability (*psychometric*) functions defined on a stimulus space. Essentially, horizontal cross-sections of the psychometric functions, made at a fixed small elevation with respect to their minima, are geometrically similar to the indicatrices attached to the stimuli at which the minima are achieved.

A different aspect of the shape of a psychometric function is related to the *global psychometric transformation*, another central concept in the theory. A transition from a stimulus to one of its "immediate" neighbors corresponds to an infinitesimal rise in the psychometric function whose minimum coincides with the original stimulus. The global psychometric transformation makes this rise *comeasurable in the small* with the suitably defined magnitude of physical transition (see the Appendix, Comment 1). We establish the operational meaning for the fundamental assumption of Fechnerian scaling, that the global psychometric transformation is, as the term indicates, global: it is one and the same for all stimuli and for all directions of stimulus change. Essentially, this assumption means that vertical cross-sections of the psychometric functions, made through their minima in various directions and considered between the minima and the horizontal cross-sections mentioned earlier, are scaled (in the horizontal dimension) replicas of each other.

All properties of the Fechnerian metrics that we consider in this paper are derived from three clearly stipulated assumptions about the shapes of the psychometric functions, when considered in very small vicinities of their minima. This is worth emphasizing: in spite of the paper's abstract mathematical style, the properties of the mathematical notions involved are not postulated, but derived from certain properties of observable entities. These properties are not guaranteed to be true, and they can be, in principle, experimentally falsified if they are de facto wrong. At the same time, the philosophy of our approach dictates that the empirical assumptions put in the foundation of Fechnerian scaling be made as weak as possible. Other assumptions can always be added to the "minimalist" list adopted in this paper, but only if warranted by empirical evidence, or if they offer an interesting theoretical development on top of the basic theory.

1.1. *Terminological notes.* The adjectives "Fechnerian" and "Fechner" attached in this paper to mathematical and psychophysical concepts are due to the suggestion made in Dzhafarov and Colonius (1999a) that the metric-from-discriminability idea constitutes the essence of Gustav Theodor Fechner's original theory (Fechner, 1851, 1860, 1877, 1887): in a unidimensional stimulus continuum, the "subjective" distance between a and b is computed as

$$G(a, b) = \int_a^b \delta(x)\, dx,$$

where $\delta(x)$ is a measure of local discriminability (that Fechner approximated by the reciprocal of a "differential threshold"). This approach can be shown (see Dzhafarov & Colonius, 1999a, for details) to be a unidimensional specialization of our definition of a Fechnerian distance, provided the discriminability measure $\delta(x)$ is computed from the probabilities with which stimulus x is discriminated from stimuli $x \pm \Delta x$, $\Delta x \to 0+$.

Geometrically, the Fechnerian metrics are identified in Dzhafarov and Colonius (1999a) as *Finsler metrics* (after Paul Finsler who proposed them in 1918; see Busemann, 1950, and Rund, 1959, for history). Because of this, one of the basic concepts of the Dzhafarov–Colonius theory is termed the "Fechner–Finsler metric function." This term is retained here for the sake of continuity, but the adjective "Finsler" in this paper refers to the *generalized* Finsler metrics, a term that we take to be synonymous with *internal metrics*. Finsler metrics *in the narrow sense* are induced by indicatrices whose shapes satisfy a strong form of convexity, which in the present context means a strong restriction imposed on the shapes of psychometric functions. This restriction may very well hold empirically, but one has no reason for postulating it in the basic theory. In abstract mathematics, by relaxing this convexity requirement in different ways one obtains various forms and levels of generalization for Finsler metrics. The level adopted in this paper (internal metrics) is achieved if one imposes no constraints on the shapes of the indicatrices at all. In the mathematical literature the terms "Finsler metrics" and "generalized Finsler metrics" do not seem to have rigidly established boundaries (compare, e.g., Asanov, 1985; Aleksandrov & Berestovskii, 1995; Busemann, 1942, 1955).

1.2. *Mathematical language of the paper.* This paper only deals with most basic aspects of the theory proposed in Dzhafarov and Colonius (1999a), but it does so in a significantly more rigorous and thorough way. A familiarity with that paper may be helpful but is not assumed.

We follow the notation and conventions adopted in Dzhafarov and Colonius (1999a): boldface letters, **x**, **u**, etc., denote vectors with components $(x^1, ..., x^n)$, $(u^1, ..., u^n)$, etc., the superscript (contravariant) and subscript (covariant) notation being used in accordance with their traditional use in differential geometry, but with no involvement of tensor algebra. A familiarity with differential geometry or calculus of variations may be helpful, but no knowledge is assumed beyond the level of standard calculus of several variables and elementary topology.

Although the paper presents a new psychophysical theory, the abstract mathematical results it contains are not entirely new from a mathematician's point of view. The theorems applicable to abstract internal metrics (rather than Fechnerian metrics specifically, related to psychometric functions) can be found in or derived without much ingenuity from the existing mathematical literature. However, the general approach, precise networking of the concepts involved and the order in which the propositions are derived, as well as the derivations themselves, significantly deviate from the literature known to us. This is one reason why we present or outline all the proofs, instead of undertaking or leaving to the reader the labor of negotiating all the differences in premises, logical order, and notation that one would encounter in trying to justify the propositions of this paper by referring to the mathematical literature. Another reason is that we want this paper to serve as a self-contained introduction to Fechnerian scaling (which, as a byproduct, makes it also a self-contained, if nonstandard, introduction to the general geometry of internal metrics).

2. PSYCHOMETRIC FUNCTIONS: BASIC ASSUMPTIONS

2.0. *Outlines.* All computations in our theory of Fechnerian scaling are based on the shapes of psychometric functions within arbitrarily small areas around the points where the functions reach their minima. A psychometric function shows the probabilities with which each (comparison) stimulus in a stimulus space is discriminated from a fixed (reference) stimulus. In this paper we do not discuss various empirical procedures by which the psychometric functions can be obtained. Nor do we discuss hypothetical psychological mechanisms underlying the decision making in any such procedure. On the present level of abstraction, the psychometric functions are taken as observable primitives of Fechnerian scaling that are assumed to satisfy certain assumptions. There are three of them.

The first assumption is that a psychometric function is a continuous function of comparison stimulus, varies continuously as a function of reference stimulus, and, in addition, reaches a single minimum at some point in the stimulus space.

The second assumption (considered to be the most fundamental assumption of Fechnerian scaling) is based on the fact that, given a psychometric function, a transition from its point of minimum to a neighboring stimulus corresponds to a rise in the value of the psychometric function. The assumption is that the (suitably defined) transition magnitudes corresponding to one and the same rise are comeasurable in the small, that is, asymptotically proportional, across all psychometric functions and directions of transition.

The third assumption is that any two oppositely directed transitions from the minimum point of a psychometric function that cause the same rise in its value are asymptotically equivalent. This assumption, rather secondary in its importance, serves to ensure that the Fechnerian distance from **a** to **b** is the same as that from **b** to **a**.

2.1. *Stimulus space and allowable paths.* Although the theory can easily be constructed by viewing a stimulus space as a general smooth manifold, we can

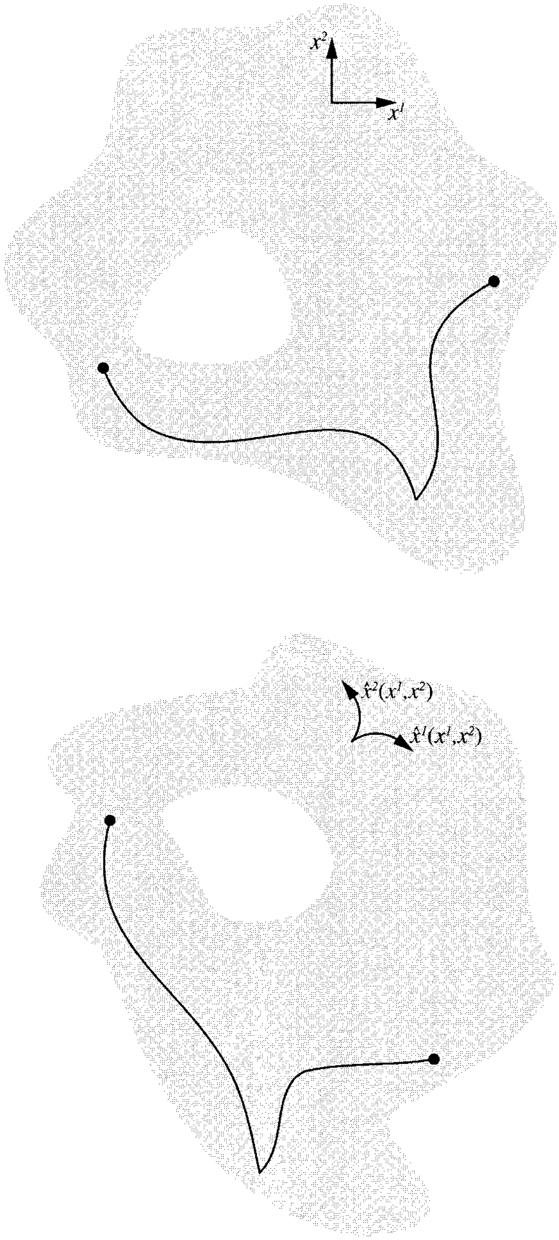

FIG. 1. A stimulus space and its diffeomorphic transformation, with the trajectory of an allowable path connecting two points.

think of no situation when the following, more specialized, definition would not be sufficient.

A *space of stimuli*, denoted by $\mathfrak{M}^{(n)}$ (Fig. 1), is an open pathwise-connected region of Re^n endowed with conventional topology (say, induced by the Euclidean or supremal metric; see the Appendix, Comment 2). The points of $\mathfrak{M}^{(n)}$ are vectors $\mathbf{x} = (x^1, ..., x^n)$ with the coordinates representing physical dimensions of the stimuli. Precisely how the stimulus dimensions are chosen is immaterial: any diffeomorphic transformation of a stimulus space $\mathfrak{M}^{(n)}$ (see the Appendix, Comment 3) is considered an equivalent reparametrization of $\mathfrak{M}^{(n)}$. The Fechnerian metric to be constructed, therefore, must be invariant with respect to all diffeomorphic transformations. As shown below, this invariance is achieved "automatically," because the values of the psychometric functions defined on the stimulus space remain invariant.

(The invariance under reparametrizations of the stimulus space also implies that our version of Fechnerian scaling has no room for any general or privileged form of a "psychophysical law" relating Fechnerian distances to stimulus coordinates. In this respect our theory radically departs from Fechner's original approach.)

The pathwise-connectedness of $\mathfrak{M}^{(n)}$ means that any two points in this stimulus space can be connected by an *allowable* (*oriented*) *path* lying entirely in $\mathfrak{M}^{(n)}$. An allowable path $\mathbf{x}(t)_a^b$ connecting stimulus $\mathbf{a} = \mathbf{x}(a)$ to $\mathbf{b} = \mathbf{x}(b)$ (where $a < b$ are some real numbers) is a continuous function $\mathbf{x}\colon [a, b] \to \mathfrak{M}^{(n)}$ whose tangent $\dot{\mathbf{x}}(t)$ is a continuous nonvanishing function on each interval $[t_{i-1}, t_i]$ of some finite partition $a = t_0 < t_1 < \cdots < t_n = b$ $(n = 1, 2, ...)$.

A path $\mathbf{z}(\tau)_\alpha^\beta$ such that $\mathbf{z}[\tau(t)] = \mathbf{x}(t)$, where $\tau(t)$ is a diffeomorphism $[a, b] \to [\alpha, \beta]$ with $\dot{\tau}(t) > 0$ (positive diffeomorphism), is considered an equivalent reparametrization of $\mathbf{x}(t)_a^b$ (Fig. 2). All path-related constructs, such as its psychometric length, defined below, must therefore be invariant under positive diffeomorphic transformations.

2.2. *Tangent spaces and line elements.* In addition to the stimuli themselves, the theory also makes prominent use of the transitions (conceptual rather than physical) from a stimulus $\mathbf{x}$ to a stimulus $\mathbf{x} + \mathbf{u}s$, $\mathbf{u} \neq \mathbf{0}$, or, put differently, from $\mathbf{x}$ in a *direction* $\mathbf{u} = (u^1, ..., u^n)$ by a (small) amount s (see the Appendix, Comment 4). As stated in the previous subsection, any diffeomorphic transformation of a stimulus space $\mathfrak{M}^{(n)}$ is considered equivalent to $\mathfrak{M}^{(n)}$. It is necessary, therefore, to know how to determine the direction of transition from $\mathbf{x}$ to $\mathbf{x} + \mathbf{u}s$ as the two stimuli undergo a diffeomorphic transformation.

If $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{x})$ is such a transformation, then (see the Appendix, Comment 5)

$$\hat{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) = \hat{\mathbf{x}}(\mathbf{x}) + \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \mathbf{u}s + o\{s\}. \tag{1}$$

where

$$\frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} = \left\{ \frac{\partial \hat{x}^i}{\partial x^j} \right\}_{i, j = 1, ..., n}$$

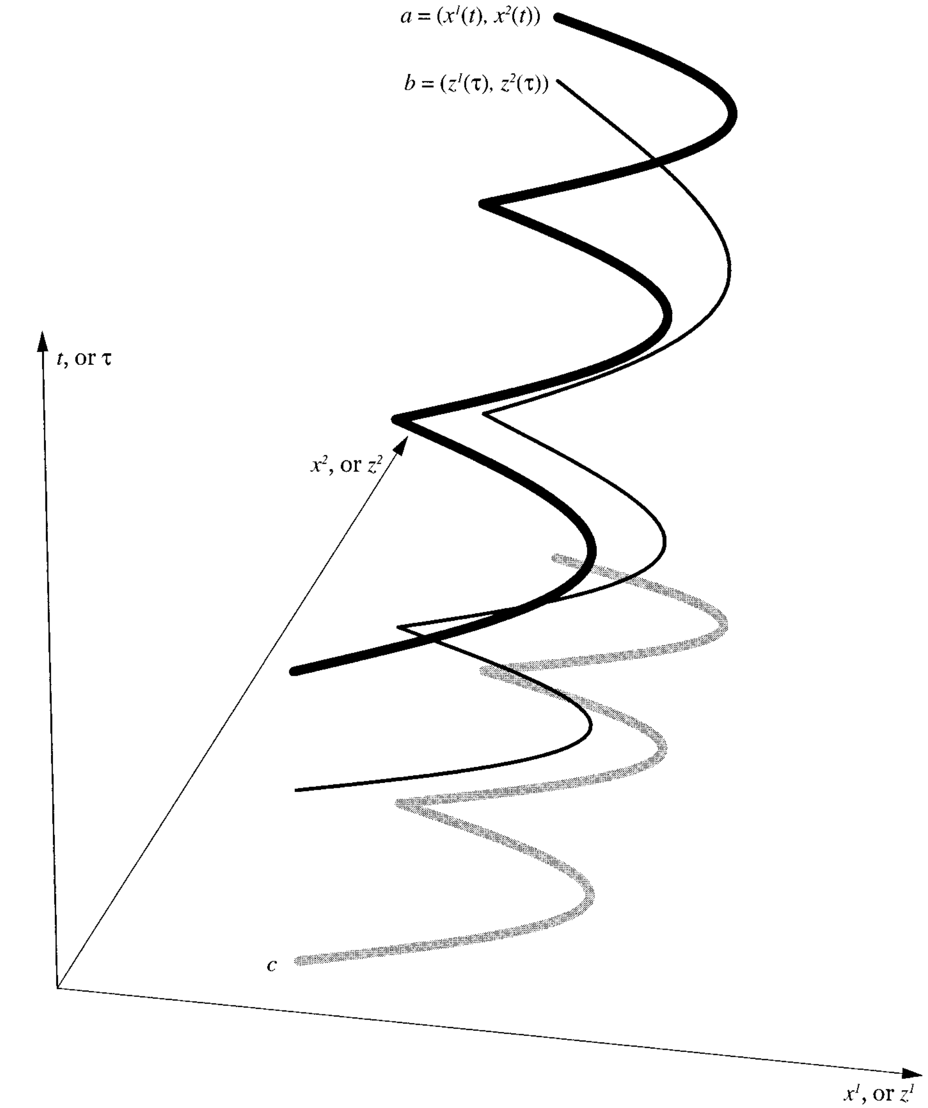

FIG. 2. An allowable path a and its diffeomorphic reparametrization b in a two-dimensional space; c is the trajectory of the path.

is the Jacobian matrix of $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{x})$ and $\mathbf{u}$ is treated as a column vector. Denote

$$\hat{\mathbf{u}} = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \times \mathbf{u}, \tag{2}$$

or, componentwise,

$$\hat{u}^i = \sum_{j=1}^{n} \frac{\partial \hat{x}^i(\mathbf{x})}{\partial x^j} u^j, \qquad i = 1, ..., n.$$

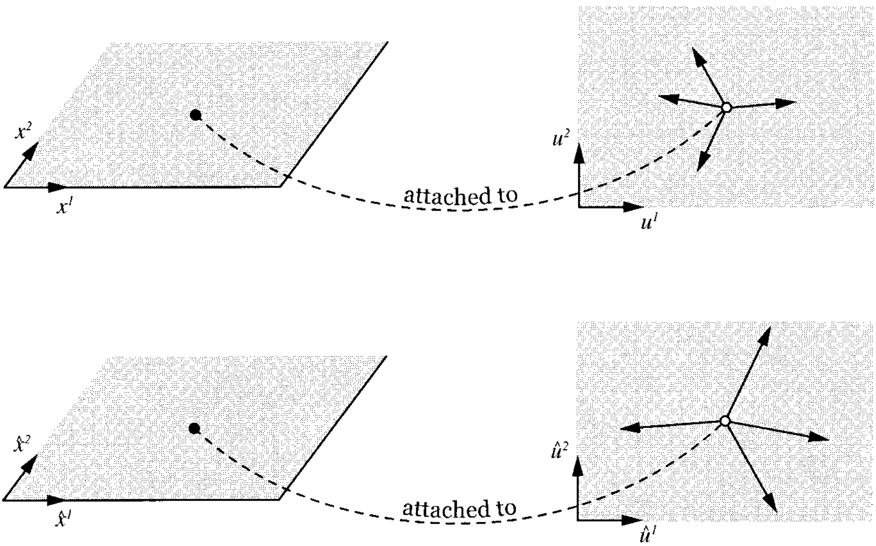

FIG. 3. (Top) A point in a stimulus space, and the tangent space attached to it. (Bottom) The same under a diffeomorphic transformation of the stimulus space.

If, under any diffeomorphic transformation $\mathbf{x} \to \hat{\mathbf{x}}$, a direction vector $\mathbf{u} \in \mathrm{Re}^n - \{\mathbf{0}\}$ transforms into $\hat{\mathbf{u}}$ according to (2), the vector $\mathbf{u}$ is called a *contravariant* vector attached to $\mathbf{x} \in \mathfrak{M}^{(n)}$. Note that, since the Jacobian matrix $\partial\hat{\mathbf{x}}/\partial\mathbf{x}$ in (2) is nondegenerate, as $\mathbf{u}$ sweeps the entire $\mathrm{Re}^n - \{\mathbf{0}\}$, so does $\hat{\mathbf{u}}$.

The space $\mathfrak{C}_{\mathbf{x}}^{(n)}$ that consists of all (nonzero) contravariant vectors attached to $\mathbf{x}$ is called the *space of directions* (endowed with conventional topology) or the *tangent space* attached to $\mathbf{x}$ (Fig. 3). The term "direction $\mathbf{u}$," therefore, always implies $\mathbf{u} \neq \mathbf{0}$ and the contravariant transformation law, but otherwise it can be any vector having the same dimensionality as $\mathfrak{M}^{(n)}$. Any stimulus-direction pair $(\mathbf{x}, \mathbf{u})$ forms a *line element*, "from point $\mathbf{x}$ in direction $\mathbf{u}$," that can be thought of as a descriptor for the transition from $\mathbf{x}$ to $\mathbf{x} + \mathbf{u}s$, as $s \to 0+$.

2.3. *Psychometric functions*: *First Assumption.* Each stimulus $\mathbf{x}$ of a stimulus space is associated with a *psychometric function*

$$\psi_{\mathbf{x}}(\mathbf{y}) = \mathrm{Prob}[\mathbf{y} \text{ is discriminated from } \mathbf{x}], \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{M}^{(n)}. \tag{3}$$

Note that the *reference stimulus* $\mathbf{x}$ is treated as the parameter (index) of a psychometric function, while the *comparison stimulus* $\mathbf{y}$ is its argument. Occasionally, however, by abuse of language, $\psi_{\mathbf{x}}(\mathbf{y})$ is taken to denote the entire indexed set of the psychometric functions,

$$\{\psi_{\mathbf{x}}(\mathbf{y})\}_{x \in \mathfrak{M}^{(n)}},$$

viewed as a single function of both $\mathbf{x}$ and $\mathbf{y}$.

