Advanced Analysis of Quantum Contextuality in a Psychophysical Double-Detection Experiment

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The results of behavioral experiments typically exhibit inconsistent connectedness, i.e., they violate the condition known as "no-signaling," "no-disturbance," or "marginal selectivity." This prevents one from evaluating these experiments in terms of quantum contextuality if the latter understood traditionally (as, e.g., in the Kochen-Specker theorem or Bell-type inequalities). The Contextuality-by-Default (CbD) theory separates contextuality from inconsistent connectedness. When applied to quantum physical experiments that exhibit inconsistent connectedness (due to context-dependent errors and/or signaling), the CbD computations reveal quantum contextuality in spite of this. When applied to a large body of published behavioral experiments, the CbD computations reveal no quantum contextuality: all context-dependence in these experiments is described by inconsistent connectedness alone. Until recently, however, experimental analysis of contextuality was confined to so-called cyclic systems of binary random variables. Here, we present the results of a psychophysical double-detection experiment that do not form a cyclic system: their analysis requires that we use a recent modification of CbD, one that makes the class of noncontextual systems more restricted. Nevertheless our results once again indicate that when inconsistent connectedness is taken into account, the system exhibits no contextuality.

KEYWORDS: contextuality, cyclic systems, double-detection, inconsistent connectedness, psychophysics.

In recent years there were many reports of behavioral experiments (Accardi, Khrennikov, Ohya, Tanaka, & Yamato, 2016; Aerts & Sozzo, 2014, 2015; Aerts, Sozzo, & Veloz, 2015; Asano, Hashimoto, Khrennikov, Ohya, & Tanaka, 2014; Bruza, Kitto, Ramm, & Sitbon, 2015; Cervantes & Dzhafarov, 2017; Dzhafarov, Zhang, & Kujala, 2015; Khrennikov, 2015; Sozzo, 2015; Wang, Solloway, Shiffrin, & Busemeyer, 2014; Zhang & Dzhafarov, 2017) aimed at (or interpretable as aimed at) revealing *contextuality* of the kind predicted by and experimentally confirmed in quantum physics (Bell, 1964; Clauser, Horne, Shimony, & Holt, 1969; Fine, 1982; Hensen et al., 2015; Klyachko, Can, Binicioğlu, & Shumovsky, 2008; Kochen & Specker, 1967; Kurzyński, Ramanathan, & Kaszlikowski, 2012; Lapkiewicz et al., 2011). All known to us behavioral data, however, violate a certain condition that makes a direct application of the traditional quantum contextuality analysis impossible. This condition is variously called "no-signaling" or "no-disturbance" in quantum physics (Bacciagaluppi, 2015, 2016; Cereceda, 2000; Kofler & Brukner, 2013; Kurzyński, Cabello, & Kaszlikowski, 2014; Popescu & Rohrlich, 1994; Ramanathan, Soeda, Kurzyński, & Kaszlikowski, 2012) and "marginal selectivity" in psychology (Dzhafarov, 2003; Townsend & Schweickert, 1989; Zhang & Dzhafarov, 2015). It is a required condition for the traditional quantum contextuality analysis, even though it is often violated in quantum mechanical experiments as well (this issue was first systematically discussed in Adenier & Khrennikov, 2007; see also Adenier & Khrennikov, 2016; Lapkiewicz et al., 2011, 2013). The Contextuality-by-Default (CbD) theory (de Barros, Dzhafarov, Kujala, & Oas, 2015; Dzhafarov, 2016; Dzhafarov & Kujala, 2014a, 2014b, 2015, 2016a, 2016b, 2017a, in press; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015) overcomes this difficulty by proposing a principled way of separating contextuality proper from *inconsistent connectedness* (the CbD term for violations of the "no-signaling" or "marginal selectivity" condition). This theory was used to reanalyze the behavioral experiments aimed at contextuality, with the conclusion that they provide no evidence for contextuality (Cervantes & Dzhafarov, 2017; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017): inconsistent connectedness is the only form of context-dependence that we have in them. By contrast, when CbD is used to reanalyze a quantum-mechanical experiment that exhibits inconsistent connectedness (Lapkiewicz et al., 2011), contextuality proper (on top of inconsistent connectedness) is established beyond doubt (Kujala, Dzhafarov, & Larsson, 2015).