The First Assumption about psychometric functions is that $\psi_{\mathbf{x}}(\mathbf{y})$ is continuous in $(\mathbf{x}, \mathbf{y})$, and that, for any given $\mathbf{x}$, it attains its single minimum at some diffeomorphically related to $\mathbf{x}$ point $\mathbf{y} = \mathbf{h}(\mathbf{x}) \in \mathfrak{M}^{(n)}$, in a vicinity of which $\psi_{\mathbf{x}}(\mathbf{y})$ increases in all

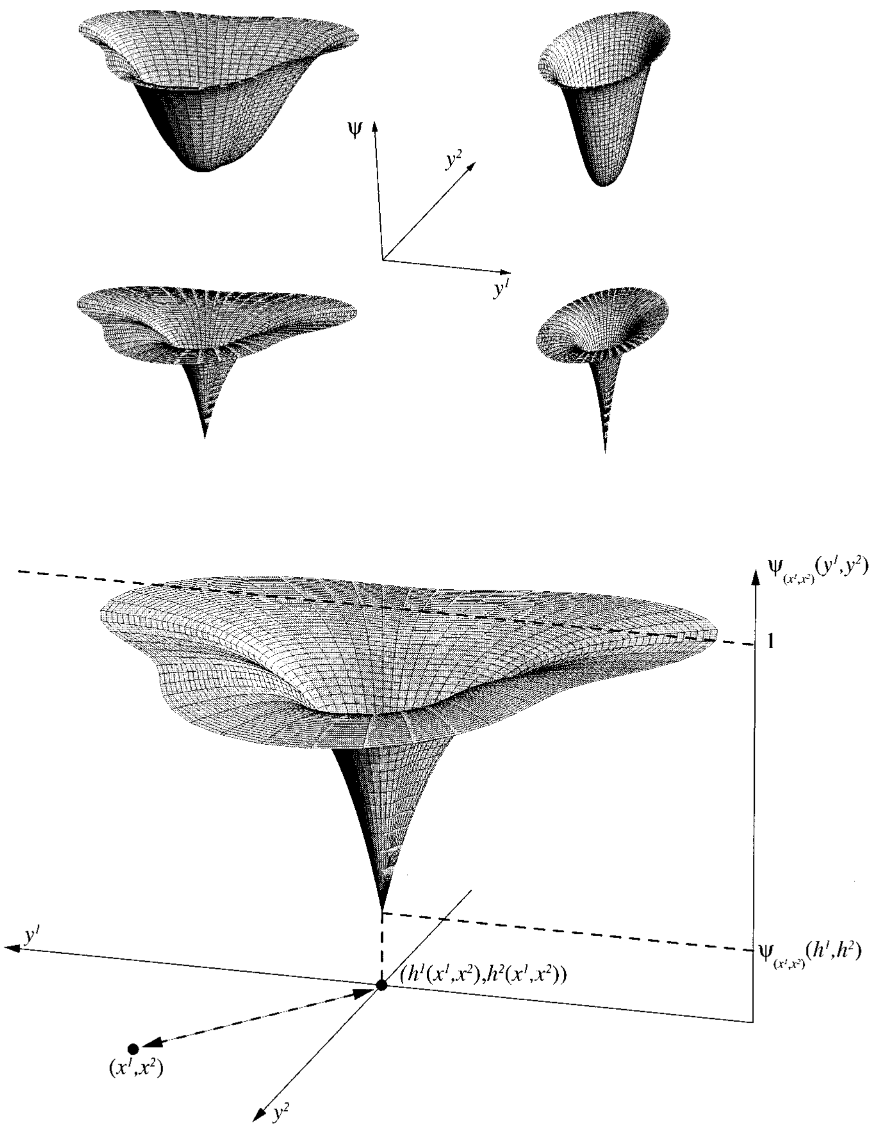

FIG. 4. Top: possible appearances of psychometric functions. Bottom: one of these functions, in detail.

directions (Fig. 4). In other words, one can find a neighborhood of $\mathbf{h}(\mathbf{x})$ within which, for any $\mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$,

$$\psi_{\mathbf{x}}[\mathbf{h}(\mathbf{x}) + \mathbf{u}s] - \psi_{\mathbf{x}}[\mathbf{h}(\mathbf{x})]$$

increases in $s > 0$; and $\psi_{\mathbf{x}}(\mathbf{y})$ has no other minima. (It is not assumed here that the minimum level of a psychometric function, $\psi_{\mathbf{x}}[\mathbf{h}(\mathbf{x})]$, must be the same for different reference stimuli, $\mathbf{x}$.)

The difference $\mathbf{h}(\mathbf{x}) - \mathbf{x}$ is traditionally (in a unidimensional case) referred to as the *constant error* of discrimination, whereas $\mathbf{h}(\mathbf{x})$ is considered the "*point of subjective equality*" for the reference stimulus $\mathbf{x}$.

One can always redefine the psychometric functions so that they are indexed by their points of minimum (i.e., the "points of subjective equality") rather than by their reference stimuli;

$$\hat{\psi}_{\mathbf{x}}(\mathbf{y}) = \text{Prob}[\mathbf{y} \text{ is discriminated from } \mathbf{h}^{-1}(\mathbf{x})], \qquad \mathbf{x}, \mathbf{y} \in \widehat{\mathfrak{M}}^{(n)} = \mathbf{h}(\mathfrak{M}^{(n)}). \quad (4)$$

It is easy to verify that the restricted stimulus space $\widehat{\mathfrak{M}}^{(n)}$ has all the properties of the original space $\mathfrak{M}^{(n)}$ (open pathwise-connected region of Re^n, with the same definition of allowable paths), and that $\hat{\psi}_{\mathbf{x}}(\mathbf{y})$ is continuous in $(\mathbf{x}, \mathbf{y})$ and attains its minimum at $\mathbf{y} = \mathbf{x} \in \widehat{\mathfrak{M}}^{(n)}$. Representation (4) is merely a reindexation of (3), except that the comparison stimuli $\mathbf{y}$ are now only considered within the restricted space $\widehat{\mathfrak{M}}^{(n)} \subseteq \mathfrak{M}^{(n)}$. This shrinkage of the domain, however, is immaterial, because Fechnerian scaling is only based on the behavior of psychometric functions in arbitrarily small neighborhoods of their minima. Besides, as follows from the procedure to be described later, all allowable paths of $\mathfrak{M}^{(n)}$ that can be of use in Fechnerian scaling would have to lie within $\widehat{\mathfrak{M}}^{(n)}$ anyway (because any point of such a path must be a minimum point of some psychometric function).

With no loss of generality, therefore, we assume for the rest of this paper that psychometric functions are (re)defined as in (4), so that $\psi_{\mathbf{x}}(\mathbf{y})$ and $\mathfrak{M}^{(n)}$ always stand for $\hat{\psi}_{\mathbf{x}}(\mathbf{y})$ and $\widehat{\mathfrak{M}}^{(n)}$, respectively.

Our First Assumption about psychometric functions can now be formulated simply: $\psi_{\mathbf{x}}(\mathbf{y})$ is continuous in $(\mathbf{x}, \mathbf{y})$, and, for any given $\mathbf{x}$, it attains its single minimum at $\mathbf{y} = \mathbf{x}$, in a vicinity of which it increases in all directions.

It may often be desirable, and innocuous from an empirical point view, to impose additional smoothness constraints on psychometric functions (this is not needed in this paper). One such constraint is that the rise in the value of a psychometric function from its minimum,

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}), \qquad \mathbf{u} \neq 0$$

(a prominent quantity in Fechnerian scaling), is infinitely differentiable in $s > 0$. Occasionally one may wish to strengthen this assumption further by requiring that $\psi_{\mathbf{x}}(\mathbf{y})$ be infinitely Fréchet-differentiable at *almost* all values of $\mathbf{x}$ and $\mathbf{y}$ (see the Appendix, Comment 6). One should be careful, however, not to omit, unless warranted by empirical evidence, the qualifier "almost." The point of minimum, for example, should always be considered a potential singularity (as in Fig. 4, bottom): the differentiability at this point is a very stringent constraint that may very well be empirically false. Also, some additional assumptions (e.g., probability summation models, not discussed in this paper) may lead to psychometric functions with sharp edges emanating from their minima (Fig. 5).

2.4. *Psychometric functions*: *Second* (*fundamental*) *Assumption.* To formulate the next assumption, we first introduce a concept that plays a prominent role in all computations involved in Fechnerian scaling. The *psychometric differential* at a stimulus $\mathbf{x} \in \mathfrak{M}^{(n)}$ in a direction $\mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$ is defined as

$$h = \psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}), \qquad s \to 0+. \quad (5)$$

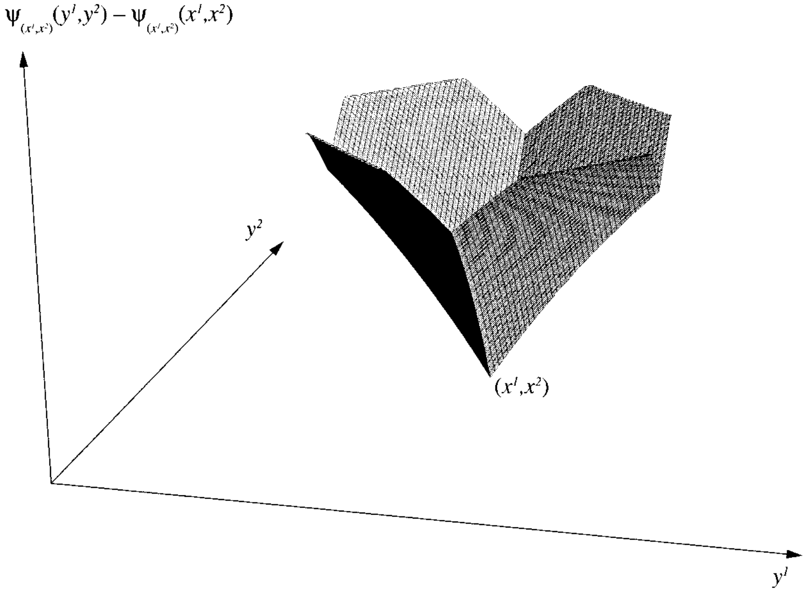

FIG. 5. A psychometric function, in the vicinity of its minimum, derived from a certain probability summation model.

This is the ray-differential (i.e., one-sided directional differential) taken at the minimum of the psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$ in the direction $\mathbf{u}$ (Fig. 6).

Plainly, the psychometric differential vanishes at $s=0$ and (due to the First Assumption) continuously increases as a function of $s>0$, at least within an interval of sufficiently small values of s. We denote this function by $\Phi^{-1}_{\mathbf{x},\mathbf{u}}(s)$, so that one has the identity

$$\Phi_{\mathbf{x},\mathbf{y}}[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]=s,$$

for sufficiently small $s \geqslant 0$. We call

$$s=\Phi_{\mathbf{x},\mathbf{u}}(h), \qquad h \to 0+, \tag{6}$$

the *stimulus differential* at $(\mathbf{x},\mathbf{u})$. Given a set of the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$, the stimulus differentials $\Phi_{\mathbf{x},\mathbf{u}}(h)$ are uniquely defined at all possible line elements $(\mathbf{x},\mathbf{u})$, and they are continuously increasing at small values of $h \geqslant 0$ and vanishing at $h=0$. Observe the symmetry: to compare two psychometric differentials, (5), one has to take them at the same value of the physical differential, (6), and vice versa.

The Second Assumption about psychometric functions is that, for some fixed $(\mathbf{x}_0,\mathbf{u}_0)$ and arbitrary $(\mathbf{x},\mathbf{u})$, the stimulus differentials $\Phi_{\mathbf{x},\mathbf{u}}(h)$ and $\Phi_{\mathbf{x}_0,\mathbf{u}_0}(h)$ are *comeasurable in the small* (i.e., asymptotically proportional),

$$0<\lim_{h\to 0+}\frac{\Phi_{\mathbf{x}_0,\mathbf{u}_0}(h)}{\Phi_{\mathbf{x},\mathbf{u}}(h)}<\infty, \tag{7}$$

and that, moreover, the asymptotic proportionality coefficient (i.e., the value of the limit) is continuous in $(\mathbf{x},\mathbf{u})$.

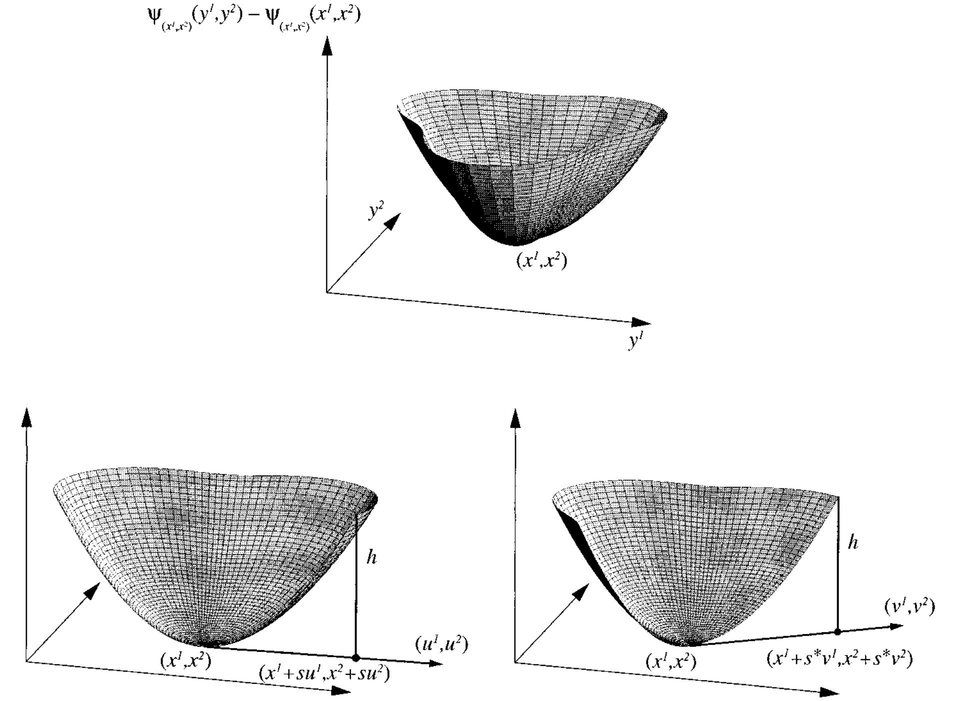

FIG. 6. A psychometric differential in two different directions, and the corresponding stimulus differentials.

The fundamental importance of the Second Assumption for Fechnerian scaling lies in the fact that (7) implies

$$\lim_{h \to 0+} \frac{\Phi_{\mathbf{x}_0, \mathbf{u}_0}(h)}{\Phi_{\mathbf{x}, \mathbf{u}}(h)} = \lim_{s \to 0+} \frac{\Phi_{\mathbf{x}_0, \mathbf{u}_0}[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{\Phi_{\mathbf{x}, \mathbf{u}}[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}$$
$$= \lim_{s \to 0+} \frac{\Phi_{\mathbf{x}_0, \mathbf{u}_0}[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s}.$$

By renaming $\Phi_{\mathbf{x}_0, \mathbf{u}_0}$ into Φ, this equation can be written as

$$F(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s}, \tag{8}$$

where the transformation Φ is *one and the same* for all $\mathbf{x}$ and $\mathbf{u}$, while $F(\mathbf{x}, \mathbf{u})$ is continuous and positive. It is easy to see that (8) determines $F(\mathbf{x}, \mathbf{u})$ and Φ essentially uniquely: to preserve (8) one can only multiply $F(\mathbf{x}, \mathbf{u})$ by some constant $k > 0$, and one can only substitute for Φ an asymptotically equivalent function multiplied by the same k. Indeed, if (8) holds together with

$$F^*(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\Phi^*[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s},$$

then

$$\frac{F^*(\mathbf{x},\mathbf{u})}{F(\mathbf{x},\mathbf{u})}=\lim_{s\to 0+}\frac{\Phi^*[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]}{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]}=\lim_{h\to 0+}\frac{\Phi^*(h)}{\Phi(h)}=k,$$

for some positive k. All these facts are summarized in

THEOREM 2.4.1 (Fundamental Theorem of Fechnerian Scaling). *There exists a transformation $\Phi(h)$, continuously increasing at small values of $h\geqslant 0$ and vanishing at $h=0$, that makes all psychometric differentials $\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})$, as $s\to 0+$, comeasurable in the small with s,*

$$\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]=F(\mathbf{x},\mathbf{u})\,s+o\{s\},\qquad s\to 0+, \tag{9}$$

with $F(\mathbf{x},\mathbf{u})$ being positive and continuous. $F(\mathbf{x},\mathbf{u})$ is determined uniquely and $\Phi(h)$ asymptotically uniquely (as $h\to 0+$), up to the multiplication with one and the same arbitrary constant $k>0$. That is, all allowable substitutions for $F(\mathbf{x},\mathbf{u})$ and $\Phi(h)$ are given by

$$\begin{aligned}&F^*(\mathbf{x},\mathbf{u})=kF(\mathbf{x},\mathbf{u})\\&\Phi^*(h)=k\Phi(h)+o\{\Phi(h)\},\,h\to 0+\end{aligned}. \tag{10}$$

We call Φ a *global psychometric transformation* on a given stimulus space $\mathfrak{M}^{(n)}$ (endowed with a given set of psychometric functions), while $F(\mathbf{x},\mathbf{u})$ is referred to as the (Fechner–Finsler) *metric function* associated with Φ. Due to (10), any function $k\Phi+o\{\Phi\}$ is a global psychometric transformation, too, for which $kF(\mathbf{x},\mathbf{u})$ is the associated metric function.

Next we look at what happens with the global psychometric transformation Φ and the metric function $F(\mathbf{x},\mathbf{u})$ under diffeomorphisms of the stimulus space. The psychometric functions, under such a diffeomorphism $\mathbf{x}\to\hat{\mathbf{x}}$, transform as

$$\hat{\psi}_{\hat{\mathbf{x}}}(\hat{\mathbf{y}})=\psi_{\mathbf{x}}(\mathbf{y}),$$

and we have

$$F(\mathbf{x},\mathbf{u})=\lim_{s\to 0+}\frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]}{s}=\lim_{s\to 0+}\frac{\Phi[\hat{\psi}_{\hat{\mathbf{x}}}[\hat{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)]-\hat{\psi}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}})]}{s}.$$

Making use of (1), with $\hat{\mathbf{u}}$ defined by (2),

$$\begin{aligned}F(\mathbf{x},\mathbf{u})&=\lim_{s\to 0+}\frac{\Phi[\hat{\psi}_{\hat{\mathbf{x}}}[\hat{\mathbf{x}}+\hat{\mathbf{u}}s+o\{s\}]-\hat{\psi}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}})]}{s}\\&=\lim_{s\to 0+}\frac{\Phi[\hat{\psi}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}}+\hat{\mathbf{u}}s)-\hat{\psi}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}})]}{s}=\hat{F}(\hat{\mathbf{x}},\hat{\mathbf{u}}).\end{aligned}$$

This proves

THEOREM 2.4.2 (Invariance under Diffeomorphisms). *The global psychometric transformation Φ and the value of the Fechner–Finsler metric function $F(\mathbf{x}, \mathbf{u})$ remain invariant under all diffeomorphisms of stimulus space $\mathfrak{M}^{(n)}$.*

2.5. *Power function version of Fechnerian scaling.* It is argued in Dzhafarov and Colonius (1999a) that, with little loss for the sphere of applicability of the theory, the global psychometric transformation in a stimulus space $\mathfrak{M}^{(n)}$ can be assumed to be a power function,

$$\Phi(h) = \sqrt[\mu]{h}, \qquad \mu > 0. \tag{11}$$

In view of (8), this is equivalent to

$$\lim_{s \to 0+} \frac{\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})}{s^{\mu}} = [F(\mathbf{x}, \mathbf{u})]^{\mu},$$

that is, all psychometric differentials $\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})$, as $s \to 0+$, are comeasurable in the small with s^{μ}. If one adopts this *power function version of Fechnerian scaling*, then the exponent μ (uniquely determined, due to the Fundamental Theorem) is referred to as the *psychometric order* of the stimulus space $\mathfrak{M}^{(n)}$. It is easy to see, for example, that if $\psi_{\mathbf{x}}(\mathbf{y})$ is analytic at $\mathbf{y} = \mathbf{x}$, that is, if

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = s \sum_{i=1}^{n} \frac{\partial \psi_{\mathbf{x}}(\mathbf{x})}{\partial x^i} u^i + \frac{s^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \psi_{\mathbf{x}}(\mathbf{x})}{\partial x^i \partial x^j} u^i u^j + \cdots,$$

then the psychometric order μ is the order of the first nonzero summand, which must be an even integer since the expansion is made at the point of minimum,

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = \frac{s^r}{r!} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \frac{\partial^r \psi_{\mathbf{x}}(\mathbf{x})}{\partial x^{i_1} \cdots \partial x^{i_r}} u^{i_1} \cdots u^{i_r} + o\{s^r\}, \qquad r = 2, 4, 6, ...$$

In this case

$$F(\mathbf{x}, \mathbf{u}) = \sqrt[r]{\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \gamma_{i_1 \ldots i_r} u^{i_1} \cdots u^{i_r}}, \qquad \Phi(h) = \sqrt[r]{h}, \mu = r.$$

(This important special case provides the main reason why it is more convenient to define the psychometric order as μ rather than $1/\mu$.)

Of course, one can think of situations in which (11) is not true for any μ, because of which $\Phi(h)$ cannot be a power function. As an example, if

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) = \varphi(\mathbf{x}, \mathbf{u})[s^{\mu} \log^{\mu}(1/s)] + o\{s^{\mu} \log^{\mu}(1/s)\},$$

then the global psychometric transformation can be presented as

$$\Phi(h) = \sqrt[\mu]{h}/\log(1/\sqrt[\mu]{h}).$$

This function is not asymptotically proportional to any power function, even though it can be shown that in this case

$$\frac{\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})}{\psi_{\mathbf{x}_0}(\mathbf{x}_0+\mathbf{u}_0 s)-\psi_{\mathbf{x}_0}(\mathbf{x}_0)} \to \frac{\varphi(\mathbf{x}, \mathbf{u})}{\varphi(\mathbf{x}_0, \mathbf{u}_0)} = \frac{F(\mathbf{x}, \mathbf{u})^{\mu}}{F(\mathbf{x}_0, \mathbf{u}_0)^{\mu}},$$

that is, the limit ratio is the same as for $\Phi(h) = \sqrt[\mu]{h}$.

2.6. *Psychometric functions*: *Third Assumption.* The Third Assumption about psychometric functions is that, for any stimulus $\mathbf{x} \in \mathfrak{M}^{(n)}$ and direction $\mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$, the two stimulus differentials $\Phi_{\mathbf{x},\mathbf{u}}(h)$ and $\Phi_{\mathbf{x},-\mathbf{u}}(h)$ are asymptotically equivalent,

$$\lim_{h\to 0+} \frac{\Phi_{\mathbf{x},\mathbf{u}}(h)}{\Phi_{\mathbf{x},-\mathbf{u}}(h)} = 1. \tag{12}$$

Since this statement is equivalent to

$$\lim_{h\to 0+} \frac{\Phi(h)}{\Phi_{\mathbf{x},\mathbf{u}}(h)} = \lim_{h\to 0+} \frac{\Phi(h)}{\Phi_{\mathbf{x},\mathbf{u}}(h)},$$

and since, by the Fundamental Theorem,

$$\lim_{h\to 0+} \frac{\Phi(h)}{\Phi_{\mathbf{x},\pm\mathbf{u}}(h)} = \lim_{s\to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}\pm\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]}{s} = F(\mathbf{x}, \pm\mathbf{u}),$$

we conclude that (12) is equivalent to

$$F(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, -\mathbf{u}), \tag{13}$$

for all line elements $(\mathbf{x}, \mathbf{u})$. In its turn, (13) is clearly equivalent to

$$\lim_{s\to 0+} \frac{\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})}{\psi_{\mathbf{x}}(\mathbf{x}-\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})} = 1. \tag{14}$$

The Third Assumption about psychometric functions only serves to ensure that the Fechnerian metrics are symmetrical: the Fechnerian distance between **a** and **b** is the same as that between **b** and **a**. This symmetry requirement, while traditionally one of the defining properties of the concept of a metric, plays a rather minor role in the theory of internal metrics in general and of Fechnerian metrics in particular: its addition to other defining properties of a metric does not seem to lead to significant new insights. In the mathematical literature it is common therefore to consider potentially asymmetric (directed) metrics and to view the symmetry requirement as optional or secondary in importance (see, e.g., Asanov, 1985; Busemann & Mayer, 1941; Rund, 1959). Without embarking on a discussion of possible interpretations of asymmetric metrics, we adopt the same approach in this paper: unless it is specifically pointed out that the Third Assumption is invoked, all our results are derived from the First and Second Assumptions only.

2.7. *Properties of Metric Function.* The Fechner–Finsler metric function $F(\mathbf{x}, \mathbf{u})$ can be viewed as the magnitude of the direction vector $\mathbf{u} \in \mathfrak{T}_{\mathbf{x}}^{(n)}$ attached to the stimulus $\mathbf{x} \in \mathfrak{M}^{(n)}$. Note that this magnitude is defined in the tangent space $\mathfrak{T}_{\mathbf{x}}^{(n)}$ rather than in the stimulus space $\mathfrak{M}^{(n)}$.

From the Fundamental Theorem we know that $F(\mathbf{x}, \mathbf{u})$ is positive and continuous. Under the Third Assumption, in addition, one has $F(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, -\mathbf{u})$. Another important property of a metric function is its Euler homogeneity, proved next.

Theorem 2.7.1 (Euler Homogeneity). *For any $k > 0$,*

$$F(\mathbf{x}, k\mathbf{u}) = kF(\mathbf{x}, \mathbf{u}). \tag{15}$$

Under the Third Assumption,

$$F(\mathbf{x}, k\mathbf{u}) = |k|\; F(\mathbf{x}, \mathbf{u}), \tag{16}$$

for any $k \neq 0$.

Proof. For $k > 0$, from the Fundamental Theorem,

$$\begin{aligned} F(\mathbf{x}, k\mathbf{u}) &= \lim_{s \to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x} + k\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s} \\ &= k \lim_{ks \to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}(ks)) - \psi_{\mathbf{x}}(\mathbf{x})]}{ks} = kF(\mathbf{x}, \mathbf{u}). \end{aligned}$$

For $k < 0$, the proof is obtained by using the symmetry property, $F(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, -\mathbf{u})$. ▌

In general, any positive, continuous, and Euler homogeneous function $\phi(\mathbf{x}, \mathbf{u})$ defined on the set of all line elements and invariant under all stimulus space diffeomorphisms can be viewed as a metric function, and, by the procedure described in the next section, it can be used to construct an internal metric.

To construct a *Finsler metric in the narrow sense*, the metric function should, in addition, be postulated to be sufficiently smooth (e.g., infinitely differentiable) and have the following regularity property: the quantities

$$g_{ij}(\mathbf{x}, \mathbf{u}) = \frac{1}{2} \frac{\partial^2 \phi(\mathbf{x}, \mathbf{u})^2}{\partial u^i \, \partial u^j}, \qquad i, j = 1, ..., n,$$

(called the components of the Finsler metric tensor) form a positive-definite matrix. Dzhafarov and Colonius (1999a) postulate this for the Fechner–Finsler metric function $F(\mathbf{x}, \mathbf{u})$. No such assumption is made in this paper.

3. FECHNERIAN METRICS AND FECHNERIAN INDICATRICES

3.0. *Outlines.* Once the notion of the Fechner–Finsler metric function $F(\mathbf{x}, \mathbf{u})$ is introduced, the procedure of Fechnerian scaling is straightforward. Since $F(\mathbf{x}, \mathbf{u})$ is interpreted as the magnitude of the direction vector $\mathbf{u}$ attached to stimulus $\mathbf{x}$, it can be used to measure the magnitude of the tangent vector $\dot{\mathbf{x}}(t)$ at a point $\mathbf{x}(t)$ of any allowable path $\mathbf{x}(t)_a^b$ connecting $\mathbf{a} = \mathbf{x}(a)$ with $\mathbf{b} = \mathbf{x}(b)$. This magnitude is $F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]$. By integrating this quantity along the path,

$$L[\mathbf{x}(t)_a^b] = \int_a^b F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt,$$

one gets what can be called the psychometric length of this path (i.e., the length derived from psychometric functions). The Fechnerian distance from $\mathbf{a}$ to $\mathbf{b}$ is computed as the infimum of the psychometric lengths of all allowable paths connecting the two points.

This construction makes the Fechnerian metric a special case of an internal metric (or generalized Finsler metric). Internal metrics can be defined through the notion of a metric function, as has just been done, or they can also be defined through the notion of an indicatrix, the set of the unit-magnitude direction vectors $\mathbf{u}$ originating at a given point $\mathbf{x}$. Although the indicatrices and the metric functions uniquely determine each other (because the indicatrix centered at $\mathbf{x}$ is described by the equation $F(\mathbf{x}, \mathbf{u}) = 1$), the introduction of the indicatrices significantly enriches the analysis of both internal metrics in general and the Fechnerian metrics in particular.

In the present context, the most important development brought forth by the notion of a Fechnerian indicatrix is that the latter, unlike the notion of the Fechner–Finsler metric function, has a direct geometric interpretation in terms of the shapes of psychometric functions: the indicatrix centered at $\mathbf{x}$ is approximated by a horizontal (i.e., parallel to the stimulus space) cross-section of the psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$ made at a very small elevation from its minimum level, $\psi_{\mathbf{x}}(\mathbf{x})$. This interpretation provides one with an unexpected theoretical bonus, a dissociation of Fechnerian indicatrices (and thereby metric functions) from the global psychometric transformation: within any compact subset of stimuli, the Fechnerian indicatrices can be ascertained without this transformation being known (although under the assumption that it exists).

The global psychometric transformation relates to another (orthogonal to the horizontal cross-sections, both logically and geometrically) aspect of the shape of psychometric functions, the unidimensional contours of the vertical cross-section of a psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$ effected by the half-planes passing through its point of minimum in all possible directions $\mathbf{u}$. The Fundamental Theorem of Fechnerian scaling amounts to the assertion that all such contours (more precisely, small portions thereof between the minima and the horizontal cross-sections made at a fixed elevation from the minima) are asymptotically identical to each other if their bases (the radii of the horizontal cross-sections) are normalized to a unity. In the power function version of Fechnerian scaling, which we believe to be of the greatest

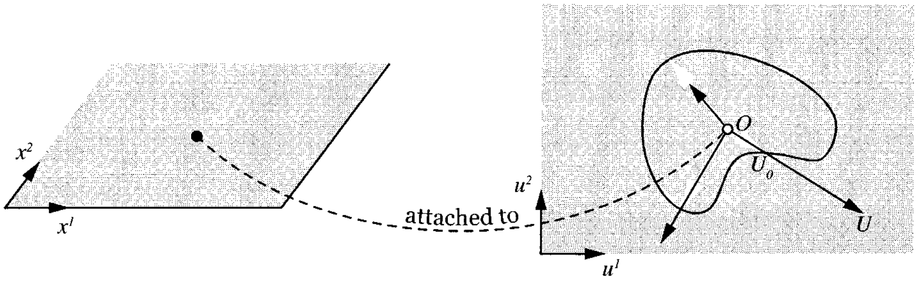

FIG. 7. A Fechnerian indicatrix (right) attached to a stimulus. The magnitude of any vector is computed as its ratio to the codirectional vector of the indicatrix.

applied importance, the Fundamental Theorem states that all the vertical cross-section contours (in the vicinity of the minima) are approximated by power functions, $b_{\mathbf{x},\mathbf{u}}s^{\mu}$, with one and the same exponent and continuously varying coefficient.

3.1. *Indicatrices.* For any stimulus $\mathbf{x}$, the set of vectors

$$\mathfrak{I}_{\mathbf{x}} = \left\{\mathbf{u} \in \mathfrak{T}_{\mathbf{x}}^{(n)} : F(\mathbf{x}, \mathbf{u}) = 1\right\} \tag{17}$$

is called the (Fechnerian) *indicatrix attached to* (*or centered at*) $\mathbf{x}$ (Fig. 7). This is a central mathematical concept of this paper, although its importance may not be apparent until the notion of a Fechnerian metric is defined and related to the shapes of psychometric functions. By abuse of language, familiar from our dealing with $\psi_{\mathbf{x}}(\mathbf{u})$, we occasionally use the term "indicatrix $\mathfrak{I}_{\mathbf{x}}$" to designate the entire set of the indicatrices indexed by the stimuli $\mathbf{x} \in \mathfrak{M}^{(n)}$.

Note that the indicatrix $\mathfrak{I}_{\mathbf{x}}$ lies within the tangent space $\mathfrak{T}_{\mathbf{x}}^{(n)}$ rather than within the stimulus space $\mathfrak{M}^{(n)}$. The endpoints of the direction vectors constituting $\mathfrak{I}_{\mathbf{x}}$ form a closed $(n-1)$-dimensional contour in $\mathfrak{T}_{\mathbf{x}}^{(n)}$. The closedness follows from the fact that for any direction vector $u \in \mathfrak{T}_{\mathbf{x}}^{(n)}$ one can find one and only one codirectional vector $\mathbf{u}_0 \in \mathfrak{I}_{\mathbf{x}}$ (the codirectionality meaning that $\mathbf{u} = \lambda \mathbf{u}_0$, $\lambda > 0$),

$$\mathbf{u} \in \mathfrak{T}_{\mathbf{x}}^{(n)} \Leftrightarrow \frac{\mathbf{u}}{F(\mathbf{x}, \mathbf{u})} = \mathbf{u}_0 \in \mathfrak{I}_{\mathbf{x}}.$$

Put differently, any vector $\mathbf{u}_0 \in \mathfrak{I}_{\mathbf{x}}$, possibly produced, intersects with the contour of $\mathfrak{I}_{\mathbf{x}}$ at one and only one point.

The *unit-vector function* $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$: $\mathfrak{T}_{\mathbf{x}}^{(n)} \to \mathfrak{I}_{\mathbf{x}}$ mapping any direction vector $\mathbf{u} \in \mathfrak{T}_{\mathbf{x}}^{(n)}$ into the codirectional vector belonging to $\mathfrak{I}_{\mathbf{x}}$,

$$\mathbf{1}_{\mathbf{x}}(\mathbf{u}) = \frac{\mathbf{u}}{F(\mathbf{x}, \mathbf{u})}, \tag{18}$$

uniquely represents the indicatrix $\mathfrak{I}_{\mathbf{x}}$, which can be viewed as the codomain of $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$. As usual, we occasionally consider $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$ to be a single function of both $\mathbf{u}$ and $\mathbf{x}$. Then the function represents the indicatrix $\mathfrak{I}_{\mathbf{x}}$ viewed as a function of $\mathbf{x}$.

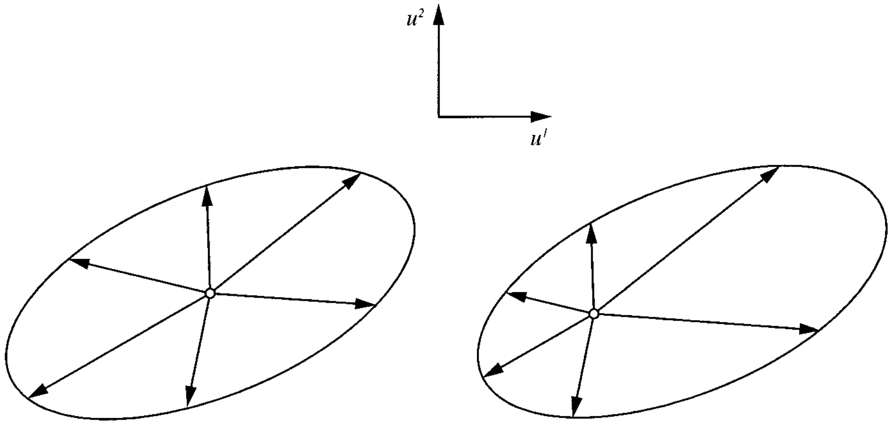

FIG. 8. The two indicatrices are different even thought they have identical contours.

THEOREM 3.1.1 (Properties of Unit-Vector Function). *The unit-vector function* $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$ *is continuous in* $(\mathbf{x}, \mathbf{u})$, *and*

$$\mathbf{1}_{\mathbf{x}}(k\mathbf{u}) = \mathbf{1}_{\mathbf{x}}(\mathbf{u}) \tag{19}$$

for any $k > 0$ (*under the Third Assumption, any* $k \neq 0$).

Proof. The proof follows from the properties of $F(\mathbf{x}, \mathbf{u})$. ▮

It is important to realize that the contour of an indicatrix $\mathfrak{I}_{\mathbf{x}}$ does not determine this indicatrix uniquely. It only does so in conjunction with the position of the center (the null vector) within the bounds of this contour (Fig. 8). A point set image of an indicatrix $\mathfrak{I}_{\mathbf{x}}$ consists, therefore, of an $(n-1)$-dimensional contour and a point within its bounds, interpreted as the center of $\mathfrak{I}_{\mathbf{x}}$ and attached to the stimulus **x**. Under the Third Assumption, however, the center of an indicatrix is, obviously, the baricenter of its contour, because of which its position is determined by the contour uniquely.

An arbitrary "freehand drawing" of a closed contour around a central point does not necessarily create an indicatrix. It is necessary, in addition, that any vector connecting the center with a point on the contour be "unobstructed," that is, that this vector does not intersect with the contour at any other points (Fig. 9). This is a point set interpretation of the uniqueness and continuity of $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$ in **u**.

In a unidimensional case, $n = 1$, the indicatrix $\mathfrak{I}_x$ attached to a point x reduces to a pair of points $u_- < 0$ and $u_+ > 0$ in Re (with $u_- = -u_+$ under the Third Assumption), provided the coordinate of the center is considered to be zero. It is interesting to note, in relation to the criticism of Fechner's original theory by Elsass (1886) and Luce and Edwards (1958), that the controversy is resolved by simply pointing out that the indicatrix $\{u_-, u_+\}$ belongs to the tangent space (here, line), rather than the space (here, line) of stimuli. We return to this issue in the Conclusion (see also Dzhafarov and Colonius, 1999a).

In a general theory of internal metrics (of which the Fechnerian metrics are a special case) one can introduce indicatrices $\mathfrak{I}_{\mathbf{x}}$ as primitives and then define the

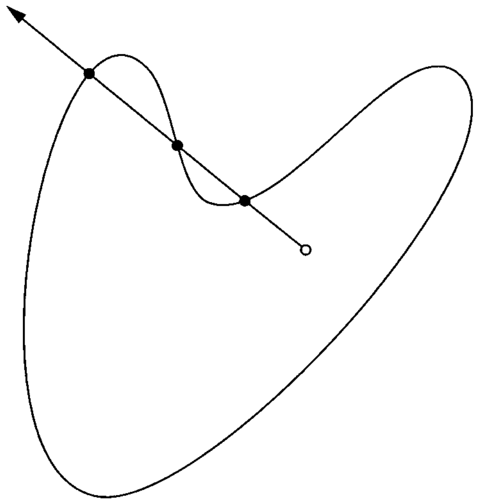

FIG. 9. This is not an indicatrix: a vector from the central point intersects the contour at more than one point.

metric function by means of (18). It is useful to spell this out (refer to Fig. 7). Given an indicatrix $\mathfrak{I}_{\mathbf{x}}$ in $\mathfrak{C}_{\mathbf{x}}^{(n)}$, one computes the magnitude of a direction vector $\overline{OU} = \mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$ attached to $\mathbf{x}$ (which stimulus is identified with O) by finding the intersection U_0 of $\overline{OU}$, possibly produced, with the contour of $\mathfrak{I}_{\mathbf{x}}$ and putting

$$F(\mathbf{x}, \mathbf{u}) = \overline{OU}/\overline{OU}_0.$$

An indicatrix is a radical generalization of a Euclidean unit sphere, which is the indicatrix of the Euclidean metric, whose corresponding metric function is the conventional Euclidean norm,

$$\begin{aligned} \tilde{F}(\mathbf{x}, \mathbf{u}) &= |\mathbf{u}| \\ \tilde{\mathbf{1}}_{\mathbf{x}}(\mathbf{u}) &= \mathbf{u}/|\mathbf{u}|. \end{aligned} \tag{20}$$

Moreover, any indicatrix $\mathfrak{I}_{\mathbf{x}}$ can be viewed as a homeomorphic transformation of a unit Euclidean sphere,

$$\mathbf{1}_{\mathbf{x}}(\mathbf{u}) = \tilde{\mathbf{1}}_{\mathbf{x}}(\mathbf{u}) \frac{|\mathbf{u}|}{F(\mathbf{x}, \mathbf{u})}.$$

One important consequence of this simple fact is that the indicatrix $\mathfrak{I}_{\mathbf{x}}$, being the codomain of $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$, is a compact set in $\mathfrak{C}_{\mathbf{x}}^{(n)}$.

3.2. *Psychometric length.* For any path $\mathbf{x}(t)$, the tangent vector $\dot{\mathbf{x}}(t)$, wherever it exists (which is everywhere except, possibly, at a finite number of t-values), is a contravariant vector attached to $\mathbf{x}(t)$, as one can easily prove by differentiating $\hat{x}^i[\mathbf{x}(t)]$, $i = 1, ..., n$. Hence $\dot{\mathbf{x}}(t) \in \mathfrak{C}_{\mathbf{x}(t)}^{(n)}$, and $[\mathbf{x}(t), \dot{\mathbf{x}}(t)]$ is always a line element (see Subsection 2.2). This line element determines the piece of the path $\mathbf{x}(t)$ between points $\mathbf{x}(t)$ and $\mathbf{x}(t+dt)=\mathbf{x}(t)+\dot{\mathbf{x}}(t)\,dt$. It follows that the function $F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]$ is well defined and can be interpreted as the length of the path $\mathbf{x}(t)$ between points $\mathbf{x}(t)$ and $\mathbf{x}(t+dt)$.

It is natural therefore to introduce the functional L, defined on the set of all allowable paths $\mathbf{x}(t)_a^b$ in $\mathfrak{M}^{(n)}$,

$$L[\mathbf{x}(t)_a^b]=\int_a^b F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\,dt, \tag{21}$$

and to call it the (*oriented*) *psychometric length* of the path $\mathbf{x}(t)_a^b$, induced by the Fechner–Finsler metric function $F(\mathbf{x}, \mathbf{u})$ or, equivalently, by the corresponding Fechnerian indicatrices $\mathfrak{J}_{\mathbf{x}}$. Note that $a<b$, but $\mathbf{x}(a)$ may coincide with $\mathbf{x}(b)$. Note also that since the metric function $F(\mathbf{x}, \mathbf{u})$ is determined up to an arbitrary scaling factor $k>0$, so is the psychometric length L.

Due to (18), the psychometric length of $\mathbf{x}(t)_a^b$ can also be written as

$$L[\mathbf{x}(t)_a^b]=\int_a^b \frac{\dot{\mathbf{x}}(t)}{\mathbf{1}_{\mathbf{x}(t)}[\dot{\mathbf{x}}(t)]}\,dt, \tag{22}$$

lending itself to the following interpretation (Fig. 10). At each point $\mathbf{x}(t)$ of the path $\mathbf{x}(t)_a^b$ there is a Fechnerian indicatrix $\mathfrak{J}_{\mathbf{x}(t)}$ attached to it. This indicatrix allows one to measure the magnitude of the tangent vector $\dot{\mathbf{x}}(t)\in\mathfrak{C}^{(n)}_{\mathbf{x}(t)}$ by the procedure discussed above (Fig. 7). When integrated along the entire path, this tangent vector magnitude yields what is natural to interpret as the length of the path. As a familiar example, in the case of the Euclidean indicatrix, (20), one gets

$$\tilde{L}[\mathbf{x}(t)_a^b]=\int_a^b |\dot{\mathbf{x}}(t)|\,dt,$$

the Euclidean length of a path.

The following theorem justifies our calling the functional L a length.

THEOREM 3.2.1 (Properties of Psychometric Length).

(i) $L[\mathbf{x}(t)_a^b]$ *is a finite positive number for any* $\mathbf{x}(t)_a^b$.

(ii) *For any* $a<b<c$, $L[\mathbf{x}(t)_a^c]=L[\mathbf{x}(t)_a^b]+L[\mathbf{x}(t)_b^c]$.

(iii) $L[\mathbf{x}(t)_a^b]$ *is invariant under all positive diffeomorphic reparametrizations of* $\mathbf{x}(t)_a^b$.

(iv) $L[\mathbf{x}(t)_a^b]$ *is invariant under all diffeomorphic transformations of* $\mathfrak{M}^{(n)}$.

Proof. (The reader is reminded that in $\mathbf{x}(t)_a^b$, $a<b$.) The continuity of $F(\mathbf{x}, \mathbf{u})$ and the allowability of the path ensure that $F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]$ is continuous in t, because of which the integral in (21) exists, in the Riemann sense. That it is positive for $a<b$ is obvious. The additivity is obvious as well. The invariance under a

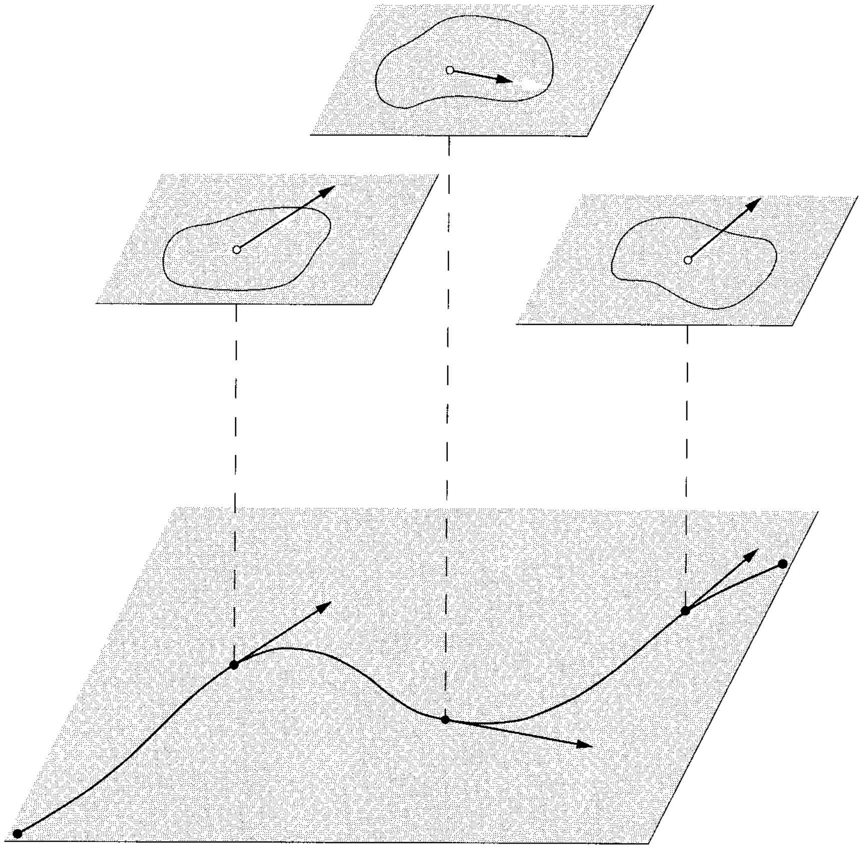

FIG. 10. The magnitude of a tangent vector taken at a point on a path is measured by the Fechnerian indicatrix attached to this point. The psychometric length of the path is the integral of this magnitude along the path.

positive diffeomorphic reparametrization $\tau(t)$ follows from the fact that if $\mathbf{z}[\tau(t)] = \mathbf{x}(t)$, then, at all points where $\dot{\mathbf{x}}(t)$ exists,

$$F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt = F\left[\mathbf{z}(\tau), \frac{d\tau}{dt}\dot{\mathbf{z}}(\tau)\right] dt = F[\mathbf{z}(\tau), \dot{\mathbf{z}}(\tau)]\, \frac{d\tau}{dt}\, dt = F[\mathbf{z}(\tau), \dot{\mathbf{z}}(\tau)]\, d\tau.$$

Finally, the invariance under all stimulus space diffeomorphisms follows from the same property of $F(\mathbf{x}, \mathbf{u})$. ∎

In view of the Third Assumption, it is convenient to introduce the concept of a path *oppositely oriented* with respect to a given path $\mathbf{x}(t)_a^b$. We refer thus to any path $\mathbf{z}(\tau)_\alpha^\beta$ such that $\mathbf{z}[\tau(t)] = \mathbf{x}(t)$, where $\tau(t)$ is a *negative* diffeomorphism $[a, b] \to [\alpha, \beta]$, $\dot{\tau}(t) < 0$. If the original path connects $\mathbf{a} = \mathbf{x}(a)$ with $\mathbf{b} = \mathbf{x}(b)$, an opposite path connects $\mathbf{b} = \mathbf{z}(\alpha)$ with $\mathbf{a} = \mathbf{z}(\beta)$ (note that $\alpha < \beta$).