Virtually all experiments aimed at revealing contextuality, both in quantum physics and in behavioral sciences, deal with a special kind of systems of random variables, called *cyclic systems* in CbD (Kujala et al., 2015). In these systems each property is measured in precisely two different contexts, and each context contains two properties being measured together. If, in addition, all random variables in the system are binary (each indicating presence or absence of a certain property), then the system is amenable to complete and exhaustive contextuality analysis (Dzhafarov & Kujala, 2016a; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015; Kujala et al., 2015). In spite of their prominence in quantum theory, however, it is highly desirable to extend contextuality analysis beyond the class of cyclic systems. Many researchers (although not the present authors) find the lack of contextuality in behavioral data to be a disappointing negative result. What if this result is due to the fact that cyclic systems in human behavior are too simple? What if it is "too easy" for a cyclic system to be noncontextual? These are valid questions, and they will have no definite answers until we have a predictive theory of (at least certain types of) human behavior on a par with quantum mechanics.

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In the absence of a predictive theory, the only, admittedly imperfect way of dealing with these considerations is to expand the experimentation and contextuality analysis to progressively broader classes of systems. In this paper we make a first step in this direction by analyzing a psychophysical experiment whose results form a non-cyclic system of random variables. This experiment was reported previously (Cervantes & Dzhafarov, 2017), but its analysis was confined to extracting from it a large number of cyclic subsystems and showing all of them to be noncontextual. It is mathematically possible, however, that a system is contextual with all its cyclic subsystems being noncontextual.

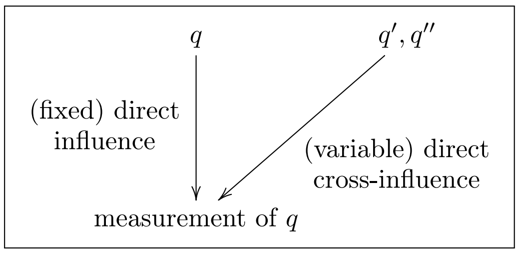
A satisfactory way to expand the contextuality analysis beyond cyclic systems was proposed in a recent modification of CbD, dubbed "CbD 2.0" (Dzhafarov & Kujala, 2017a, in press): it is essentially the original CbD in which the measurements of the same property (say, responses to the same stimulus) are analyzed in pairs only. This modification has compelling reasons behind it, The main one is that in the modified theory a subsystem of a noncontextual system is always noncontextual. Another reason is that contextuality analysis is reduced to the problem of compatibility of two *uniquely defined* sets of distributions: the empirically known distributions of context-sharing random variables and the distributions of the "multimaximal couplings" of the random variables measuring the same property in different contexts. All of this is clarified below (Section 2). The modification in question does not affect the theory of cyclic systems, so the results mentioned earlier remain unchanged. However, when it comes to non-cyclic systems, the modification makes the requirements that a system should satisfy to be noncontextual more stringent.

The plan of the paper is as follows. In the next two sections we present the basics of the CbD theory, in the "CbD 2.0" version. The discussion is primarily confined to systems of binary random variables (dichotomic measurements), both for simplicity and because the double-detection experiment to be analyzed involves only dichotomic judgments. In Section 3 we apply this theory to the results of our double-detection experiment. Our conclusion is that in spite of the notion of noncontextuality we use being more restrictive than in the original version of the CbD theory, the double detection experiment does not exhibit any contextuality.

1. INTRODUCTION TO CONTEXTUALITY

Every experiment results in a system of random variables. In most physics experiments these random variables are interpreted as measurements of properties, in most behavioral experiments they are interpreted as responses to stimuli, such as questions. For brevity we will use the term "measurement" in both meanings (because responding to a stimulus can always be viewed as a form of measurement). *What* is being measured therefore is part of the identity of a random variable representing a measurement. It is referred to as the *content* of the random variable. The content, however, does not specify a random variable uniquely, because one and the same content can be measured under different conditions, referred to as *contexts*. For instance, if a content q is measured simultaneously with measurements of other contents, in some cases q' and in other cases q'', then in the former cases the context is $c = (q, q')$ and in the latter ones it is $c' = (q, q'')$. As in Dzhafarov and Kujala (2016a, 2017a), we will write "conteXt" and "conteNt" to prevent their confusion in reading. The conteXt and conteNt of a random variable uniquely identify it within a given system of random variables. So each random variable in a system is double-indexed, R_q^c.