ADDENDUM TO THEOREM 3.2.1 (Symmetry of Length). *Under the Third Assumption, $L[\mathbf{x}(t)_a^b]$ is invariant under all negative diffeomorphic reparametrizations of $\mathbf{x}(t)_a^b$.*

Proof. Using $\mathbf{z}[\tau(t)] = \mathbf{x}(t)$, as introduced in the previous paragraph,

$$L[\mathbf{x}(t)_a^b] = \int_a^b F[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt$$

$$= \int_a^b F\left[\mathbf{z}(\tau), \frac{d\tau}{dt}\dot{\mathbf{z}}(\tau)\right] dt = \int_a^b F[\mathbf{z}(\tau), \dot{\mathbf{z}}(\tau)]\left(-\frac{d\tau}{dt}\right) dt$$

$$= \int_\alpha^\beta F[\mathbf{z}(\tau), \dot{\mathbf{z}}(\tau)]\, d\tau = L[\mathbf{z}(\tau)_\alpha^\beta]. \quad \blacksquare$$

Since the stimulus space $\mathfrak{M}^{(n)}$ possesses conventional topology that can be viewed as induced by the Euclidean metrization of $\mathfrak{M}^{(n)}$, it is important to understand the constraints imposed on the psychometric length of a path that lies within a small Euclidean ball. A Euclidean ball $\mathfrak{B}(\mathbf{a}, R) \subseteq \mathrm{Re}^n$, centered at $\mathbf{a}$ with radius R, is considered *allowable* if it lies entirely within $\mathfrak{M}^{(n)}$. Consider a path $\mathbf{x}(t)_a^b$ lying entirely within an allowable ball $\mathfrak{B}(\mathbf{a}, R)$,

$$|\mathbf{x}(t) - \mathbf{a}| \leqslant R.$$

It is well-known that $\mathbf{x}(t)_a^b$ permits what is called its *natural parametrization* (e.g., Kreyszig, 1968, p. 29), that is, it can be parametrized as $\mathbf{z}(\tau)_0^E$, where τ is the Euclidean length of $\mathbf{x}(t)_a^b$ between $\mathbf{x}(a)$ and $\mathbf{x}(t)$,

$$\tau(t) = \int_a^t |\dot{\mathbf{x}}(t)|\, dt.$$

Thus E is the Euclidean length of the entire path $\mathbf{z}(\tau)_0^E$. It is easy to show (essentially, by the Pythagorean theorem applied in the small; see Kreyszig, 1968, p. 31) that in this parametrization all tangents of $\mathbf{z}(\tau)_0^E$ (that exist and change continuously at all but a finite number of points) have a unit Euclidean norm,

$$|\dot{\mathbf{z}}(\tau)| = 1. \tag{23}$$

The set of all line elements $(\mathbf{z}, \tilde{\mathbf{u}})$ such that $\mathbf{z} \in \mathfrak{B}(\mathbf{a}, R)$ and $|\tilde{\mathbf{u}}| = 1$ is compact (including the case $R = 0$), because of which $F(\mathbf{z}, \tilde{\mathbf{u}})$ attains on this set its precise minimum, $F_{\min}(\mathbf{a}, R) > 0$, and its precise maximum, $F_{\max}(\mathbf{a}, R) > 0$. It follows, due to (23), that

$$\int_0^E F[\mathbf{z}(\tau), \dot{\mathbf{z}}(\tau)]\, d\tau \leqslant F_{\max}(\mathbf{a}, R)\, E$$

and

$$\int_0^E F[\mathbf{z}(\tau), \dot{\mathbf{z}}(\tau)]\, d\tau \geqslant F_{\min}(\mathbf{a}, R)\, E \geqslant F_{\min}(\mathbf{a}, R)\, |\mathbf{z}(0) - \mathbf{z}(E)|.$$

These results are summarized in

LEMMA 3.2.1 (Euclidean Bounds for Psychometric Length). *For any allowable ball* $\mathfrak{B}(\mathbf{a}, R)$, $R \geqslant 0$, *one can find two numbers* $F_{\min}(\mathbf{a}, R) > 0$ *and* $F_{\max}(\mathbf{a}, R) > 0$, *such that, for any path* $\mathbf{x}(t)_a^b$ *never leaving* $\mathfrak{B}(\mathbf{a}, R)$,

$$L[\mathbf{x}(t)_a^b] \geqslant F_{\min}(\mathbf{a}, R)\, |\mathbf{x}(a) - \mathbf{x}(b)|, \tag{24}$$

$$L[\mathbf{x}(t)_a^b] \leqslant F_{\max}(\mathbf{a}, R)\, E, \tag{25}$$

where E is the Euclidean length of the path.

In particular, if $\mathbf{x}(t)_a^b = \mathbf{s}(t)_a^b$ is a straight line segment,

$$\mathbf{s}(t)_a^b = \frac{b-t}{b-a}\, \mathbf{x}(a) + \frac{t-a}{b-a}\, \mathbf{x}(b),$$

then (25) becomes

$$L[\mathbf{s}(t)_a^b] \leqslant F_{\max}(\mathbf{a}, R)\, |\mathbf{x}(a) - \mathbf{x}(b)|. \tag{26}$$

3.3. *Fechnerian distance.* We approach the final step in the construction of a Fechnerian metric. Denote by $\{\mathbf{a} \mapsto \mathbf{b}\}$ the class of all allowable oriented paths $\mathbf{x}(t)_a^b$ with $\mathbf{a} = \mathbf{x}(a)$, $\mathbf{b} = \mathbf{x}(b)$, and put

$$G(\mathbf{a}, \mathbf{b}) = \inf_{\mathbf{x}(t)_a^b \in \{\mathbf{a} \mapsto \mathbf{b}\}} L[\mathbf{x}(t)_a^b]. \tag{27}$$

This infimum must exist, since all psychometric lengths are nonnegative. We say that $G(\mathbf{a}, \mathbf{b})$ is the *Fechnerian metric* induced by the (Fechner–Finsler) metric function $F(\mathbf{x}, \mathbf{u})$ or, equivalently, by the corresponding Fechnerian indicatrices $\mathfrak{I}_{\mathbf{x}}$. Recall that the psychometric length L is determined up to an arbitrary scaling factor $k > 0$, because of which the same applies to $G(\mathbf{a}, \mathbf{b})$.

Lemma 3.2.1 leads to the following important

LEMMA 3.3.1 (Euclidean Bounds for Fechnerian Metric). *For any* $\mathbf{a} \in \mathfrak{M}^{(n)}$ *and any allowable Euclidean ball* $\mathfrak{B}(\mathbf{a}, R)$, *if* $\mathbf{b} \in \mathfrak{B}(\mathbf{a}, R)$, *then*

$$G(\mathbf{a}, \mathbf{b}) \leqslant F_{\max}(\mathbf{a}, R)\, |\mathbf{a} - \mathbf{b}|, \tag{28}$$

$$G(\mathbf{a}, \mathbf{b}) \geqslant F_{\min}(\mathbf{a}, R)\, |\mathbf{a} - \mathbf{b}|. \tag{29}$$

Proof. The first inequality follows from (26) on observing that $G(\mathbf{a}, \mathbf{b}) \leqslant L[\mathbf{s}(t)_a^b]$. Due to (24), the infimum of $L[\mathbf{x}(t)_a^b]$ for all paths lying entirely within $\mathfrak{B}(\mathbf{a}, R)$ cannot fall below $F_{\min}(\mathbf{a}, R)\, |\mathbf{x}(a) - \mathbf{x}(b)| = F_{\min}(\mathbf{a}, R)\, |\mathbf{a} - \mathbf{b}|$, while the infimum of $L[\mathbf{x}(t)_a^b]$ for all paths puncturing the sphere of $\mathfrak{B}(\mathbf{a}, R)$ cannot fall below $F_{\min}(\mathbf{a}, R)\, R$. Since $R \geqslant |\mathbf{x}(a) - \mathbf{x}(b)|$, this proves the second inequality. ∎

Clearly, the analogues of (28) and (29) for the *terminal* point, $\mathbf{b}$, also hold: if $\mathbf{a} \in \mathfrak{B}(\mathbf{b}, R)$, then

$$G(\mathbf{a}, \mathbf{b}) \leqslant F_{\max}(\mathbf{b}, R)\, |\mathbf{a}-\mathbf{b}|, \tag{30}$$

$$G(\mathbf{a}, \mathbf{b}) \geqslant F_{\min}(\mathbf{b}, R)\, |\mathbf{a}-\mathbf{b}|. \tag{31}$$

The following theorem justifies our calling $G(\mathbf{a}, \mathbf{b})$ a metric.

THEOREM 3.3.1 (Metric Properties). *$G(\mathbf{a}, \mathbf{b})$ is a continuous oriented distance on $\mathfrak{M}^{(n)}$, that is,*

(i) $\mathbf{a} \neq \mathbf{b} \Rightarrow G(\mathbf{a}, \mathbf{b}) > 0$,

(ii) $G(\mathbf{a}, \mathbf{a}) = 0$,

(iii) $G(\mathbf{a}, \mathbf{b}) \leqslant G(\mathbf{a}, \mathbf{x}) + G(\mathbf{x}, \mathbf{b})$,

(iv) *$G(\mathbf{a}, \mathbf{b})$ is continuous in* $(\mathbf{a}, \mathbf{b})$ (*with respect to the conventional topology on* $\mathfrak{M}^{(n)}$),

(v) *under the Third Assumption*, $G(\mathbf{a}, \mathbf{b}) = G(\mathbf{b}, \mathbf{a})$, and

(vi) *$G(\mathbf{a}, \mathbf{b})$ is invariant under all diffeomorphic transformations of $\mathfrak{M}^{(n)}$.*

Proof. Proposition (i) follows from (24), by considering an allowable ball $\mathfrak{B}(\mathbf{a}, R)$ with $R \leqslant |\mathbf{a}-\mathbf{b}|$: any path originating at $\mathbf{a}$ and reaching $\mathbf{b}$ should puncture the sphere of this ball, because of which it is bounded from below by $F_{\min}(\mathbf{a}, R)\, R$.

The proofs for (ii), (iii), (v), and (vi) are trivial.

To prove (iv), assume $\mathbf{a}_k \to \mathbf{a}$, $\mathbf{b}_k \to \mathbf{b}$ ($k = 1, 2, ...$). Using the triangle inequality, (iii), one derives

$$-G(\mathbf{a}_k, \mathbf{a}) - G(\mathbf{b}, \mathbf{b}_k) \leqslant G(\mathbf{a}, \mathbf{b}) - G(\mathbf{a}_k, \mathbf{b}_k) \leqslant G(\mathbf{a}, \mathbf{a}_k) + G(\mathbf{b}_k, \mathbf{b}).$$

Beginning with some k, all $\mathbf{a}_k$ and $\mathbf{b}_k$ must be confined to allowable Euclidean balls $\mathfrak{B}(\mathbf{a}, R)$ and $\mathfrak{B}(\mathbf{b}, R)$, respectively. Applying (28) to $G(\mathbf{a}, \mathbf{a}_k)$ and $G(\mathbf{b}, \mathbf{b}_k)$, and applying (30) to $G(\mathbf{a}_k, \mathbf{a})$ and $G(\mathbf{b}_k, \mathbf{b})$, one gets

$$\begin{aligned} -F_{\max}(\mathbf{a}, R)\, &|\mathbf{a}-\mathbf{a}_k| - F_{\max}(\mathbf{b}, R)\, |\mathbf{b}-\mathbf{b}_k| \\ &\leqslant G(\mathbf{a}, \mathbf{b}) - G(\mathbf{a}_k, \mathbf{b}_k) \\ &\leqslant F_{\max}(\mathbf{a}, R)\, |\mathbf{a}-\mathbf{a}_k| + F_{\max}(\mathbf{b}, R)\, |\mathbf{b}-\mathbf{b}_k|. \end{aligned}$$

Since the bounds tend to zero, (iv) is proved. ▮

A metric in which the distance between two points is defined as the infimum of the lengths of all paths connecting these points is called an *internal*, or *generalized*, *Finsler* metric. The Fechnerian metric on the stimulus space is, therefore, an internal metric.

In this paper we are not concerned with establishing conditions under which the infimum in (27) is the minimum, that is, when the set $\{\mathbf{a} \mapsto \mathbf{b}\}$ contains a *Fechnerian geodesic*, an allowable path whose psychometric length equals $G(\mathbf{a}, \mathbf{b})$.

Nor are we concerned with the existence of any (not necessarily allowable) path whose length, appropriately defined, equals $G(\mathbf{a}, \mathbf{b})$ (called a *shortest*, or *Hilbert*, path). Thorough discussions of the mathematical issues involved in these problems can be found in Busemann (1955), Busemann & Mayer (1941), and Caratheodory (1982, Chap. 16).

For several reasons, however, it is important to investigate the problem of *Fechnerian geodesics in the small*, that is, the existence and properties of an allowable path $\mathbf{a}+\mathbf{x}(t)_a^b\, s$ connecting $\mathbf{a}$ to $\mathbf{b}=\mathbf{a}+\mathbf{u}s$, whose psychometric length tends to the Fechnerian distance $G(\mathbf{a}, \mathbf{a}+\mathbf{u}s)$ as $s \to 0+$. It should be mentioned, although with no intention of elaborating this further, that the issue of the Fechnerian geodesics in the small has a practical significance for computing Fechnerian distances, especially in the case of "brute force" techniques utilizing fine-mesh discretizations of a stimulus space. More importantly, however, within the logic of this paper this issue serves as a bridge to the subsequent analysis of the shapes of the Fechnerian indicatrices and of their relationship to Fechnerian distances. It should be noted that this approach deviates from the existing mathematical tradition.

3.4. *Min-metric function and Fechnerian geodesics in the small.* Given an indicatrix $\mathfrak{I}_{\mathbf{x}}$ (associated with the metric function $F(\mathbf{x}, \mathbf{u})$) and a direction vector $\mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$, we call a sequence of (not necessarily distinct) direction vectors $\mathbf{u}_1 \in \mathfrak{C}_{\mathbf{x}}^{(n)}, ..., \mathbf{u}_n \in \mathfrak{C}_{\mathbf{x}}^{(n)}$ a *minimizing chain for* $\mathbf{u}$ *at* $\mathfrak{I}_{\mathbf{x}}$, if $\mathbf{u}=\mathbf{u}_1+\cdots+\mathbf{u}_n$, and

$$F(\mathbf{x}, \mathbf{u}_1)+\cdots+F(\mathbf{x}, \mathbf{u}_n) \leqslant F(\mathbf{x}, \mathbf{v}_1)+\cdots+F(\mathbf{x}, \mathbf{v}_m), \tag{32}$$

for any sequence $\mathbf{v}_1 \in \mathfrak{C}_{\mathbf{x}}^{(n)}, ..., \mathbf{v}_m \in \mathfrak{C}_{\mathbf{x}}^{(n)}$, $m \leqslant n$, such that $\mathbf{u}=\mathbf{v}_1+\cdots+\mathbf{v}_m$. That a minimizing chain (not necessarily unique) exists for any $\mathbf{u}$ at any $\mathfrak{I}_{\mathbf{x}}$ is proved by the following reasoning.

Recall that $\mathfrak{C}_{\mathbf{x}}^{(n)}$ does not include the null vector, so all the vectors in (32) are nonvanishing. It is convenient, however, to extend the metric function $F(\mathbf{x}, \mathbf{v})$ to vanishing vectors by putting

$$F^*(\mathbf{x}, \mathbf{v})=\begin{cases} F(\mathbf{x}, \mathbf{v}) & \text{if } \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{v}=\mathbf{0} \end{cases}.$$

Then the problem can be reformulated as that of finding the minimum value of $F^*(\mathbf{x}, \mathbf{v}_1)+\cdots+F^*(\mathbf{x}, \mathbf{v}_n)$ across all vector n-tuples $(\mathbf{v}_1, ..., \mathbf{v}_n)$ such that $\mathbf{u}=\mathbf{v}_1+\cdots+\mathbf{v}_n$. One can confine the analysis to only those $(\mathbf{v}_1, ..., \mathbf{v}_n)$ for which $F^*(\mathbf{x}, \mathbf{v}_i) \leqslant F(\mathbf{x}, \mathbf{u})$, $i=1, ..., n$. Indeed, since $\mathbf{u}=(\mathbf{u}/n)+\cdots+(\mathbf{u}/n)$, the minimum value of $F^*(\mathbf{x}, \mathbf{v}_1)+\cdots+F^*(\mathbf{x}, \mathbf{v}_n)$, if it exists, cannot exceed $nF(\mathbf{x}, \mathbf{u}/n)=F(\mathbf{x}, \mathbf{u})$. The set of all $(\mathbf{v}_1, ..., \mathbf{v}_n)$ such that $\mathbf{u}=\mathbf{v}_1+\cdots+\mathbf{v}_n$ and $F^*(\mathbf{x}, \mathbf{v}_i) \leqslant F(\mathbf{x}, \mathbf{u})$ is compact, as it describes the intersection of an $n(n-1)$-dimensional hyperplane with the n^2-dimensional body formed by the Cartesian product of compact sets $\{\mathbf{v}_i\colon F^*(\mathbf{x}, \mathbf{v}_i) \leqslant F(\mathbf{x}, \mathbf{u})\}$, $i=1, ..., n$. The function $F^*(\mathbf{x}, \mathbf{v}_1)+\cdots+F^*(\mathbf{x}, \mathbf{v}_n)$, being continuous on this set, attains its precise minimum, $\widehat{F}(\mathbf{x}, \mathbf{u})$, at some $(\mathbf{v}_1, ..., \mathbf{v}_n)$. Since $\mathbf{v}_1+\cdots+\mathbf{v}_n=\mathbf{u} \neq \mathbf{0}$, some of these vectors, say, $\mathbf{u}_1=\mathbf{v}_{i_1}, ..., \mathbf{u}_k=\mathbf{v}_{i_k}$, $k \leqslant n$, are nonzero. One has, therefore, a number $k \leqslant n$ and vectors $\mathbf{u}_1 \in \mathfrak{C}_{\mathbf{x}}^{(n)}, ..., \mathbf{u}_k \in \mathfrak{C}_{\mathbf{x}}^{(n)}$ such that

$$F(\mathbf{x}, \mathbf{u}_1) + \cdots + F(\mathbf{x}, \mathbf{u}_k) = \widehat{F}(\mathbf{x}, \mathbf{u}) \leqslant F(\mathbf{x}, \mathbf{v}_1) + \cdots + F(\mathbf{x}, \mathbf{v}_m)$$

for all possible vector m-tuples $(\mathbf{v}_1 \in \mathfrak{C}_{\mathbf{x}}^{(n)}, ..., \mathbf{v}_m \in \mathfrak{C}_{\mathbf{x}}^{(n)})$ such that $\mathbf{u} = \mathbf{v}_1 + \cdots + \mathbf{v}_m$. It remains to observe that one can always replace $\mathbf{u}_1$ with the sum of $n-k+1$ identical vectors $\mathbf{u}_1/(n-k+1)$ to obtain a minimizing chain $\mathbf{u}_1 \in \mathfrak{C}_{\mathbf{x}}^{(n)}, ..., \mathbf{u}_n \in \mathfrak{C}_{\mathbf{x}}^{(n)}$ satisfying (32). We have proved

THEOREM 3.4.1 (Existence of Minimizing Chains). *Given an indicatrix $\mathfrak{I}_{\mathbf{x}}$, there is a minimizing chain $(\mathbf{u}_1, ..., \mathbf{u}_n)$ for any $\mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$ at $\mathfrak{I}_{\mathbf{x}}$, with the minimum value $\widehat{F}(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, \mathbf{u}_1) + \cdots + F(\mathbf{x}, \mathbf{u}_n)$ that does not exceed $F(\mathbf{x}, \mathbf{u})$.* ▌

For reasons made apparent in the next subsection, we refer to the function $\widehat{F}(\mathbf{x}, \mathbf{u})$, the magnitude of a minimizing chain for $\mathbf{u}$ at $\mathfrak{I}_{\mathbf{x}}$, as the *min-metric function* associated with $\mathfrak{I}_{\mathbf{x}}$. Clearly, $\widehat{F}(\mathbf{x}, k\mathbf{u}) = k\widehat{F}(\mathbf{x}, \mathbf{u})$, for any $k > 0$. Because of this one can confine the discussion of minimizing chains to vectors $\mathbf{u}$ with a fixed norm, for example, $\mathbf{u} = \bar{\mathbf{u}}$, where $F(\mathbf{x}, \bar{\mathbf{u}}) = 1$. The following geometric interpretation of $\widehat{F}(\mathbf{x}, \mathbf{u})$ plays a useful role in the analysis of indicatrices (Fig. 11). If $\bar{\mathbf{u}} = \mathbf{v}_1 + \cdots + \mathbf{v}_n$, each $\mathbf{v}_i$ can be presented as $F(\mathbf{x}, \mathbf{v}_i)\, \bar{\mathbf{v}}_i$, $\bar{\mathbf{v}}_i \in \mathfrak{I}_{\mathbf{x}}$, $i = 1, ..., n$. Then

$$\delta\bar{\mathbf{u}} = \gamma^1 \bar{\mathbf{v}}_1 + \cdots + \gamma^n \bar{\mathbf{v}}_n, \tag{33}$$

where

$$\delta = \frac{1}{F(\mathbf{x}, \mathbf{v}_1) + \cdots + F(\mathbf{x}, \mathbf{v}_n)}, \qquad \gamma^i = \frac{F(\mathbf{x}, \mathbf{v}_i)}{F(\mathbf{x}, \mathbf{v}_1) + \cdots + F(\mathbf{x}, \mathbf{v}_n)}, \; i = 1, ..., n.$$

Consider the contour of the indicatrix $\mathfrak{I}_{\mathbf{x}}$ with its center O attached to $\mathbf{x}$. Denoting $\overline{OU} = \bar{\mathbf{u}}$, $\overline{OU'} = \delta\bar{\mathbf{u}}$, and $\overline{OV_i} = \bar{\mathbf{v}}_i$, $i = 1, ..., n$, (33) says that the interior of the n-gon $\overline{V_1 \cdots V_n}$ (not necessarily distinct) intersects $\overline{OU}$, possibly produced, at the point U'. If $(\mathbf{v}_1, ..., \mathbf{v}_n) = (\mathbf{u}_1, ..., \mathbf{u}_n)$ is a minimizing chain for $\mathbf{u}$ at $\mathfrak{I}_{\mathbf{x}}$, the value of δ is at its maximum,

$$\delta(\mathbf{x}, \bar{\mathbf{u}}) = \frac{1}{F(\mathbf{x}, \mathbf{u}_1) + \cdots + F(\mathbf{x}, \mathbf{u}_n)} = \frac{1}{\widehat{F}(\mathbf{x}, \bar{\mathbf{u}})}. \tag{34}$$

We see that $\widehat{F}(\mathbf{x}, \bar{\mathbf{u}})$ corresponds to the maximal possible extension $\overline{OU'} = \delta\bar{\mathbf{u}}$ of $\overline{OU} = \bar{\mathbf{u}}$ for which one can still find an n-gon $\overline{V_1 \cdots V_n}$, with all its vertices (not necessarily distinct) on the contour of $\mathfrak{I}_{\mathbf{x}}$, that contains U' within its interior (Fig. 11, right). For a general $\mathbf{u} \in \mathfrak{C}_{\mathbf{x}}^{(n)}$,

$$\widehat{F}(\mathbf{x}, \mathbf{u}) = \frac{F(\mathbf{x}, \mathbf{u})}{\delta(\mathbf{x}, \bar{\mathbf{u}})}. \tag{35}$$

Observe that $\delta(\mathbf{x}, \bar{\mathbf{u}}) \geqslant 1$, because $\widehat{F}(\mathbf{x}, \mathbf{u}) \leqslant F(\mathbf{x}, \mathbf{u})$, by Theorem 3.4.1.

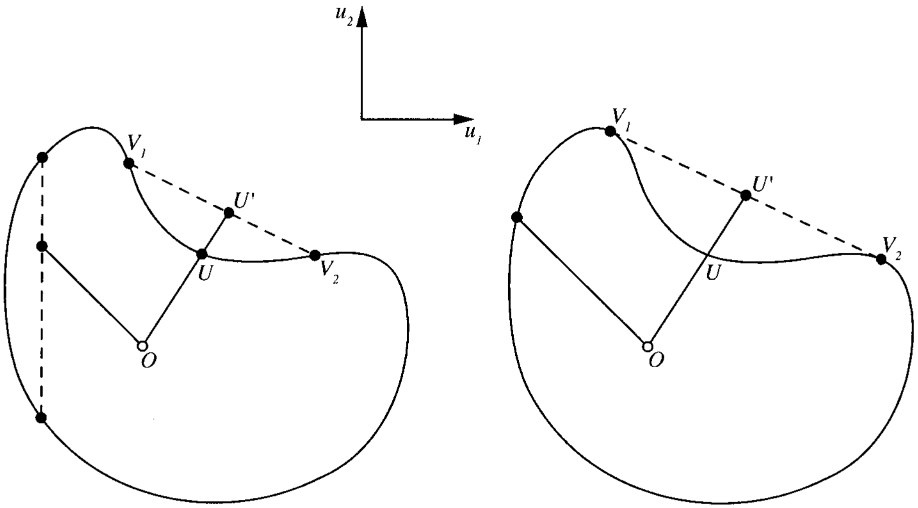

FIG. 11. A geometric interpretation of the min-metric function for a vector $\overline{OU}$ of an indicatrix. Considering all possible polygons (here, straight line segments) $\overline{V_1 V_2}$ whose interiors intersect the vector, possibly produced (left panel), one finds its maximal extension for which such a polygon exists (right panel). When this happens, $\overline{OU}/\overline{OU'}$ equals the min-metric function. Observe that for the second, unlabeled direction shown, the maximal extension is achieved when $V_1 = V_2 = U = U'$.

Theorem 3.4.2 (Global Minimization). *For any* $(\mathbf{v}_1, ..., \mathbf{v}_m)$ *such that* $\mathbf{u} = \mathbf{v}_1 + \cdots + \mathbf{v}_m$, $\widehat{F}(\mathbf{x}, \mathbf{u}) \leqslant F(\mathbf{x}, \mathbf{v}_1) + \cdots + F(\mathbf{x}, \mathbf{v}_m)$.

Proof. If $m \leqslant n$, this is true by definition. Suppose therefore that $m > n$. With no loss of generality, let $\mathbf{u} = \bar{\mathbf{u}} \in \mathfrak{I}_{\mathbf{x}}$. By the same geometric construction as above, the endpoint of some extension $\overline{OU'} = \delta\bar{\mathbf{u}}$ of $\overline{OU} = \bar{\mathbf{u}}$ lies within the interior of the m-gon $\overline{V_1 \cdots V_m}$ formed by the endpoints of $\bar{\mathbf{v}}_1 = \mathbf{1}_{\mathbf{x}}(\mathbf{v}_1), ..., \bar{\mathbf{v}}_m = \mathbf{1}_{\mathbf{x}}(\mathbf{v}_m)$, not necessarily distinct,

$$\delta\bar{\mathbf{u}} = \alpha^1\bar{\mathbf{v}}_1 + \cdots + \alpha^n\bar{\mathbf{v}}_m,$$

where

$$\delta = \frac{1}{F(\mathbf{x}, \mathbf{v}_1) + \cdots + F(\mathbf{x}, \mathbf{v}_m)}.$$

If this m-gon lies within a hyperplane of dimensionality $r < n$ (Fig. 12, left), then, by considering all possible n-gons with their vertices chosen from $(V_1, ..., V_m)$, with replacement, we should find at least one (say, $\overline{V_1 \cdots V_k}$) that contains U' within its interior. Then (33) holds for some vectors $(\mathbf{v}'_1, ..., \mathbf{v}'_n)$ that are codirectional with $(\mathbf{v}_1, ..., \mathbf{v}_n)$, and it holds with the same value of δ. Therefore

$$F(\mathbf{x}, \mathbf{v}_1) + \cdots + F(\mathbf{x}, \mathbf{v}_m) = \frac{1}{\delta} = F(\mathbf{x}, \mathbf{v}'_1) + \cdots + F(\mathbf{x}, \mathbf{v}'_n) \geqslant \widehat{F}(\mathbf{x}, \bar{\mathbf{u}}).$$

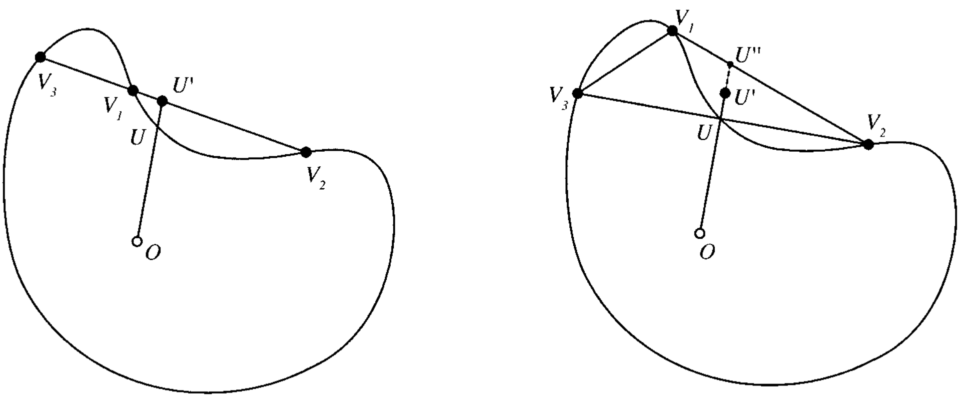

FIG. 12. An illustration for the Global Minimization Theorem.

The remaining case is when the m-gon $\overline{V_1 \cdots V_m}$ is an n-dimensional polyhedron, and U' lies within it (Fig. 12, right). Then one can produce $\overline{OU'}$ farther, until it hits an $(n-1)$-dimensional face of the polyhedron at some point U'', falling within its interior, or the interior of one of its subpolygons. Clearly,

$$\delta = \overline{OU'}/\overline{OU} \leqslant \overline{OU''}/\overline{OU} \leqslant \delta(\mathbf{x}, \bar{\mathbf{u}}),$$

and this case reduces to the previous one. ∎

Theorem 3.4.3 (Invariance under diffeomorphisms). *The min-metric function $\widehat{F}(\mathbf{x}, \mathbf{u})$ is invariant under diffeomorphisms of the stimulus space.*

Proof. Under a diffeomorphism $\mathbf{x} \to \hat{\mathbf{x}}$, all direction vectors in $\mathfrak{C}_{\mathbf{x}}^{(n)}$ undergo the linear transformation (2) that does not change linear relations among these vectors. In particular, if $\mathbf{u} = \mathbf{u}_1 + \cdots + \mathbf{u}_n$, then $\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 + \cdots + \hat{\mathbf{u}}_n$. Since $\widehat{F}(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, \mathbf{u}_1) + \cdots + F(\mathbf{x}, \mathbf{u}_n)$, the invariance in question follows from the invariance of the (Fechner–Finsler) metric function F. ∎

We are finally prepared to prove the main result of this subsection. Consider any allowable path $\mathbf{a} + \mathbf{x}(t)_0^1 s$ with $\mathbf{x}(0) = \mathbf{0}$, $\mathbf{x}(1) = \mathbf{u}$; this path connects $\mathbf{a}$ to $\mathbf{a} + \mathbf{u}s$. Let $(\mathbf{u}_1, ..., \mathbf{u}_n)$ be a minimizing chain for $\mathbf{u}$ at $\mathfrak{I}_{\mathbf{a}}$. We compare the psychometric length

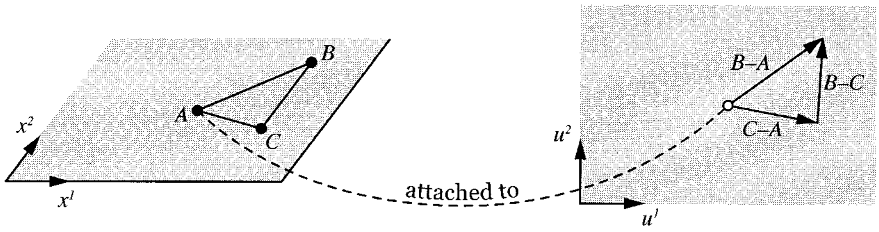

FIG. 13. Fechnerian geodesic in the small: As point B tends to point A along a given direction, the Fechnerian distance between them tends to the sum of (here) two segments whose directions form a minimizing chain for the direction from A to B at the indicatrix attached to A.

of $\mathbf{a}+\mathbf{x}(t)_0^1 s$ with the length of the path formed by the straight line segments (Fig. 13)

$$\mathbf{s}_1(t)_0^s=\mathbf{a}+t\mathbf{u}_1,\quad \mathbf{s}_2(t)_0^s=\mathbf{a}+s\mathbf{u}_1+t\mathbf{u}_2, ..., \mathbf{s}_n(t)_0^s=\mathbf{a}+s\mathbf{u}_1+\cdots+s\mathbf{u}_{n-1}+t\mathbf{u}_n,$$

each segment being parametrized separately by $0\leqslant t\leqslant s$. The first segment connects $\mathbf{a}$ to $\mathbf{a}+s\mathbf{u}_1$ the second $\mathbf{a}+s\mathbf{u}_1$ to $\mathbf{a}+s(\mathbf{u}_1+\mathbf{u}_2)$, and so on, until the last segment connects $\mathbf{a}+s(\mathbf{u}_1+\cdots+\mathbf{u}_{n-1})$ to $\mathbf{a}+s(\mathbf{u}_1+\cdots+\mathbf{u}_n)=\mathbf{a}+s\mathbf{u}$. We have

$$\frac{\int_0^1 F[\mathbf{a}+\mathbf{x}(t)\,s, \dot{\mathbf{x}}(t)\,s]\,dt}{\sum_{i=1}^{n}\int_0^s F\left(\mathbf{a}+s\sum_{j=1}^{i-1}\mathbf{u}_j+\mathbf{u}_i t, \mathbf{u}_i\right)dt}=\frac{\sum_{i=1}^{k}\left[F\left(\mathbf{a}+s\sum_{j=1}^{i-1}\mathbf{v}_j, \mathbf{v}_i\right)s+o\{s\}\right.}{\sum_{i=1}^{n}\left[F\left(\mathbf{a}+s\sum_{j=1}^{i-1}\mathbf{u}_j, \mathbf{u}_i\right)s+o\{s\}\right]}, \tag{36}$$

where the interval $[0, 1]$ for the numerator has been partitioned by $0=t_0<t_1<\cdots<t_k=1$ in such a way that $\dot{\mathbf{x}}(t)$ is continuous on any $[t_i, t_{i+1}]$ and

$$\mathbf{v}_i=\frac{\mathbf{x}(t_i)-\mathbf{x}(t_{i-1})}{t_i-t_{i-1}},\qquad i=1, ..., k.$$

In other words, the (Riemann) integral in the numerator of (36) has been approximated by the values of $\mathbf{x}(t)$ at points $\mathbf{a}$, $\mathbf{a}+s\mathbf{v}_1$, $\mathbf{a}+s(\mathbf{v}_1+\mathbf{v}_2)$, ..., $\mathbf{a}+s(\mathbf{v}_1+\cdots+\mathbf{v}_k)=\mathbf{a}+s\mathbf{u}$, with $\dot{\mathbf{x}}(t)$ being continuous in between. As $s\to 0+$, (36) tends to

$$\frac{\sum_{i=1}^{k}F(\mathbf{a}, \mathbf{v}_i)}{\sum_{i=1}^{n}F(\mathbf{a}, \mathbf{u}_i)}=\frac{\sum_{i=1}^{k}F(\mathbf{a}, \mathbf{v}_i)}{\widehat{F}(\mathbf{x}, \mathbf{u})}\geqslant 1, \tag{37}$$

where the inequality follows from Theorem 3.4.2. Since this is true for any path $\mathbf{a}+\mathbf{x}(t)_0^1 s$, and since the sequence of the segments $\mathbf{a}+t\mathbf{u}_1$, $\mathbf{a}+s\mathbf{u}_1+t\mathbf{u}_2$, ..., $\mathbf{a}+s\mathbf{u}_1+\cdots+s\mathbf{u}_{n-1}+t\mathbf{u}_n$ forms an allowable path from $\mathbf{a}$ to $\mathbf{a}+\mathbf{u}s$, this sequence forms a Fechnerian geodesic in the small. Conversely, if it is known that, for some sequence of segments $\mathbf{a}+t\mathbf{u}_1$, $\mathbf{a}+s\mathbf{u}_1+t\mathbf{u}_2$, ..., $\mathbf{a}+s\mathbf{u}_1+\cdots+s\mathbf{u}_{n-1}+t\mathbf{u}_n$ connecting $\mathbf{a}$ to $\mathbf{a}+\mathbf{u}s$, the limit of the ratio in (36) does not fall below 1 for any allowable path $\mathbf{a}+\mathbf{x}(t)_0^1 s$ connecting the same two points, then the inequality (37) must hold. Because of this $(\mathbf{u}_1, ..., \mathbf{u}_n)$ must be a minimizing chain for $\mathbf{u}\in\mathfrak{C}_{\mathbf{x}}^{(n)}$ at $\mathfrak{I}_{\mathbf{x}}$. We summarize these results in

Theorem 3.4.4 (Fechnerian Geodesics in the Small). *An n-tuple of vectors* $(\mathbf{u}_1\in\mathfrak{C}_{\mathbf{x}}^{(n)}, ..., \mathbf{u}_n\in\mathfrak{C}_{\mathbf{x}}^{(n)})$ *is a minimizing chain for* $\mathbf{u}\in\mathfrak{C}_{\mathbf{x}}^{(n)}$ *at* $\mathfrak{I}_{\mathbf{x}}$ *if and only if the sequence of segments* $\mathbf{s}_1(t)_0^s=\mathbf{a}+t\mathbf{u}_1$, $\mathbf{s}_2(t)_0^s=\mathbf{a}+s\mathbf{u}_1+t\mathbf{u}_2$, ..., $\mathbf{s}_n(t)_0^s=\mathbf{a}+s\mathbf{u}_1+\cdots+s\mathbf{u}_{n-1}+t\mathbf{v}_n$ *forms a Fechnerian geodesic in the small from* $\mathbf{a}$ *to* $\mathbf{a}+\mathbf{u}s$,

$$\lim_{s\to 0+}\frac{G(\mathbf{a}, \mathbf{a}+\mathbf{u}s)}{L[\mathbf{s}_1(t)_0^s]+\cdots+L[\mathbf{s}_n(t)_0^s]}=\lim_{s\to 0+}\frac{G(\mathbf{a}, \mathbf{a}+\mathbf{u}s)}{\widehat{F}(\mathbf{x}, \mathbf{u})\,s}=1.$$

This theorem has a corollary in many respects more important than the theorem itself.

COROLLARY TO THEOREM 3.4.4 (Differentiability of Fechnerian Distance). *$G(\mathbf{x}, \mathbf{x}+\mathbf{u}s)$ is differentiable at $s=0+$, the derivative being equal to the min-metric function $\widehat{F}(\mathbf{x}, \mathbf{u})$:*

$$\left.\frac{dG(\mathbf{x}, \mathbf{x}+\mathbf{u}s)}{ds}\right|_{s=0+} = \lim_{s\to 0+} \frac{G(\mathbf{x}, \mathbf{x}+\mathbf{u}s)}{s} = \widehat{F}(\mathbf{x}, \mathbf{u}).$$

3.5. *Shapes of Indicatrices and Convex Closure.* Refer to the geometric construction illustrated in Fig. 11: given a direction vector $\overline{OU}=\mathbf{u}\in\mathfrak{C}_{\mathbf{x}}^{(n)}$, there is its maximal possible extension $\overline{OU'}=\delta(\mathbf{x}, \bar{\mathbf{u}})\,\bar{\mathbf{u}}$, $\delta(\mathbf{x}, \bar{\mathbf{u}})\geqslant 1$, for which one can still find an n-gon $\overline{V_1\cdots V_n}$, with all its vertices on the contour of $\mathfrak{I}_{\mathbf{x}}$, that contains U' within its interior (the existence of this maximal extension is guaranteed by (34) and the existence of minimizing chains). For convenience, we refer to n-gons $\overline{V_1\cdots V_n}$ with the properties just mentioned as *terminal n-gons in the direction* $\mathbf{u}$. Both the value of $\delta(\mathbf{x}, \bar{\mathbf{u}})$ and the dimensionality of the terminal n-gons $\overline{V_1\cdots V_n}$ can be used for classification purposes.

Any terminal n-gon $\overline{V_1\cdots V_n}$ in the direction $\mathbf{u}$, if produced, forms a hyperplane of dimensionality $0\leqslant r\leqslant n-1$. We refer to the maximum value of r, across all possible terminal n-gons $\overline{V_1\cdots V_n}$ in a given direction, as the *order of flatness* of the indicatrix $\mathfrak{I}_{\mathbf{x}}$ in this direction (see the Appendix, Comment 7). Zero order, for example, indicates that the only terminal n-gon $\overline{V_1\cdots V_n}$ for $\mathbf{u}$ is a single point (it is easy to see then that this point can only be U, the endpoint of $\mathbf{u}$, i.e., $\delta(\mathbf{x}, \bar{\mathbf{u}})$ can only be 1); the order 1 indicates that some of the terminal n-gons $\overline{V_1\cdots V_n}$ for $\mathbf{u}$ are (collinear) straight line segments, while others in the same direction may only be single-points. The maximal order, $n-1$, indicates that some terminal n-gon $\overline{V_1\cdots V_n}$ is an $(n-1)$-dimensional hyperplanar face (Figs. 14, 15).

Figures 14 and 15 also illustrate the next notion. The indicatrix $\mathfrak{I}_{\mathbf{x}}$ is called *convex* or *concave* in the direction $\mathbf{u}$ according to whether $\delta(\mathbf{x}, \bar{\mathbf{u}})=1$ or $\delta(\mathbf{x}, \bar{\mathbf{u}})>1$, respectively. If $\mathfrak{I}_{\mathbf{x}}$ is convex in the direction $\mathbf{u}$, the order of flatness in this direction can be any number from zero to $n-1$: the case of convexity with zero order of flatness corresponds to the traditional notion of *strict convexity*. If $\mathfrak{I}_{\mathbf{x}}$ is concave in the direction $\mathbf{u}$, the order of flatness in this direction can be any number from 1 to $n-1$.