According to the CbD theory's main principle (Dzhafarov, 2016; Dzhafarov & Kujala, 2014a, 2016a, 2016b, in press; Dzhafarov, Kujala, & Cervantes, 2016), two random variables R_q^c and $R_{q'}^{c'}$ are jointly distributed if and only if $c = c'$, i.e., if and only if they are recorded in the same conteXt. Otherwise they are *stochastically unrelated*, i.e., joint probabilities for them are undefined. This means, in particular, that any two R_q^c and $R_q^{c'}$ with the same conteNt in different conteXts are stochastically unrelated (which implies, among other things, that they can never be considered to be one and the same random variable). Their individual distributions may be the same but they need not be. If these distributions are different, the system exhibits a form of context-dependence. However, in CbD, this context-dependence by itself does not say that the system is contextual in the sense related to how this term is used in quantum mechanics. Rather the difference in the distributions is treated as manifestation of information/energy flowing to the measurements of conteNt q from elements of the contexts c, c' other than q. We will refer to this transfer of information/energy as *direct cross-influences*. Thus, if $c = (q, q')$ and $c' = (q, q'')$, the conteNt q does, of course, directly influence its measurement, but, with q fixed, the second conteNt in the pair can also affect this measurement. This can sometimes be attributed to some physical action of q' or q'' upon the process measuring q, or (as another form of information transfer) it can be a form of contextual bias, a change in the procedure by which q is measured depending on what else is being measured.



The difference between the distributions of R_q^c and $R_q^{c'}$ (equivalently, the strength of the direct cross-influences responsible for this difference) is measured in CbD by the probability with which R_q^c and $R_q^{c'}$ could be made to coincide *if they were jointly distributed.* This means that we consider all *couplings* of $R_q^c, R_q^{c'}$, i.e., the jointly distributed pairs of random variables $T_q^c, T_q^{c'}$ whose respective individual distributions are the same as those of $R_q^c, R_q^{c'}$, and among these pairs we find the one(s) with the maximal possible probability of $T_q^c = T_q^{c'}$. The larger this maximal probability, the closer the two distributions to each other, and the weaker the direct cross-influences by conteXts c, c' upon the measurement of q. This maximal probability is 1 if and only if the two distributions are identical, and it is 0 if an only if the two distributions have disjoint supports.

Consider now an experiment represented by a system of random variables R_q^c with varying c and q, and suppose that we have computed the maximal probability just described for each pair of random variables that share a conteNt. And we know (or can empirically estimate) the joint distributions of all random variables that share a conteXt. Intuitively, quantum contextuality is about whether these computed maximal probabilities and these empirically defined joint distributions are mutually compatible. If they are not, then one can say that conteXts force the random variables sharing conteNts to be more dissimilar than they are made by direct cross-influences alone. The system then can be considered *contextual*.

To understand this without conceptual and technical complications, consider first a cyclic system of binary random variables (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2015; Kujala et al., 2015). It is depicted in Fig. 1. The conteXts and conteNts are such that, with appropriate enumeration, in conteXt c_i one measures precisely two cyclically-successive conteNts $q_i, q_{i\oplus 1}$ (where $i = 1, \ldots, n$; $i \oplus 1 = i + 1$ for $i < n$; and $n \oplus 1 = 1$):

$$\begin{tikzcd} q_1 \arrow[r, "c_1"] & q_2 \arrow[r, "c_2"] & \cdots \arrow[r, "c_{n-2}"] & q_{n-1} \arrow[r, "c_{n-1}"] & q_n, \arrow[llll, bend left, "c_n"] \end{tikzcd}$$

Each pair $R_i^i, R_{i\oplus 1}^i$ ($i = 1, \ldots, n$) of random variables sharing a conteXt (within a row in Fig. 1) are jointly distributed. Since all the measurements in the system are binary (± 1), the joint distribution of R_j^i is uniquely determined by three probabilities,

$$\begin{gathered} p_i^i = \Pr\left[R_i^i = 1\right],\ p_{i\oplus 1}^i = \Pr\left[R_{i\oplus 1}^i = 1\right], \\ p^i = \Pr\left[R_i^i = R_{i\oplus 1}^i = 1\right]. \end{gathered} \tag{1}$$