THEOREM 3.5.1 (Invariance under Diffeomorphisms). *Given an indicatrix $\mathfrak{I}_{\mathbf{x}}$ and a vector $\bar{\mathbf{u}}\in\mathfrak{I}_{\mathbf{x}}$, the value of $\delta(\mathbf{x}, \bar{\mathbf{u}})$ and the order of flatness in the direction $\bar{\mathbf{u}}$ are invariant under diffeomorphisms of the stimulus space.*

Proof. The invariance of $\delta(\mathbf{x}, \bar{\mathbf{u}})$ follows from (34) and from the invariance of $\widehat{F}(\mathbf{x}, \mathbf{u})$ (Theorem 3.4.3). The invariance of the order of flatness follows from the fact that the transformation (2) preserves all linear and incidence relations among the direction vectors. ▮

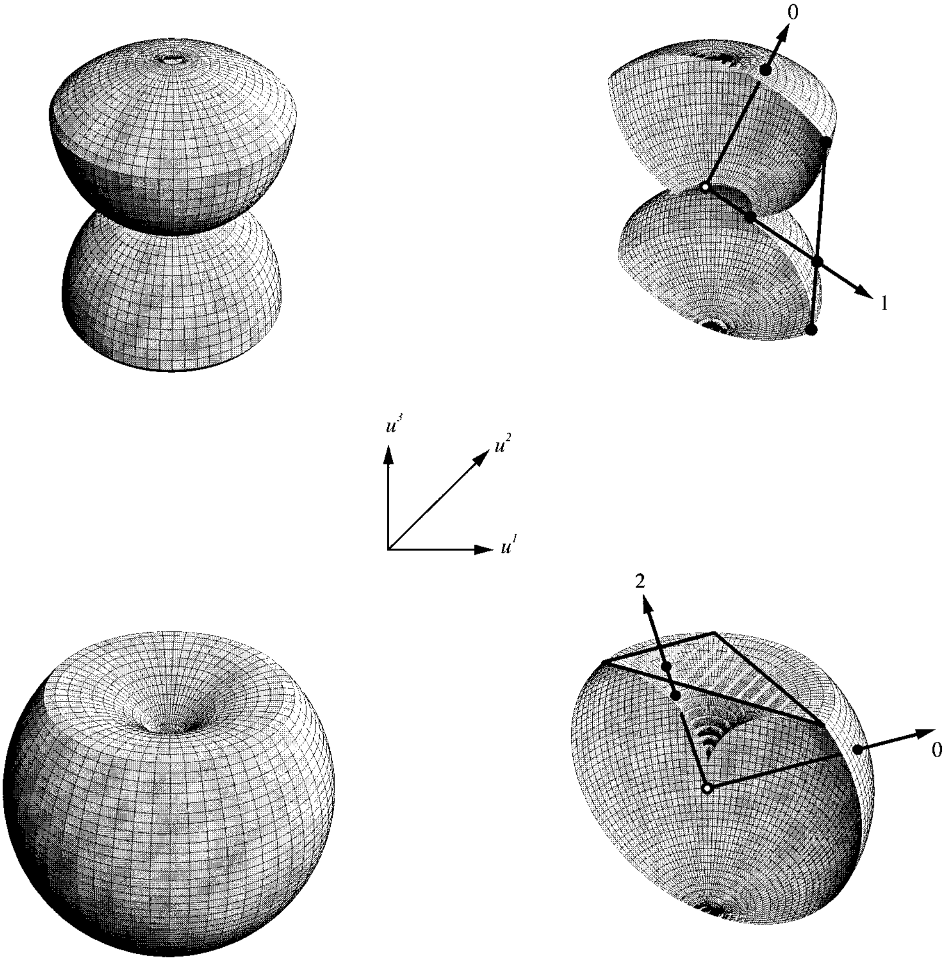

FIG. 14. Two three-dimensional indicatrices (left) with their cross-sections (right) are shown. The number attached to a direction vector indicates the order of flatness in this direction. Zero flatness is always associated with (strict) convexity. In the other directions shown, with the order of flatness 1 and 2, the indicatrices are concave.

THEOREM 3.5.2 (Convexity and Minimizing Chains). *Given an indicatrix* $\mathfrak{I}_{\mathbf{x}}$ *with a corresponding metric function* $F(\mathbf{x}, \mathbf{u})$ *and a direction vector* $\mathbf{u}$, *the following statements are equivalent*:

(i) $\mathfrak{I}_{\mathbf{x}}$ *is convex in the direction* $\mathbf{u}$;

(ii) $\widehat{F}(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, \mathbf{u})$;

(iii) $(\mathbf{u}_1 = \mathbf{u}/n, ..., \mathbf{u}_n = \mathbf{u}/n)$ *is a minimizing chain for* $\mathbf{u}$ *at* $\mathfrak{I}_{\mathbf{x}}$; *and*

(iv) *the segment* $\mathbf{s}(t)_0^s = \mathbf{x} + t\mathbf{u}$ *is a Fechnerian geodesic in the small from* $\mathbf{x}$ *to* $\mathbf{x} + \mathbf{u}s$.

Proof. Statement (ii) is equivalent to (i) because of (35). That statement (iii) is equivalent to (ii) follows from the definitions of the minimizing chains and the min-metric function. The equivalence of (iii) and (iv) follows from Theorem 3.4.4. ∎

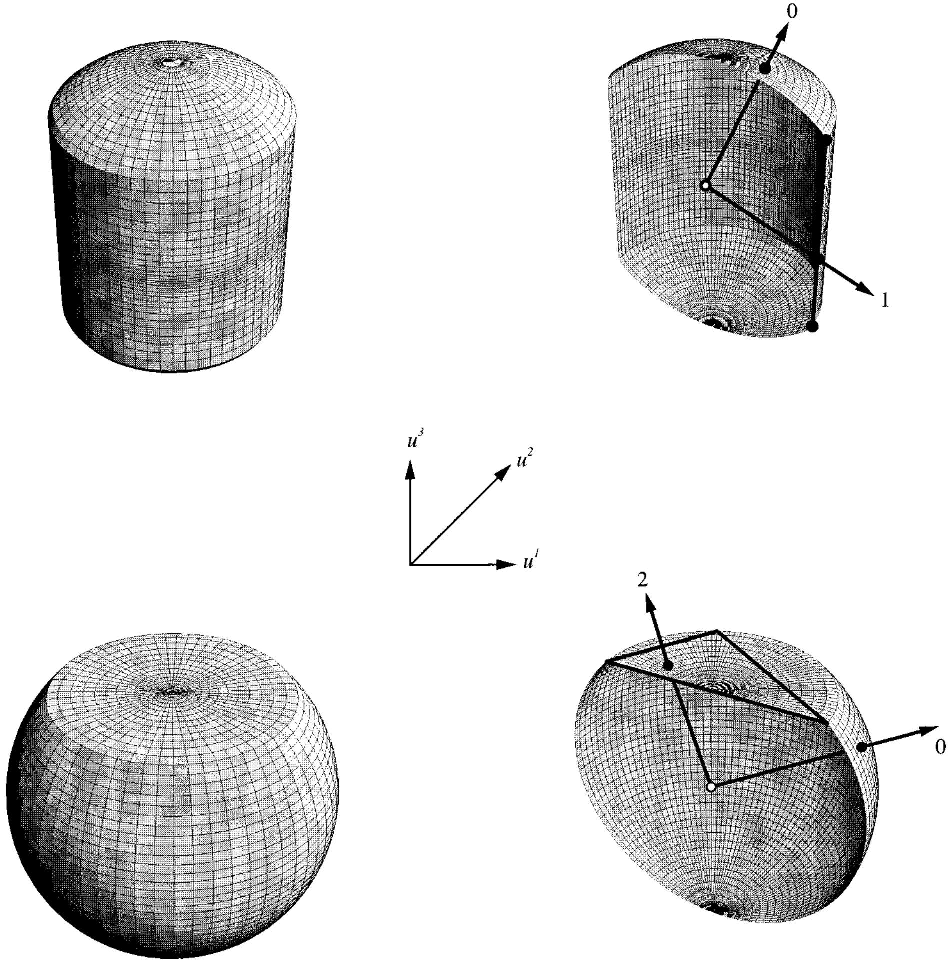

FIG. 15. The same as in Fig. 14, except that the indicatrices here are convex in the directions with the order of flatness 1 and 2.

The significance of the implication (i) → (iv) above is worth emphasizing. If the Fechnerian indicatrix $\mathfrak{I}_{\mathbf{x}}$ is convex in a direction $\mathbf{u}$, then the Fechnerian geodesic in the small that connects $\mathbf{x}$ to $\mathbf{x}+\mathbf{u}s$ is a straight line segment. It follows that we have

Corollary to Theorem 3.5.2 (Fechnerian Geodesics in the Small under Convexity of Indicatrices). *If all Fechnerian indicatrices $\mathfrak{I}_{\mathbf{x}}$ are convex in all directions, then, for any $(\mathbf{x}, \mathbf{u})$, the segment $\mathbf{s}(t)_0^s = \mathbf{x} + t\mathbf{u}$ is a Fechnerian geodesic in the small from $\mathbf{x}$ to $\mathbf{x}+\mathbf{u}s$, $s \to 0+$.*

The vector set

$$\widehat{\mathfrak{I}}_{\mathbf{x}} = \{\mathbf{u}\colon \mathbf{u} = \delta(\mathbf{x}, \bar{\mathbf{u}})\, \bar{\mathbf{u}}, \bar{\mathbf{u}} \in \mathfrak{I}_{\mathbf{x}}\}$$

is called the *convex closure* of the indicatrix $\mathfrak{I}_{\mathbf{x}}$ (Fig. 16): it is obtained by extending each vector $\bar{\mathbf{u}}$ of the indicatrix $\mathfrak{I}_{\mathbf{x}}$ by the factor of $\delta(\mathbf{x}, \bar{\mathbf{u}})$. Since $\mathbf{u}$ can always be

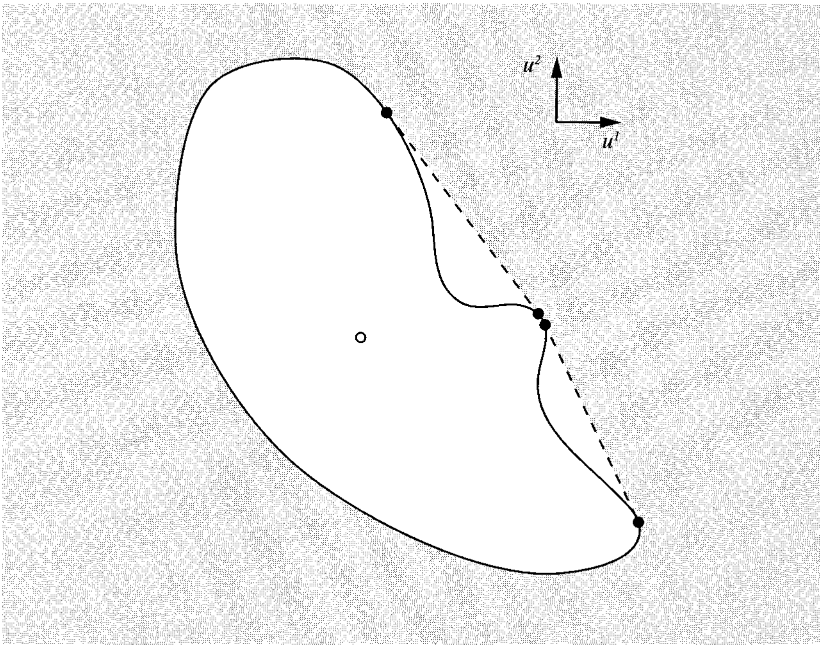

FIG. 16. Convex closure of an indicatrix ($n=2$) is shown. For $n=3$, the indicatrices shown in Fig. 15 are convex closures of the correspondingly placed indicatrices in Fig. 14.

presented as $F(\mathbf{x}, \mathbf{u})\, \bar{\mathbf{u}}$, the equation $\mathbf{u} = \delta(\mathbf{x}, \bar{\mathbf{u}})\, \bar{\mathbf{u}}$ is equivalent to $F(\mathbf{x}, \mathbf{u}) = \delta(\mathbf{x}, \bar{\mathbf{u}})$, and, on taking into account (35), it is equivalent to $\widehat{F}(\mathbf{x}, \mathbf{u}) = 1$. The convex closure of the indicatrix $\mathfrak{I}_\mathbf{x}$ can therefore be presented as

$$\widehat{\mathfrak{I}}_\mathbf{x} = \{\mathbf{u} \in \mathfrak{C}_\mathbf{x}^{(n)} : \widehat{F}(\mathbf{x}, \mathbf{u}) = 1\}. \tag{38}$$

The similarity with the definition of the indicatrix $\mathfrak{I}_\mathbf{x}$ itself, (17), is more than superficial. It turns out that

Theorem 3.5.3 (Convex Closure Theorem). *The min-metric function $\widehat{F}(\mathbf{x}, \mathbf{u})$ associated with an indicatrix $\mathfrak{I}_\mathbf{x}$ is a metric function: that is, $\widehat{F}(\mathbf{x}, \mathbf{u})$ is a positive, continuous, and Euler homogeneous function defined on the set of all line elements and invariant under all stimulus space diffeomorphisms. The indicatrix corresponding to $\widehat{F}(\mathbf{x}, \mathbf{u})$ is the convex closure $\widehat{\mathfrak{I}}_\mathbf{x}$ of the indicatrix $\mathfrak{I}_\mathbf{x}$.*

Proof. The positiveness and Euler homogeneity of $\widehat{F}(\mathbf{x}, \mathbf{u})$ directly follow from the fact that $\widehat{F}(\mathbf{x}, \mathbf{u})$ is $F(\mathbf{x}, \mathbf{u}_1) + \cdots + F(\mathbf{x}, \mathbf{u}_n)$, for some direction vectors $\mathbf{u}_1, ..., \mathbf{u}_n$. The invariance under stimulus space diffeomorphisms is proved in Theorem 3.4.2. It remains to prove the continuity of $\widehat{F}(\mathbf{x}, \mathbf{u})$ in $(\mathbf{x}, \mathbf{u})$.

Suppose that $\mathbf{u}_k \to \mathbf{u}$, $(\mathbf{u}_1, ..., \mathbf{u}_n)$ is a minimizing chain for $\mathbf{u}$, and $(\mathbf{u}_{1k}, ..., \mathbf{u}_{nk})$ is a minimizing chain for $\mathbf{u}_k$, $k = 1, 2, ...$. Obviously, one can find a sequence of vector n-tuples $(\mathbf{v}_{1k}, ..., \mathbf{v}_{nk})$ such that $\mathbf{v}_{1k} + \cdots + \mathbf{v}_{nk} = \mathbf{u}_k$ and $(\mathbf{v}_{1k}, ..., \mathbf{v}_{nk}) \to (\mathbf{u}_1, ..., \mathbf{u}_n)$; and one can find a sequence of vector n-tuples $(\mathbf{w}_{1k}, ..., \mathbf{w}_{nk})$ such that $\mathbf{w}_{1k} + \cdots + \mathbf{w}_{nk} = \mathbf{u}$ and $|(\mathbf{w}_{1k}, ..., \mathbf{w}_{nk}) - (\mathbf{u}_{1k}, ..., \mathbf{u}_{nk})| \to 0$. Due to the continuity of the metric function, if $\mathbf{x}_k \to \mathbf{x}$, then

$$F(\mathbf{x}_k, \mathbf{v}_{1k}) + \cdots + F(\mathbf{x}_k, \mathbf{v}_{nk}) \to F(\mathbf{x}, \mathbf{u}_1) + \cdots + F(\mathbf{x}, \mathbf{u}_n) = \widehat{F}(\mathbf{x}, \mathbf{u}),$$

$$|[F(\mathbf{x}, \mathbf{w}_{1k}) + \cdots + F(\mathbf{x}, \mathbf{w}_{nk})] - [F(\mathbf{x}_k, \mathbf{u}_{1k}) + \cdots + F(\mathbf{x}_k, \mathbf{u}_{nk})]|$$
$$= |[F(\mathbf{x}, \mathbf{w}_{1k}) + \cdots + F(\mathbf{x}, \mathbf{w}_{nk})] - \widehat{F}(\mathbf{x}_k, \mathbf{u}_k)| \to 0.$$

From the first of these limit statements we deduce that, for any $\varepsilon > 0$, one can choose k_ε so that

$$\widehat{F}(\mathbf{x}_k, \mathbf{u}_k) \leqslant \widehat{F}(\mathbf{x}, \mathbf{u}) \pm \varepsilon, \qquad k > k_\varepsilon.$$

Analogously, we deduce from the second limit statement that

$$\widehat{F}(\mathbf{x}, \mathbf{u}) \leqslant \widehat{F}(\mathbf{x}_k, \mathbf{u}_k) \pm \varepsilon, \qquad k > k_\varepsilon.$$

Since ε can be made arbitrarily small, it follows that, as $\mathbf{u}_k \to \mathbf{u}$ and $\mathbf{x}_k \to \mathbf{x}$,

$$\widehat{F}(\mathbf{x}_k, \mathbf{u}_k) \to \widehat{F}(\mathbf{x}, \mathbf{u}).$$

That the indicatrix corresponding to $\widehat{F}(\mathbf{x}, \mathbf{u})$ is the convex closure $\widehat{\mathfrak{I}}_\mathbf{x}$ of the indicatrix $\mathfrak{I}_\mathbf{x}$ is true by definition, (38). ▮

The proof of the following theorem trivially follows from the construction of $\widehat{\mathfrak{I}}_\mathbf{x}$.

THEOREM 3.5.4 (Properties of Convex Closure).

(i) *The convex closure $\widehat{\mathfrak{I}}_\mathbf{x}$ of an indicatrix $\mathfrak{I}_\mathbf{x}$ is convex in all directions.*

(ii) *For any given direction, $\widehat{\mathfrak{I}}_\mathbf{x}$ has the same degree of flatness as the indicatrix* $=\mathfrak{I}_\mathbf{x}$.

(iii) *An indicatrix coincides with its convex closure, $\widehat{\mathfrak{I}}_\mathbf{x} = \mathfrak{I}_\mathbf{x}$, if and only if $\mathfrak{I}_\mathbf{x}$ is convex in all directions. In particular, $\widehat{\widehat{\mathfrak{I}}}_\mathbf{x} = \widehat{\mathfrak{I}}_\mathbf{x}$.*

(iv) *$\widehat{\mathfrak{I}}_\mathbf{x}$ is contained within any convex body that contains $\mathfrak{I}_\mathbf{x}$.*

For completeness, having shown that $\widehat{\mathfrak{I}}_\mathbf{x}$ is an indicatrix corresponding to a metric function $\widehat{F}(\mathbf{x}, \mathbf{u})$, it is natural to introduce also the corresponding unit-vector function,

$$\widehat{\mathbf{1}}_\mathbf{x}(\mathbf{u}) = \frac{\mathbf{u}}{\widehat{F}(\mathbf{x}, \mathbf{u})} = \frac{\mathbf{1}_\mathbf{x}(\mathbf{u})}{\delta(\mathbf{x}, \mathbf{1}_\mathbf{x}(\mathbf{u}))}. \tag{39}$$

This function, which is convenient to refer to as the *convex closure of the unit-vector function* $\mathbf{1}_\mathbf{x}(\mathbf{u})$, uniquely represents the indicatrix $\widehat{\mathfrak{I}}_\mathbf{x}$.

Recall now the Corollary to Theorem 3.4.4: the min-metric function $\widehat{F}(\mathbf{x}, \mathbf{u})$ associated with an indicatrix $\mathfrak{I}_\mathbf{x}$ can be obtained by differentiating, at $s = 0+$, the Fechnerian distance $G(\mathbf{x}, \mathbf{x} + \mathbf{u}s)$ induced by the same indicatrix $\mathfrak{I}_\mathbf{x}$. One comes to a very interesting mathematical situation. By means of one indicatrix, $\mathfrak{I}_\mathbf{x}$, or, equivalently, the corresponding metric function $F(\mathbf{x}, \mathbf{u})$, one uniquely computes a certain Fechnerian metric $G(\mathbf{a}, \mathbf{b})$, following the procedure described earlier. Then from this metric $G(\mathbf{a}, \mathbf{b})$ one uniquely obtains another indicatrix, $\widehat{\mathfrak{I}}_\mathbf{x}$, that corresponds to the min-metric function $\widehat{F}(\mathbf{x}, \mathbf{u})$. Clearly, the indicatrix $\widehat{\mathfrak{I}}_{\mathbf{x}}$ or, equivalently, $\widehat{F}(\mathbf{x}, \mathbf{u})$, induces on the stimulus space a certain internal metric. Denoting this metric by $\widehat{G}(\mathbf{a}, \mathbf{b})$, one is naturally led to the question: How does $\widehat{G}(\mathbf{a}, \mathbf{b})$ relate to the Fechnerian metric $G(\mathbf{a}, \mathbf{b})$? The answer is remarkable.

Theorem 3.5.5 (The Busemann–Mayer Identity). $\widehat{G}(\mathbf{a}, \mathbf{b}) = G(\mathbf{a}, \mathbf{b})$.

Proof. (This is the main theorem in Busemann & Mayer, 1941, proved there in a very different way.) Since $\widehat{F}(\mathbf{x}, \mathbf{u}) \leqslant F(\mathbf{x}, \mathbf{u})$, it follows that $\widehat{L}[\mathbf{x}(t)_a^b] \leqslant L[\mathbf{x}(t)_a^b]$ and $\widehat{G}(\mathbf{a}, \mathbf{b}) \leqslant G(\mathbf{a}, \mathbf{b})$. At the same time, by the Corollary to Theorem 3.4.3,

$$\begin{aligned}\widehat{L}[\mathbf{x}(t)_a^b] &= \int_a^b \widehat{F}[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\,dt = \int_a^b \left.\frac{dG[\mathbf{x}(t), \mathbf{x}(t) + \dot{\mathbf{x}}(t)\,s]}{ds}\right|_{s=0+} dt \\ &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{dG[\mathbf{x}(t), \mathbf{x}(t) + \dot{\mathbf{x}}(t)\,dt]}{dt}\,dt \\ &= \int_a^b G[\mathbf{x}(t), \mathbf{x}(t+dt)] \geqslant G[\mathbf{x}(a), \mathbf{x}(b)],\end{aligned}$$

where the partition $a = t_0 < t_1 < \cdots < t_n = b$ is chosen so that $\dot{\mathbf{x}}(t)$ exists on each of the intervals $[t_{i-1}, t_i]$. It follows that $\widehat{G}(\mathbf{a}, \mathbf{b}) \geqslant G(\mathbf{a}, \mathbf{b})$, hence $\widehat{G}(\mathbf{a}, \mathbf{b}) = G(\mathbf{a}, \mathbf{b})$. ▮

Thus, the Fechnerian metric induced by a Fechnerian indicatrix (equivalently, a Fechner–Finsler metric function) is also induced by the convex closure of this indicatrix (equivalently, the min-metric function associated with the Fechnerian indicatrix). Some of the immediate consequences of this statement are as follows.

Corollary 1 to Theorem 3.5.5 (Fechnerian Triad). *The Fechnerian metric* $G(\mathbf{a}, \mathbf{b})$, *the convex closure* $\widehat{\mathfrak{I}}_{\mathbf{x}}$ *of the Fechnerian indicatrix* $\mathfrak{I}_{\mathbf{x}}$, *and the min-metric function* $\widehat{F}(\mathbf{x}, \mathbf{u})$ *associated with* $\mathfrak{I}_{\mathbf{x}}$ *determine each other uniquely.*

Corollary 2 to Theorem 3.5.5 (Equivalence of Indicatrices). *Different Fechnerian indicatrices induce one and the same Fechnerian metric* $G(\mathbf{a}, \mathbf{b})$ *if and only if they have one and the same convex closure.*

It must be clear now why the function $\widehat{F}(\mathbf{x}, \mathbf{u})$ is referred to as the min-metric function: among all possible metric functions generating a given Fechnerian metric (Fig. 17), $\widehat{F}(\mathbf{x}, \mathbf{u})$ has the smallest possible value for any $(\mathbf{x}, \mathbf{u})$.

In view of the Corollary to Theorem 3.5.2, we also have

Corollary 3 to Theorem 3.5.5 (Geodesics in the Small under Convex Closure). *For any Fechnerian metric* $G(\mathbf{a}, \mathbf{b})$, *if the indicatrix* $\mathfrak{I}_{\mathbf{x}}$ *that induces it is replaced with its convex closure,* $\widehat{\mathfrak{I}}_{\mathbf{x}}$, *then, for any* $(\mathbf{x}, \mathbf{u})$, *the segment* $\mathbf{s}(t)_0^s = \mathbf{x} + t\mathbf{u}$ *is a Fechnerian geodesic in the small from* $\mathbf{x}$ *to* $\mathbf{x} + \mathbf{u}s$, $s \to 0+$.

We conclude this subsection by a remark on the issue of metric symmetry, $G(\mathbf{a}, \mathbf{b}) = G(\mathbf{b}, \mathbf{a})$. The symmetry is guaranteed by the Third Assumption, but, as shown in Fig. 18, it is possible that $G(\mathbf{a}, \mathbf{b}) = G(\mathbf{b}, \mathbf{a})$ while the Third Assumption

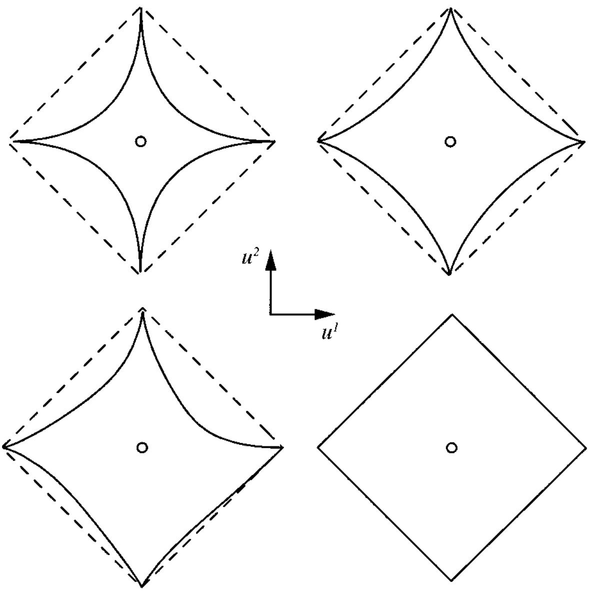

FIG. 17. All the indicatrices shown correspond to the same min-metric function and locally induce the same Fechnerian metric. The last indicatrix coincides with its convex closure and corresponds to the min-metric function.

does not hold (which is another reason for considering this assumption to be secondary in importance). One can now formulate the necessary and sufficient conditions for the metric symmetry.

THEOREM 3.5.6 (Symmetry). *A Fechnerian metric is symmetrical,* $G(\mathbf{a}, \mathbf{b}) = G(\mathbf{b}, \mathbf{a})$, *if and only if the min-metric function inducing this metric is symmetrical,* $\widehat{F}(\mathbf{x}, \mathbf{u}) = \widehat{F}(\mathbf{x}, -\mathbf{u})$.

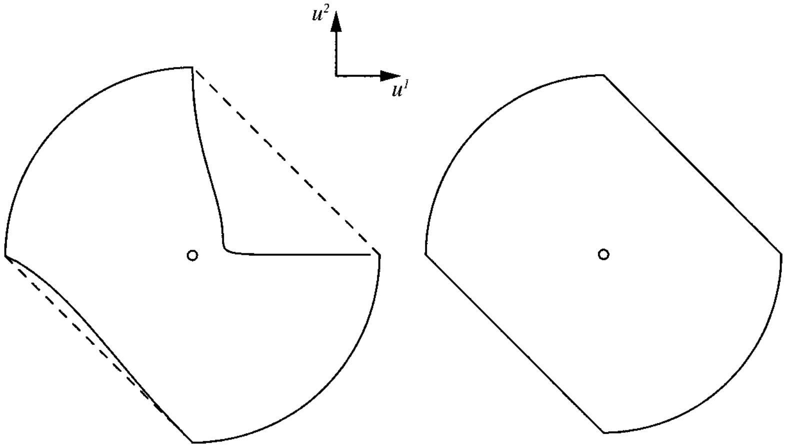

FIG. 18. The left indicatrix is not symmetrical but it has a symmetrical convex closure (the right indicatrix), locally inducing thereby a symmetrical Fechnerian metric.

Proof. The sufficiency is obvious. The necessity follows from the Corollary to Theorem 3.4.4, on observing that

$$\left.\frac{dG(\mathbf{x}+\mathbf{u}s,\mathbf{x})}{ds}\right|_{s=0+}=\lim_{s\to 0+}\frac{G[(\mathbf{x}+\mathbf{u}s),(\mathbf{x}+\mathbf{u}s)-\mathbf{u}s]}{s}$$
$$=\lim_{s\to 0+}\widehat{F}(\mathbf{x}+\mathbf{u}s,\ -\mathbf{u})=\widehat{F}(\mathbf{x},\ -\mathbf{u}).\quad\blacksquare$$

3.6. *Indicatrices and psychometric functions.* In spite of the abstract character and great generality of the results established so far, one should keep in mind that the notion and properties of the Fechner–Finsler metric function $F(\mathbf{x},\mathbf{u})$, upon which all the results of this section are based, are derived from properties of psychometric functions (Section 2). Here, we return to an explicit analysis of the psychometric functions in order to establish the empirical meaning of the principal notions involved.

Given a psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$, its *horizontal cross-section at an elevation* $h>0$ *from its minimum* (Fig. 19) is defined as the set of stimuli

$$\mathfrak{F}_{\mathbf{x},h}=\{\mathbf{y}\colon \psi_{\mathbf{x}}(\mathbf{y})-\psi_{\mathbf{x}}(\mathbf{x})=h\},\tag{40}$$

considered together with its *center* $\mathbf{x}$. Even though the set $\mathfrak{F}_{\mathbf{x},h}$ lies within the stimulus space $\mathfrak{M}^{(n)}$, it can be put in a linear correspondence with $\mathfrak{C}_{\mathbf{x}}^{(n)}$,

$$\mathbf{y}\in\mathfrak{F}_{\mathbf{x},h}\leftrightarrow\mathbf{u}\in\mathfrak{C}_{\mathbf{x}}^{(n)}\qquad\text{if and only if}\quad \mathbf{y}-\mathbf{x}=\mathbf{u}.\tag{41}$$

This correspondence allows one to establish the geometric relationship of $\mathfrak{F}_{\mathbf{x},h}$ to the Fechnerian indicatrix $\mathfrak{I}_{\mathbf{x}}$ attached to the center of $\mathfrak{F}_{\mathbf{x},h}$.

Due to the First Assumption, one should be able, by choosing $h^*(\mathbf{x})$ sufficiently small, to ensure that for all direction vectors $\mathbf{u}\in\mathfrak{C}_{\mathbf{x}}^{(n)}$, the psychometric differential in (5),

$$h=\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x}),$$

increases in $s\geqslant 0$ whenever $h<h^*(\mathbf{x})$. This implies that, for any direction vector $\mathbf{u}$ and any $h<h^*(\mathbf{x})$, one can uniquely find the stimulus $\mathbf{y}\in\mathfrak{F}_{\mathbf{x},h}$ such that $\mathbf{z}=\mathbf{y}-\mathbf{x}$ is codirectional with $\mathbf{u}$. The cross-section $\mathfrak{F}_{\mathbf{x},h}$, therefore, can be represented by a vector function of direction,

$$\mathbf{y}-\mathbf{x}=\mathbf{z}_{\mathbf{x},h}(\mathbf{u})\tag{42}$$

(that $h<h^*(\mathbf{x})$ is hereafter assumed tacitly). As s in the psychometric differential is an increasing function of $h>0$, vanishing at $h=0$, (6), we have

$$\mathbf{z}_{\mathbf{x},h}(\mathbf{u})=\mathbf{u}\Phi_{\mathbf{x},\mathbf{u}}(h).\tag{43}$$

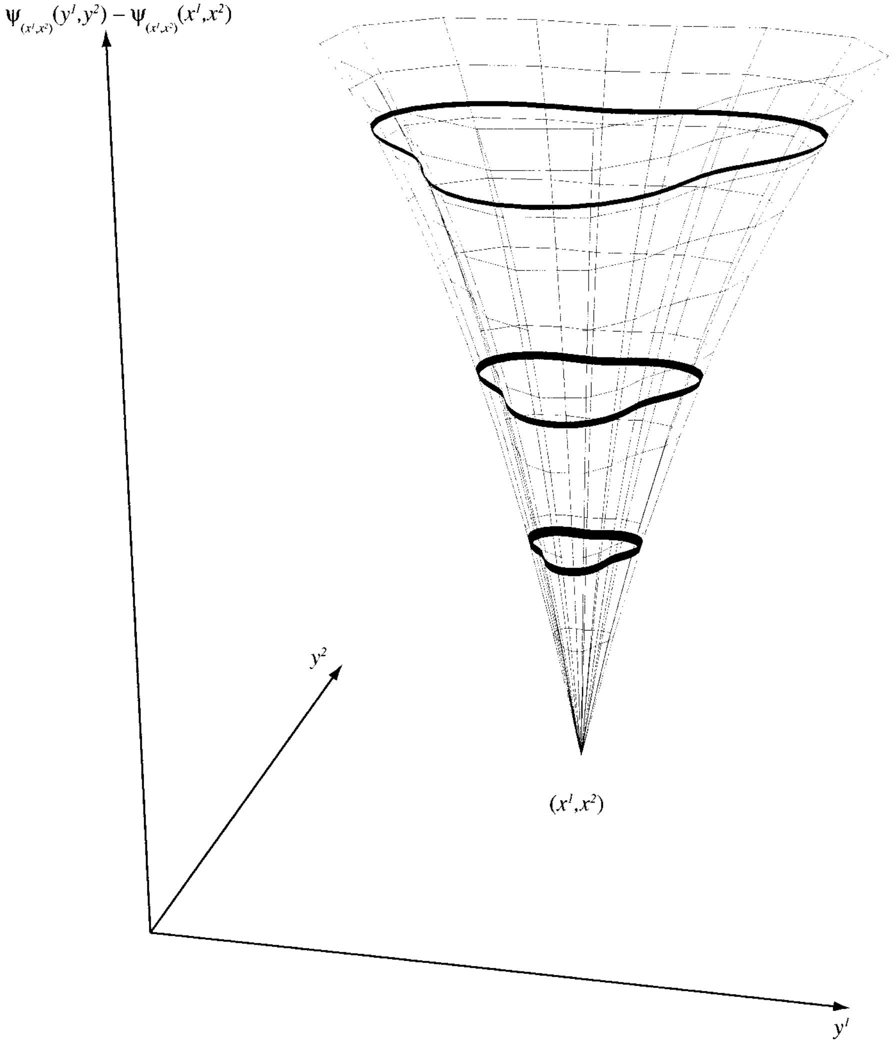

FIG. 19. Horizontal cross-sections of a psychometric function at three elevations from its minimum.

Clearly, for any $k > 0$,

$$\mathbf{z}_{\mathbf{x},\,h}(k\mathbf{u}) = \mathbf{z}_{\mathbf{x},\,h}(\mathbf{u}), \tag{44}$$

$$\Phi_{\mathbf{x},\,k\mathbf{u}}(h) = \frac{1}{k}\,\Phi_{\mathbf{x},\,\mathbf{u}}(h). \tag{45}$$

Consider now the indicatrix $\mathfrak{I}_{\mathbf{x}}$, attached to the center of $\mathfrak{F}_{\mathbf{x},\,h}$, and the associated unit-vector function $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$. By definition,

$$\lim_{s \to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{1}_{\mathbf{x}}(\mathbf{u})\,s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s} = F[\mathbf{x}, \mathbf{1}_{\mathbf{x}}(\mathbf{u})] = 1.$$

This can also be written as

$$\lim_{s \to 0+} \frac{s}{\Phi[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{1}_{\mathbf{x}}(\mathbf{u})\, s) - \psi_{\mathbf{x}}(\mathbf{x})]} = 1.$$

Viewing s as a function of h, (6), we have

$$\lim_{h \to 0+} \frac{\Phi_{\mathbf{x},\, \mathbf{1}_{\mathbf{x}}(\mathbf{u})}(h)}{\Phi(h)} = 1.$$

Due to (43), this is equivalent to

$$\lim_{h \to 0+} \frac{\mathbf{z}_{\mathbf{x},\, h}[\mathbf{1}_{\mathbf{x}}(\mathbf{u})]}{\Phi(h)} = \lim_{h \to 0+} \frac{\mathbf{1}_{\mathbf{x}}(\mathbf{u})\, \Phi_{\mathbf{x},\, \mathbf{1}_{\mathbf{x}}(\mathbf{u})}(h)}{\Phi(h)} = \mathbf{1}_{\mathbf{x}}(\mathbf{u}),$$

which, in view of (44), can be written as

$$\lim_{h \to 0+} \frac{1}{\Phi(h)}\, \mathbf{z}_{\mathbf{x},\, h}(\mathbf{u}) = \mathbf{1}_{\mathbf{x}}(\mathbf{u}). \tag{46}$$

On recalling (42), we have proved (see the Appendix, Comment 8)

THEOREM 3.6.1 (Homothety between Indicatrix and Horizontal Cross-section). *The horizontal cross-section* $\mathfrak{F}_{\mathbf{x},\, h}$, *represented* (*for sufficiently small* h) *by a vector function* $\mathbf{z}_{\mathbf{x},\, h}(\mathbf{u})$, *and the Fechnerian indicatrix* $\mathfrak{I}_{\mathbf{x}}$, *represented by the unit-vector function* $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$, *are asymptotically homothetic*,

$$\frac{\mathbf{z}_{\mathbf{x},\, h}(\mathbf{u})}{\mathbf{1}_{\mathbf{x}}(\mathbf{u})} = \Phi(h) + o\{\Phi(h)\}, \qquad h \to 0+, \tag{47}$$

with the coefficient of homothety $\Phi(h)$ *equal to the global psychometric transformation* (*and therefore one and the same for all* $\mathbf{x}$ *and* $\mathbf{u}$).

Recall that the Fechner–Finsler metric function and, hence, the Fechnerian indicatrices, too, are determined up to an arbitrary scaling factor $k > 0$. Because of this, assuming that one is able to choose $h = h_0$ so small that $o\{\Phi(h_0)\}/\Phi(h_0)$ in (47) is negligible, one can replace $\Phi(h_0)$ with any arbitrary constant, say, unity. In other words, one can take the cross-section $\mathfrak{F}_{\mathbf{x},\, h_0}$ as a direct approximation for the concentric Fechnerian indicatrix (Fig. 20),

$$\frac{\mathbf{z}_{\mathbf{x},\, h_0}(\mathbf{u})}{\mathbf{1}_{\mathbf{x}}(\mathbf{u})} \approx 1. \tag{48}$$

The dependence of the error value $o\{\Phi(h)\}/\Phi(h)$ on the direction vector $\mathbf{u}$ is immaterial, because both $\mathbf{z}_{\mathbf{x},\, h_0}(\mathbf{u})$ and $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$ are invariant under positive scaling of $\mathbf{u}$, (44), and (19). The ratio in (47), therefore, is a continuous function of $\mathbf{1}_{\mathbf{x}}(\mathbf{u})$, because of which $o\{\Phi(h)\}/\Phi(h)$ is bounded from above for any given h. As a result,

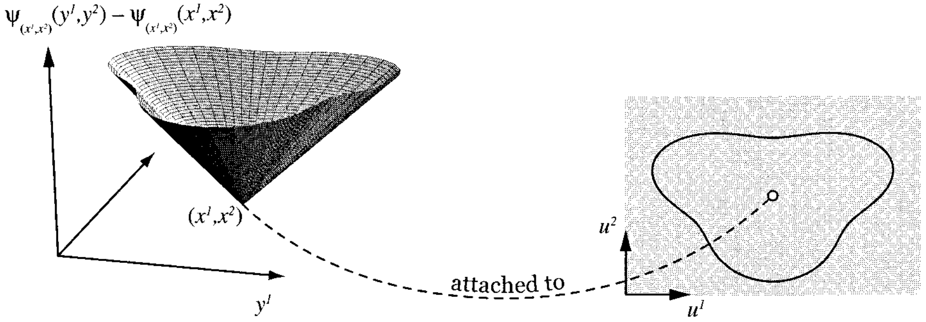

FIG. 20. A horizontal cross-section at a very small elevation is homothetic to and can be identified with the Fechnerian indicatrix attached to the minimum point of the psychometric function.

the value $h = h_0$ at which $o\{\Phi(h_0)\}/\Phi(h_0)$ is negligible can be chosen the same for all $\mathbf{u}$, for any given $\mathbf{x}$.

Unless additional assumptions are introduced (which we are not willing to do in the basic theory), however, one cannot deal similarly with the dependence of the error value on $\mathbf{x}$, as well as the dependence on $\mathbf{x}$ of the bound $h^*(\mathbf{x})$ brought up above. One has to keep in mind, therefore, that the approximation (48), with one and the same elevation h_0 used to cross-section more than one psychometric function, requires that the stimuli $\mathbf{x}$ be confined to a compact (but otherwise arbitrarily large) subset of the stimulus space $\mathfrak{M}^{(n)}$. In practice, this constraint is not stringent, as the stimulus space has to be approximated by a finite mesh of stimuli anyway, with the indicatrices for the remaining values being subsequently interpolated and extrapolated from those computed on this mesh.

The operational meaning of saying that h_0 is sufficiently small to warrant the approximation (48) can be presented in the following way. It follows from (47) that

$$\frac{\mathbf{z}_{\mathbf{x},\,h_2}(\mathbf{u})}{\mathbf{z}_{\mathbf{x},\,h_1}(\mathbf{u})}=\frac{\Phi(h_2)+o\{\Phi(h_2)\}}{\Phi(h_1)+o\{\Phi(h_1)\}}=\frac{\Phi(h_2)}{\Phi(h_1)}+o\left\{\frac{\Phi(h_2)}{\Phi(h_1)}\right\}, \qquad h_2<h_1\to 0. \tag{49}$$

Unlike (47), this ratio involves only observable quantities. The approximation error

$$\frac{o\{\Phi(h_2)/\Phi(h_1)\}}{\Phi(h_2)/\Phi(h_1)}$$

can also be expressed in terms of the observables, as

$$\frac{\mathbf{z}_{\mathbf{x},\,h_2}(\mathbf{u})/\mathbf{z}_{\mathbf{x},\,h_1}(\mathbf{u})-\mathbf{z}_{\mathbf{x}_0,\,h_2}(\mathbf{u}_0)/\mathbf{z}_{\mathbf{x}_0,\,h_1}(\mathbf{u}_0)}{\mathbf{z}_{\mathbf{x}_0,\,h_2}(\mathbf{u}_0)/\mathbf{z}_{\mathbf{x}_0,\,h_1}(\mathbf{u}_0)}, \tag{50}$$

where $(\mathbf{x}_0, \mathbf{u}_0)$ is an arbitrarily chosen line element. Then an elevation h_0 can be considered sufficiently small to warrant (48), if (50) is shown to be smaller than a preset level of error, whenever $h_2 < h_1 < h_0$, for all directions and all stimuli within a compact subset of $\mathfrak{M}^{(n)}$ (Fig. 21).

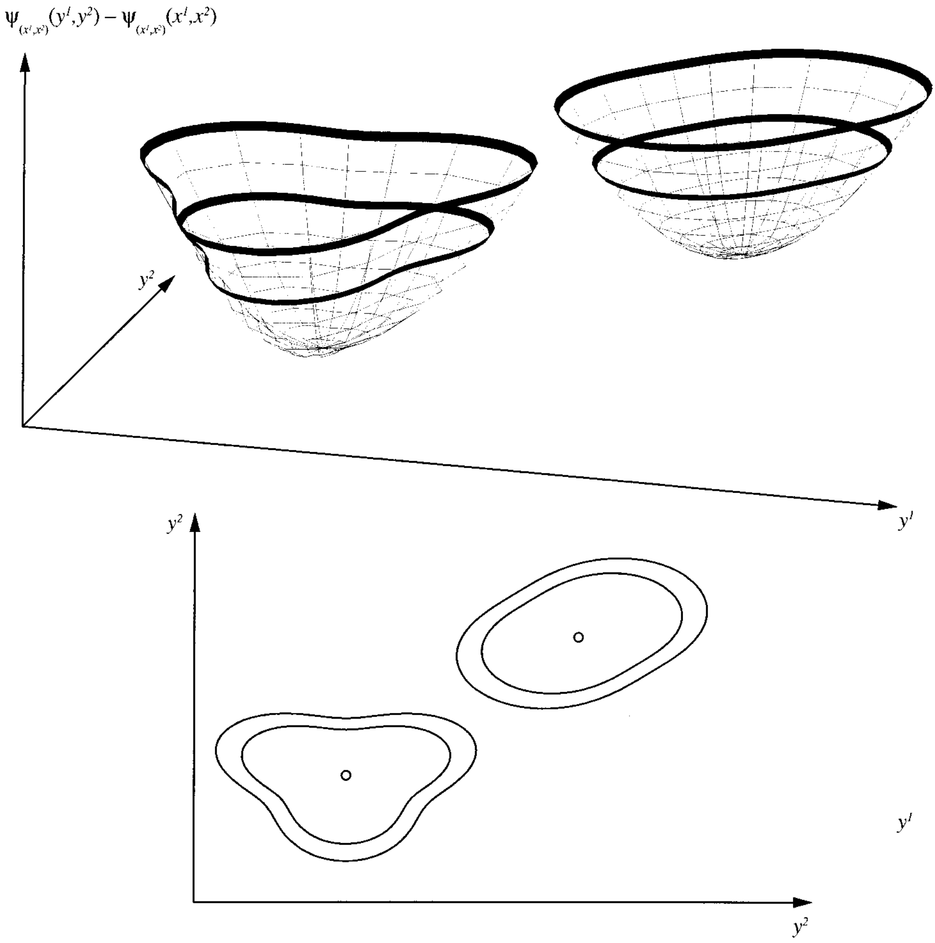

FIG. 21. If the horizontal cross-sections made at two small elevations are homothetic, with one and the same coefficient of homothety across a compact set of stimuli, then the elevations can be considered sufficiently small to warrant the identification of the Fechnerian indicatrices with the cross-sections made at either of the two elevations.