Random variables $R_i^{i\ominus 1}, R_i^i$ within a column share a conteNt, and we compute for each such a pair the magnitude of direct cross-influences, $\max \Pr\left[T_i^i = T_i^{i\ominus 1}\right]$, across all couplings $\left(T_i^{i\ominus 1}, T_i^i\right)$ of $R_i^{i\ominus 1}, R_i^i$: in this case the couplings are the pairs $\left(T_i^{i\ominus 1}, T_i^i\right)$ with all possible values of $\Pr\left[T_i^i = T_i^{i\ominus 1} = 1\right]$ and with

$$\Pr\left[T_i^i = 1\right] = p_i^i,\ \Pr\left[T_i^{i\ominus 1} = 1\right] = p_i^{i\ominus 1}. \tag{2}$$

Here, $i = 1, \ldots, n$; $i \ominus 1 = i - 1$ for $i > 1$; and $1 \ominus 1 = n$. The coupling $\left(T_i^{i\ominus 1}, T_i^i\right)$ with this property is called *maximal coupling*. It is easy to show (Thorisson, 2000) that this maximal coupling always exists and is defined by complementing (2) with

$$p_i = \Pr\left[T_i^i = T_i^{i\ominus 1} = 1\right] = \min\left\{p_i^i, p_i^{i\ominus 1}\right\}. \tag{3}$$

The probabilities (1) and (3) are shown in Fig. 2. Note that (2) and (3) uniquely define the joint distribution of the two random variables $T_i^{i\ominus 1}, T_i^i$ within each column of the matrix, in the same way as (1) uniquely define the joint distribution of $R_i^i, R_{i\oplus 1}^i$ within each row of the matrix. The only difference is that the row-wise joint distributions are empirical reality, whereas the column-wise

⋆	⋆			$\cdots$			c_1
	⋆	⋆		$\cdots$			c_2
		⋆	⋆	$\cdots$			c_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
				$\cdots$	⋆	⋆	c_{n-1}
⋆				$\cdots$		⋆	c_n
q_1	q_2	q_3	q_4	$\cdots$	q_{n-1}	q_n	$\boxed{\text{CYC}}$

FIG. 1: A cyclic system (shown here for a sufficiently large n, although n can be as small as 2 or 3). The system involves n conteNts $q_1, \ldots, q_n$ and n conteXts $c_1, \ldots, c_n$. The star symbol in the (c_i, q_j)-cell indicates that conteNt q_j was measured in conteXt c_i, and the result of the measurement is random variable R_j^i; otherwise q_j was not measured in c_i and the cell is left empty. All R_j^i are binary random variables, with possible values denoted $+1$ and -1.

joint distributions are constructed artificially to depict the direct cross-influences. Contextuality in CbD is all about the compatibility of these column-wise and row-wise joint distributions: the system is considered noncontextual if all these probabilities can be achieved within a jointly distributed set of $2n$ random variables. In other word, we seek a set of jointly distributed random variables S_j^i replacing the star symbols in Fig. 1, such that

$$\begin{aligned} &(i) \quad \Pr\left[S_i^i = 1\right] = p_i^i,\ \Pr\left[S_{i\oplus 1}^i = 1\right] = p_{i\oplus 1}^i, \\ &(ii) \quad \Pr\left[S_i^i = S_{i\oplus 1}^i = 1\right] = p^i, \\ &(iii) \quad \Pr\left[S_i^i = S_i^{i\ominus 1} = 1\right] = p_i = \min\left\{p_i^i, p_i^{i\ominus 1}\right\} \end{aligned} \tag{4}$$

The equations (i) and (ii) in (4) tell us that the set of the S_j^i-variables we seek is a coupling of the original random variables R_j^i arranged row-wise in Fig. 1: in each row the variables R_j^i have a well-defined joint distribution, but different rows are stochastically unrelated, so the coupling "saws them together" in a single joint distribution. The equations (i) and (iii) in (4) tell us that the set of the S_j^i-variables is a coupling for the column-wise maximal couplings T_j^i: in each of the columns the variables T_j^i have a well-defined joint distribution, but different columns are stochastically unrelated because the maximal couplings were computed for each column separately; so the coupling "saws the columns together" in a single joint distribution. It is easy to see that each of these two couplings (of the rows and of the columns) exists, because the random variables in the different rows do not overlap, and the same is true for different columns. In a typical case, each of the two couplings can be constructed in an infinity of ways, and the question is whether a jointly distributed set of $2n$ random variables can be simultaneously a coupling for the rows and for the columns. If the answer to this question is negative, the conteXts intervene beyond the effect of the direct cross-influences.