We come to a remarkable conclusion: the Fechnerian indicatrices can be computed, to any desired degree of approximation, with no knowledge of the global psychometric transformation involved. Since the Fechnerian indicatrices determine a Fechnerian metric uniquely, one can also say that no knowledge of the global psychometric transformation is required to compute the Fechnerian metric, provided that the indicatrices can be with a sufficient precision extrapolated from a compact subset of $\mathfrak{M}^{(n)}$ to the entire stimulus space. The existence of the global psychometric transformation, however, one and the same for all stimuli and directions of change, remains the fundamental assumption underlying the entire enterprise of Fechnerian scaling, including the approximations just described.

The homothetic relationship between $\mathfrak{F}_{\mathbf{x},h}$ and the Fechnerian indicatrix $\mathfrak{I}_{\mathbf{x}}$ stated in Theorem 3.6.1 is established by linearly projecting $\mathfrak{F}_{\mathbf{x},h}$ on the tangent space $\mathfrak{T}_{\mathbf{x}}^{(n)}$, by means of (41). It is also of interest, however, to simply take $\mathfrak{F}_{\mathbf{x},h}$ for what it is, a set of stimuli, and to find out Fechnerian distances between the central point $\mathbf{x}$ and these stimuli. In other words, we are interested in the value of

$$G(\mathbf{x}, \mathbf{y}) = G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})], \qquad h \to 0+.$$

Due to (47),

$$G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})] = G[\mathbf{x}, \mathbf{x} + \mathbf{1}_{\mathbf{x}}(\mathbf{u})(\Phi(h) + o\{\Phi(h)\})],$$

and

$$\lim_{h \to 0+} \frac{G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})]}{\Phi(h)} = \lim_{\Phi(h) \to 0+} \frac{G[\mathbf{x}, \mathbf{x} + \mathbf{1}_{\mathbf{x}}(\mathbf{u})(\Phi(h) + o\{\Phi(h)\})]}{\Phi(h)}$$
$$= \lim_{s \to 0+} \frac{G[\mathbf{x}, \mathbf{x} + \mathbf{1}_{\mathbf{x}}(\mathbf{u}) s]}{s}.$$

Then, by the Corollary to Theorem 3.4.4,

$$\lim_{h \to 0+} \frac{G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})]}{\Phi(h)} = \lim_{s \to 0+} \frac{G[\mathbf{x}, \mathbf{x} + \mathbf{1}_{\mathbf{x}}(\mathbf{u}) s]}{s}$$
$$= \widehat{F}[\mathbf{x}, \mathbf{1}_{\mathbf{x}}(\mathbf{u})] = \widehat{F}\left[\mathbf{x}, \frac{\mathbf{u}}{F(\mathbf{x}, \mathbf{u})}\right] = \frac{\widehat{F}(\mathbf{x}, \mathbf{u})}{F(\mathbf{x}, \mathbf{u})},$$

and we have proved

THEOREM 3.6.2 (Center-to-Contour Distances for Horizontal Cross-section). *The Fechnerian distance from* $\mathbf{x}$ *to the contour of the horizontal cross-section* $\mathfrak{F}_{\mathbf{x}, h}$ *of* $\psi_{\mathbf{x}}(\mathbf{y})$ *at an elevation* h *is*

$$G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})] = \frac{\widehat{F}(\mathbf{x}, \mathbf{u})}{F(\mathbf{x}, \mathbf{u})} \Phi(h) + o\{\Phi(h)\}.$$

We see that $\mathfrak{F}_{\mathbf{x}, h}$ generally is not a Fechnerian sphere: the stimuli on its contour are not (asymptotically) equidistant from the center. It is easy to see that we have

COROLLARY TO THEOREM 3.6.2 (Convex Horizontal Cross-sections as Fechnerian Spheres). *The Fechnerian distances* $G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})]$ *in different directions* $\mathbf{u}$ *are asymptotically equal to each other,*

$$G[\mathbf{x}, \mathbf{x} + \mathbf{z}_{\mathbf{x}, h}(\mathbf{u})] = \Phi(h) + o\{\Phi(h)\},$$

if and only if the Fechnerian indicatrix $\mathfrak{I}_{\mathbf{x}}$ *is convex in all directions.*

None of the assumptions made in this paper guarantees that the horizontal cross-sections of psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ have convex contours at sufficiently small elevations h (see the Appendix, Comment 9). Due to Theorem 3.5.5, however, to compute a Fechnerian metric one can always replace the Fechnerian indicatrix $\mathfrak{I}_{\mathbf{x}}$ with its convex closure $\widehat{\mathfrak{I}}_{\mathbf{x}}$. If one uses (47), or (48), to estimate $\mathfrak{I}_{\mathbf{x}}$, then, by convexly closing the contour of $\mathfrak{F}_{\mathbf{x},h}$, one obviously gets an estimate of $\widehat{\mathfrak{I}}_{\mathbf{x}}$. By the Corollary to Theorem 3.6.2 and by Corollary 3 to Theorem 3.5.5, the convex closure of $\mathfrak{F}_{\mathbf{x},h}$ is a Fechnerian sphere in the small with straight line radii: as these radii get smaller, their psychometric lengths get closer to each other.

3.7. *Global psychometric transformation.* As shown in the previous subsection, Fechnerian indicatrices can be estimated, to any desired degree of approximation, by horizontal cross-sections of psychometric functions, and this estimation does not require any knowledge of the global psychometric transformation. This remarkable fact becomes less surprising when one realizes that the global psychometric transformation, compared to a horizontal cross-section, pertains to a different, orthogonal (both logically and geometrically) aspect of psychometric functions. Namely, the psychometric differential $\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})$ describes the *vertical cross-section* of the psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$, made in the direction $\mathbf{u}$. Geometrically, this is the contour of the intersection of $\psi_{\mathbf{x}}(\mathbf{y})$ with the (two-dimensional) half-plane swept by the ray $\mathbf{x}+\mathbf{u}s$, $s>0$, moving orthogonally to the stimulus space (Fig. 22). Here, we discuss the operational meaning of the global psychometric transformation, Φ, and the fundamental assumption of Fechnerian scaling, that Φ is the same for all stimuli and all directions of change. It is tacitly assumed that the consideration is restricted to a compact subset of stimuli, as discussed in the previous subsection.

For a small elevation h_0 and the corresponding horizontal cross-section $\mathfrak{F}_{\mathbf{x},h_0}$, represented by $\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})$, we have

$$\lim_{s\to 0+}\frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})s)-\psi_{\mathbf{x}}(\mathbf{x})]}{s}=F[\mathbf{x},\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})]. \tag{51}$$

Due to (47),

$$\begin{aligned}F[\mathbf{x},\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})]&=F[\mathbf{x},\mathbf{1}_{\mathbf{x}}(\mathbf{u})(\Phi(h_0)+o\{\Phi(h_0)\})]\\&=F[\mathbf{x},\mathbf{1}_{\mathbf{x}}(\mathbf{u})][\Phi(h_0)+o\{\Phi(h_0)\}],\qquad h_0\to 0+,\end{aligned}$$

and, since $F[\mathbf{x},\mathbf{1}_{\mathbf{x}}(\mathbf{u})]=1$, (51) becomes

$$\lim_{s\to 0+}\frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})s)-\psi_{\mathbf{x}}(\mathbf{x})]}{s}=[\Phi(h_0)+o\{\Phi(h_0)\}],\qquad h_0\to 0+.$$

Equivalently,

$$\begin{aligned}&\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})s)-\psi_{\mathbf{x}}(\mathbf{x})]\\&\quad=[\Phi(h_0)+o\{\Phi(h_0)\}]s+o\{s\},\qquad h_0\to 0+,\quad s\to 0+.\end{aligned} \tag{52}$$

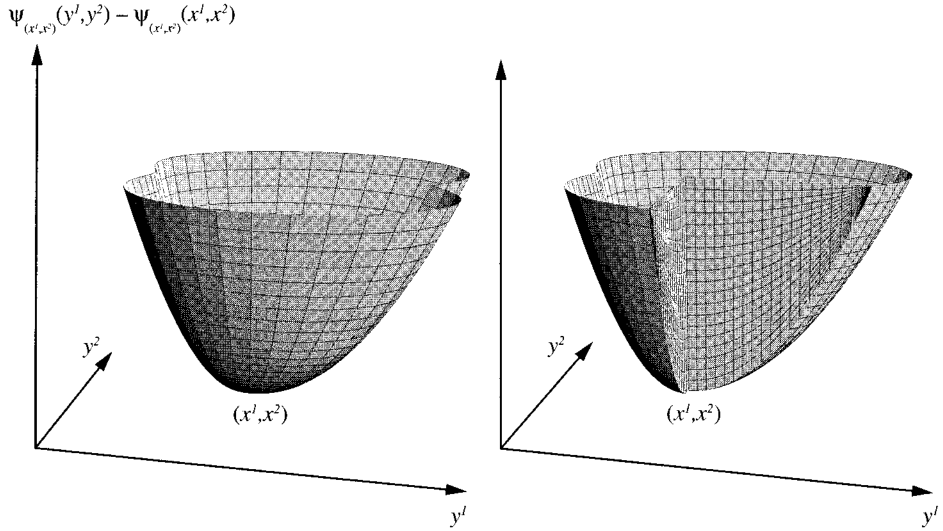

FIG. 22. Two vertical cross-sections of a psychometric function, considered between its minimum and a low-elevation horizontal cross-section.

Since s and $h = \psi_{\mathbf{x}}(\mathbf{x} + \mathbf{z}_{\mathbf{x}, h_0}(\mathbf{u}) s) - \psi_{\mathbf{x}}(\mathbf{x})$ are asymptotically proportional, we can write

$$o\{s\} = o\{\Phi(h)\},$$

and, as $h \leqslant h_0$ (for $s \leqslant 1$),

$$o\{s\} = o\{\Phi(h_0)\}.$$

Then (52) can be written as

$$\Phi[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{z}_{\mathbf{x}, h_0}(\mathbf{u}) s) - \psi_{\mathbf{x}}(\mathbf{x})] = \Phi(h_0) s + o\{\Phi(h_0)\}, \qquad h_0 \to 0+. \tag{53}$$

Assuming now that an elevation h_0 is chosen sufficiently small by the criteria discussed in the previous subsection, one can replace $\Phi(h_0)$ with unity, as in (48), and obtain

$$\Phi[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{z}_{\mathbf{x}, h_0}(\mathbf{u}) s) - \psi_{\mathbf{x}}(\mathbf{x})] \approx s. \tag{54}$$

This approximation should hold, to a desired degree of precision (that determines the choice of h_0), for all $h \leqslant h_0$ (i.e., for all $s \leqslant 1$), across different $\mathbf{u}$ and $\mathbf{x}$.

We arrive now at the operational meaning of the global psychometric transformation and the fundamental assumption of Fechnerian scaling. For any stimulus $\mathbf{x}$ and direction vector $\mathbf{u}$, one estimates the transformation Φ by plotting the value of s, between 0 and 1, against the corresponding value of $\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{z}_{\mathbf{x}, h_0}(\mathbf{u}) s) - \psi_{\mathbf{x}}(\mathbf{x})$, ranging from 0 to h_0; according to the fundamental assumption, this function must be (approximately) the same for all $\mathbf{x}$ and $\mathbf{u}$ (Fig. 23).

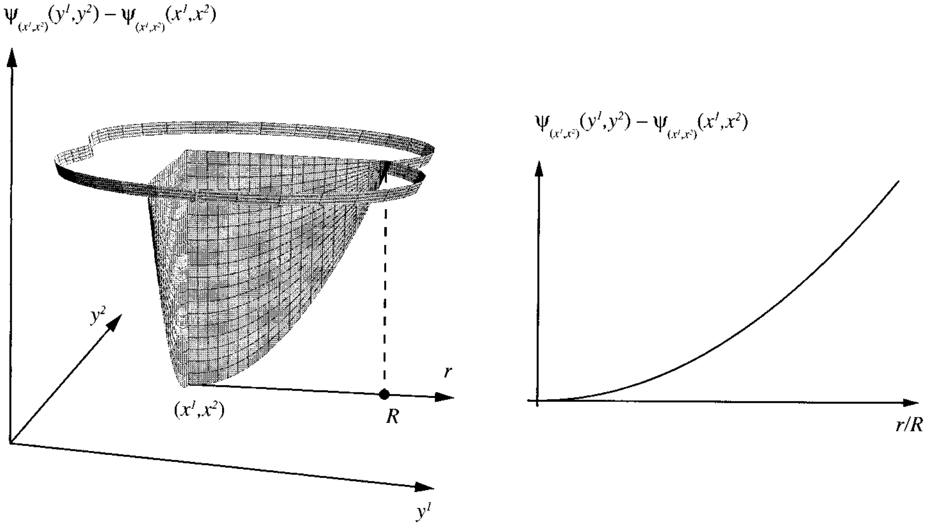

FIG. 23. If the radii (R) of the horizontal cross-section that serve as bases for the vertical cross-sections are normalized to a unity, then all vertical cross-sections have the same shape (across different directions, as shown, but also across different stimuli).

This interpretation acquires an especially simple form if one adopts the power function version of Fechnerian scaling, that is, if one assumes that the global psychometric transformation is a power function, (11), possibly multiplied by a positive constant. Then, by plotting $\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})s)-\psi_{\mathbf{x}}(\mathbf{x})$ against s, between 0 and 1, one should be able to uniquely find a value of μ, such that, for all line elements $(\mathbf{x}, \mathbf{u})$,

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{z}_{\mathbf{x},h_0}(\mathbf{u})s)-\psi_{\mathbf{x}}(\mathbf{x}) \approx h_0 s^{\mu}.$$

Conversely, given a Fechnerian indicatrix $\mathfrak{I}_{\mathbf{x}}$ and a global psychometric transformation Φ, one can reconstruct, essentially uniquely, the shape of the psychometric function $\psi_{\mathbf{x}}(\mathbf{y})$ in a very small vicinity of its minimum, $\mathbf{y}=\mathbf{x}$,

$$\psi_{\mathbf{x}}(\mathbf{y})-\psi_{\mathbf{x}}(\mathbf{x}) \approx \Phi^{-1}\left[\frac{\mathbf{y}-\mathbf{x}}{\mathbf{z}_{\mathbf{x},h_0}(\mathbf{y}-\mathbf{x})}\right],$$

where Φ has been multiplied with a positive constant that ensures $\Phi(h_0)=1$.

For the power function version of Fechnerian scaling,

$$\psi_{\mathbf{x}}(\mathbf{y})-\psi_{\mathbf{x}}(\mathbf{x}) \approx h_0\left[\frac{\mathbf{y}-\mathbf{x}}{\mathbf{z}_{\mathbf{x},h_0}(\mathbf{y}-\mathbf{x})}\right]^{\mu}.$$

4. CONCLUSION

As stated in the opening paragraph of this paper, Fechnerian scaling is motivated by the vague belief that Fechnerian distances could lie in the foundation of other behavioral measurements. A prominent feature of Fechnerian scaling that contributes to this belief is that Fechnerian distances are computed from strictly local (in-the-small) considerations: we know that this is how metrics are derived in physics. At present, however, as this vague belief has been neither tested nor formulated more rigorously, one can only take Fechnerian scaling for what it undoubtedly is, a powerful mathematical language for psychophysics, and develop it by relating it to as broad a variety of psychophysical problems and approaches as possible. It is demonstrated in Dzhafarov and Colonius (1999a) that sometimes a mere formulation of a long-standing problem in the language of Fechnerian scaling is sufficient to solve it (we refer here to the problem of Fechnerian distances between isosensitivity curves).

Another famous historical example is the internal inconsistency found in Fechner's original theory by Elsass (1886) and Luce and Edwards (1958). Recall that in a unidimensional case the indicatrix $\mathfrak{I}_{\mathbf{x}}$ attached to a point x is (under the Third Assumption) a pair of points $\{-u(x), u(x)\}$, $u>0$. Stated in terms of the present theory, the criticism in question is as follows: If $u(x)$ is claimed to be "subjectively constant" across different values of x (*Fechner's Postulate*), so that its subjective magnitude can be equated to 1, and if one uses (22) and (27) to compute the Fechnerian ("subjective") distances $G(a, b)$, then, the argument goes, $G[a, a+u(a)]$ must be constant (in fact, equal to 1) across different values of a. This is not, however, the case. The application of (22) and (27) in this situation leads to

$$G[a, a+u(a)] = \int_{a}^{a+u(a)} \frac{dx}{u(x)},$$

which varies with a in all cases except when $u(x)=cx$ (Luce & Edwards's argument), and even in the latter case

$$G[a, a+ca] = \frac{1}{c} \log(1+c) \neq 1$$

(Elsass's argument). As mentioned in Subsection 3.1, this controversy is resolved by simply pointing out that the indicatrix $\{-u(x), u(x)\}$ belongs to the tangent space, rather than the stimulus space. Quite aside from the issue of how the value of u is estimated (in Fechner's theory, by measuring "differential thresholds"), the subjective constancy of $u(x)$ postulated by Fechner should be understood as the constancy of the magnitude of the indicatrix radii. This constancy holds by the definition of an indicatrix, whereas the constancy of distances $G[a, a+u(a)]$ in the stimulus space can only be true approximately, as discussed in Subsection 3.6, and as Fechner (1887, p. 167) pointed out in his rejoinder to Elsass.

Many empirical and theoretical problems are being led to by Fechnerian scaling in a natural fashion. It remains an open question, for example, whether the same Fechnerian distances can be derived from different kinds of psychometric functions, say, those obtained from direct "same–different" judgments and those obtained from forced choices between $(\mathbf{x}, \mathbf{x})$ and $(\mathbf{x}, \mathbf{y})$ pairs. The theory of Fechnerian scaling does not predict that these distances should be the same, but it would simply become less interesting if they turn out to be completely unrelated. This is an empirical issue. An important theoretical problem is how the discrimination probabilities $\psi_{\mathbf{x}}(\mathbf{y})$ are related to Fechnerian distances $G(\mathbf{x}, \mathbf{y})$: In particular, must $\psi_{\mathbf{x}}(\mathbf{y})$ be a function of $G(\mathbf{x}, \mathbf{y})$ if it is a function of *some* distance between $\mathbf{x}$ and $\mathbf{y}$? Another important theoretical problem is that of determining the constraints imposed on Fechnerian indicatrices and on global psychometric transformations by the assumption that psychometric functions are derived from a model of a Thurstonian kind, the model in which stimuli are represented by random events in a hypothetical perceptual space and two stimuli are judged to be different if and only if their perceptual representations differ by more than a critical distance, in some suitably defined perceptual metric. Yet another important problem is that of the relationship between the Fechnerian metric in a stimulus space $\mathfrak{M}^{(n)}$ and the Fechnerian metrics in the lower-dimensionality subspaces of $\mathfrak{M}^{(n)}$. One aspect of this problem involves such issues as perceptual separability and stochastic independence in the perception of different physical attributes (Ashby & Townsend, 1986; Thomas, 1996). Some preliminary results obtained when Fechnerian scaling is applied to these problems are described in Dzhafarov and Colonius (1999b).

It should be emphasized that although Fechnerian scaling can be viewed as a mathematical language for psychophysics, it is also an empirical theory, based on assumptions that may or may not be true. Due to the fact that all but one of these assumptions deal with asymptotic (limit) properties of psychometric functions (the one exception being the single-minimum part of the First Assumption), one might be inclined to think that they cannot be refuted experimentally even if they are de facto wrong. This opinion is unsubstantiated. In the context of response time decompositions, for example, the possibility of choosing empirically among competing assumptions involving asymptotic properties is demonstrated in Dzhafarov (1992) and Dzhafarov and Rouder (1996). Colonius (1995) shows how certain asymptotic properties of theoretical response time distributions can be determined from the tail behavior of a sample distribution function. In the present context, consider, as a possible scenario, that one finds that a certain elevation h_0 for horizontal cross-sections of several psychometric functions is sufficiently small by the criteria established in Subsection 3.6, in the discussion related to (49) and (50); one also finds that each of the vertical cross-sections of these psychometric functions, made in several different directions, can be well approximated by power functions (see Subsection 3.7); but one finds that the exponents of these functions are significantly different for different psychometric functions, or different directions of change. This would constitute a convincing empirical refutation for the fundamental Second Assumption, invalidating thereby the entire enterprise of Fechnerian scaling.

APPENDIX: TECHNICAL COMMENTS

1. The term "in the small" is used throughout this paper as a synonym for "local" or "in the limit" and indicates that the quantities considered tend to zero or

the areas considered tend to points. A specific definition is given every time the term is used in a given context for the first time.

2. This means that the convergence of points in the stimulus space can be defined through vanishing Euclidean or supremal distances, among other metrics: $\mathbf{x}_k \to \mathbf{x}$, $k = 1, 2, \ldots$, if and only if $|\mathbf{x}_k - \mathbf{x}| \to 0$, where $|\cdots|$ is the Euclidean norm, or, equivalently, $\mathbf{x}_k \to \mathbf{x}$ if and only if $\max_{i=1,\ldots,n}(|x_k^i - x^i|) \to 0$.

3. A diffeomorphic transformation $\hat{\mathbf{x}}(\mathbf{x})$ is a one-to-one (onto) continuously differentiable transformation whose inverse $\mathbf{x}(\hat{\mathbf{x}})$ is also continuously differentiable. The Jacobian of a diffeomorphism, $\det[\partial\hat{\mathbf{x}}/\partial\mathbf{x}]$, never vanishes.

4. Calling s the amount (or magnitude) of stimulus change is a somewhat subtle point. Observe that if the corresponding components of $\mathbf{x}$ and $\mathbf{u}$ are measured in the same units, s is dimensionless. Observe also that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are codirectional (i.e., $\mathbf{u}_2 = \lambda\mathbf{u}_1$, $\lambda > 0$) they are nevertheless considered different direction vectors, and the changes from $\mathbf{x}$ by the same amount s in the directions $\mathbf{u}_1$ and $\mathbf{u}_2$ are different changes arriving at different stimuli.

5. The term $o\{s\}$ in (1) and hereafter denotes a quantity of a higher order of infinitesimality than $s \to 0+$; in other words, $o\{s\}/s \to 0$ as $s \to 0+$.

6. Fréchet differentiability is a vectorial generalization of the conventional differentiability. The reader not familiar with the concept need not be concerned, as it is not used in the subsequent development.

7. The minimum possible value of r ("lower order of flatness") also has a classificatory value, but we refrain here from discussing these issues in greater detail. A more comprehensive presentation should also include theorems linking our notions to the traditional definitions of convexity and strong convexity of metric functions (Busemann, 1955, pp. 99–100).

8. The term "homothety" in the formulation of the theorem means geometric similarity (proportionality) of two concentric contours. The proportionality factor is called the coefficient of homothety.

9. The regularity property mentioned at the end of Section 2 does imply, however, that the Fechnerian indicatrices are strictly convex (moreover, have positive Gaussian curvature). To induce a Finsler metric in the narrow sense, therefore, psychometric functions must have horizontal cross-sections satisfying this property at small elevations.

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Received: March 1, 2000; published online April 4, 2001