2. CONTEXTUALITY IN ARBITRARY SYSTEMS OF BINARY MEASUREMENTS

Let us discuss now how the analysis just presented extends beyond cyclic systems. We will continue to assume that all the random variables in play are binary.

p_1^1	p_2^1			$\cdots$			p^1
	p_2^2	p_3^2		$\cdots$			p^2
		p_3^3	p_4^3	$\cdots$			p^3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
				$\cdots$	p_{n-1}^{n-1}	p_n^{n-1}	p^{n-1}
p_1^{n-1}				$\cdots$		p_n^n	p^n
p_1	p_2	p_3	p	$\cdots$	p_{n-1}	p_n	CYC

FIG. 2: The probability values that characterize the cyclic system in Fig. 1 in accordance with (1) and (3). The system is noncontextual if there is a set of $2n$ jointly distributed random variables $\left(S_j^i : i = 1, \ldots, n; j = i \text{ or } j = i \oplus 1\right)$ with $\Pr\left[S_j^i = 1\right] = p_j^i$, $\Pr\left[S_i^i = S_{i\oplus 1}^i = 1\right] = p^i$, and $\Pr\left[S_i^i = S_i^{i\ominus 1} = 1\right] = p_i = \min\left\{p_i^i, p_i^{i\ominus 1}\right\}$.

⋆	⋆			c_1
	⋆	⋆		c_2
		⋆	⋆	c_3
⋆			⋆	c_4
q_1	q_2	q_3	q_4	CYC_4

⋆	⋆		⋆	c_1
	⋆	⋆		c_2
		⋆	⋆	c_3
⋆			⋆	c_4
q_1	q_2	q_3	q_4	X

FIG. 3: A cyclic system with $n = 4$ (CYC_4) and a system X obtained from CYC_4 by adding to it the random variable R_4^1.

Consider Fig. 3. The system X is not cyclic, as it has three random variables in the first row (conteXt c_1) and three random variables in the fourth column (conteNt q_4). The number and arrangement of the random variables in a row, however, is immaterial for the logic of the contextuality analysis. The joint distribution of R_1^1, R_2^1, R_4^1 in the first row of X is uniquely defined empirically. It simply requires more probabilities than in (1) to be described:

$$p_1^1 = \Pr\left[R_1^1 = 1\right], \; p_2^1 = \Pr\left[R_2^1 = 1\right], \; p_4^1 = \Pr\left[R_4^1 = 1\right],$$

$$\begin{array}{c} p_{12}^1 = \Pr\left[R_1^1 = R_2^1 = 1\right], \\ p_{24}^1 = \Pr\left[R_2^1 = R_4^1 = 1\right], \\ p_{14}^1 = \Pr\left[R_1^1 = R_4^1 = 1\right], \end{array} \tag{5}$$

$$p_{124}^1 = \Pr\left[R_1^1 = R_2^1 = R_4^1 = 1\right].$$

Nor does anything change in how one treats the pairs of the conteNt-sharing random variables in the first three columns: one computes the maximal coupling for each of these columns. One faces choices, however, when dealing with the three random variables in the fourth column. What is the right way of generalizing the maximal coupling in this case? There is a compelling reason (Dzhafarov & Kujala, 2017a, in press) to consider the three conteNt-sharing random variables one pair at a time, and to compute maximal couplings for them separately. This means finding a jointly distributed triple $\left(T_4^1, T_4^3, T_4^4\right)$ whose elements are distributional copies of, respectively, R_4^1, R_4^3, R_4^4, i.e.,

$$\begin{array}{c} \Pr\left[T_4^1 = 1\right] = p_4^1, \; \Pr\left[T_4^3 = 1\right] = p_4^3, \\ \Pr\left[T_4^4 = 1\right] = p_4^4, \end{array} \tag{6}$$

such that $\left(T_4^1, T_4^3\right)$ is the maximal coupling of R_4^1, R_4^3, $\left(T_4^3, T_4^4\right)$ is the maximal coupling of R_4^3, R_4^4, and $\left(T_4^1, T_4^4\right)$ is the maximal coupling of R_4^1, R_4^4. In terms of probability values,

$$\begin{array}{c} \Pr\left[T_4^1 = T_4^3 = 1\right] = \min\left\{p_4^1, p_4^3\right\}, \\ \Pr\left[T_4^3 = T_4^4 = 1\right] = \min\left\{p_4^3, p_4^4\right\}, \\ \Pr\left[T_4^1 = T_4^4 = 1\right] = \min\left\{p_4^1, p_4^4\right\}. \end{array} \tag{7}$$

As shown in Dzhafarov and Kujala (2017a, in press), such a coupling (called *multimaximal* in CbD) always exists, and it is unique (as all the random variables here are binary). The above-mentioned compelling reason for maximizing the couplings pairwise is that then, if the system is noncontextual, it will remain noncontextual after one deletes from it one or more random variables. In other words, any subsystem of a noncontextual system is noncontextual. This would not be true, for instance, if we only maximized the value of $\Pr\left[T_1^1 = T_2^1 = T_4^1 = 1\right]$. At the same time, the maximization of $\Pr\left[T_1^1 = T_2^1 = T_4^1 = 1\right]$ is achieved "automatically" if (7) is satisfied. Moreover, one of the equalities in (7) is redundant as it can be derived from the other two: if, e.g., $p_4^3 \leq p_4^1 \leq p_{44}$, then the redundant equality in (7) is the second one.Generalizing, we have the following theorem.

Theorem 2.1 (Dzhafarov & Kujala, 2017a, in press)**.** *Let $R_q^1, \ldots, R_q^k$, $k > 1$, be binary (± 1) random variables with conteXts enumerated so that*

$$p_q^1 = \Pr\left[R_q^1 = 1\right] \leq \ldots \leq \Pr\left[R_q^k = 1\right] = p_q^k.$$

Then there is a unique set of jointly distributed $\left(T_q^1, \ldots, T_q^k\right)$ such that $\left(T_q^i, T_q^{i+1}\right)$ is the maximal coupling of R_q^i, R_q^{i+1}, for $i = 1, \ldots, k-1$. The coupling $\left(T_q^1, \ldots, T_q^k\right)$ has the following properties.

(i) For any subset $\{i_1, \ldots, i_m\} \subseteq (1, \ldots, k)$ with $m \leq k$, $\left(T_q^{i_1}, \ldots, T_q^{i_m}\right)$ is the maximal coupling of $R_q^{i_1}, \ldots, R_q^{i_m}$, i.e., $\Pr\left[T_q^{i_1} = \ldots = T_q^{i_m}\right]$ has the maximal possible value among all couplings of $R_q^{i_1}, \ldots, R_q^{i_m}$. In particular, for any $i, j \in (1, \ldots, k)$, $\left(T_q^i, T_q^j\right)$ is the maximal coupling of $R_q^i, \ldots, R_q^j$.

(ii) The distribution of $\left(T_q^{i_1}, \ldots, T_q^{i_m}\right)$ is defined by

$$\Pr\left[T_q^1 = \ldots = T_q^k = 1\right] = p_1,$$

$$\Pr\left[T_q^1 = \ldots = T_q^l = -1\,;\, T_q^{l+1} = \ldots = T_q^k = 1\right] = p_{l+1} - p_l, \quad (\text{for } l = 1, \ldots, k-1)$$

$$\Pr\left[T_q^1 = \ldots = T_q^k = -1\right] = 1 - p_k, \tag{8}$$

with all other combinations of values having probability zero.

Now we can formulate the generalization of the definition of contextuality given in the previous section.

Definition 2.2. A system of binary random variables R_q^c is noncontextual if there exists a jointly distributed set of (correspondingly labeled) random variables S_q^c such that (i) for every conteXt c, the joint distribution of all S_q^c with this value of c is identical to the joint distribution of the corresponding R_q^c; and (ii) for every conteNt q, the joint distribution of all S_q^c with this value of q forms the (unique) multimaximal coupling of the corresponding R_q^c.

The notion of contextuality is, once again, about compatibility of the uniquely determined row-wise and column-wise distributions. The row distributions are empirically given, the column distributions are computed as multimaximal couplings, and the question is whether it is possible to find a single coupling for both the rows and the columns. Once again, the logic of the approach is that if the coupling in question does not exist, it means that the conteXts force some pairs of the random variables measuring the same conteNt to be more dissimilar than they are made by direct cross-influences alone — and the system is therefore contextual.

If a system of random variables turns out to be contextual, one can compute the degree of its contextuality as the smallest possible total variation of *quasi-couplings* of this system. A quasi-coupling differs from a coupling in that the probabilities for its values are replaced with arbitrary real numbers (not necessarily nonnegative) that sum to 1. The existence of quasi-couplings for any system and the uniqueness of the minimum total variation are proved in Dzhafarov and Kujala (2016a). We need not discuss this otherwise important topic further because the experimental results reported below reveal no contextuality.

3. DOUBLE-DETECTION EXPERIMENT

We now apply the theory just described to the results of a double-detection experiment. We remind the reader that this experiment was previously described in Cervantes and Dzhafarov (2017), but to keep this paper self-sufficient we recapitulate the procedural details below. In Cervantes and Dzhafarov (2017) the system formed by the data was analyzed by extracting from it a multitude of cyclic subsystems. In this paper we analyze the system in its entirety.

The double-detection experiment is one of only two contextuality-aimed experiments known to us that uses a within-subject design, i.e., with probabilities estimated from the responses of a single person to multiple replications of stimuli. (The other such experiment is the psychophysical matching one described in Dzhafarov, Zhang, & Kujala, 2015, and Zhang & Dzhafarov, 2017.) Most experiments use aggregation of responses obtained from many persons. The double detection paradigm suggested in Dzhafarov and Kujala (2012) and Dzhafarov and Kujala (2017b) provides a framework where both (in)consistent connectedness and contextuality can be studied in a manner very similar to how they are studied in quantum-mechanical systems (or could be studied, because consistent connectedness in quantum physics is often assumed rather than documented).

3.1. Method

3.1.1. Participants

The participants were three volunteers, graduate students at Purdue University, two females and one male (the first author of this paper), aged around 30, with normal or corrected to normal vision. The experimental program was regulated by the Purdue University's IRB protocol #1202011876. The participants are identified as $P1 - P3$ in the text below.

3.1.2. Equipment

A personal computer was used with an Intel® Core™ processor running Windows XP, a 24-in. monitor with a resolution of 1920×1200 pixels (px), and a standard US 104-key keyboard. The participant's head was steadied in a chin-rest with forehead support at 90 cm distance from the monitor; at this distance a pixel on the screen subtended 62 sec arc.

3.1.3. Stimuli

The stimuli presented on the computer screen consisted of two brightly grey colored circles (RGB 100-100-100) on a black background, with their centers 320 px apart horizontally, each circle having the radius of 135 px and circumference 4 px wide. Each circle contained a dot of 4 px in diameter in its center or 4 px away from it, in the upward or downward direction. An example of the stimuli (in reversed contrast and scaled) is shown in Figure 4.

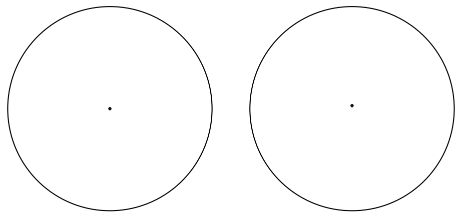


FIG. 4: An example of the stimulus in the double-detection experiment. In the left circle the dot is in the center, in the right one it is shifted 4 px upwards. The participant's task was to say, for each of the two circles, whether the dot was in the center (the answer coded 1) or off-center (the answer coded -1), irrespective of whether it was shifted up or down.

3.1.4. Procedure

In each trial the participant was asked to indicate, for each circle, whether the dot was in its center or not in the center (irrespective of in what direction). The responses were given by pressing in any order and holding together two designated keys, one for each circle, and the stimuli were displayed until both keys were pressed. Then, the dots in each circle disappeared, and a "Press the space bar to continue" message appeared above the circles. Pressing the space bar removed the message, and the next pair of dots appeared 400 ms later. (Response times were recorded but not used in the data analysis.)

Each participant completed nine experimental sessions, each lasting 30 minutes and containing about 560 trials recorded and used for the analysis, preceded by several practice trials. In each practice trial the participants received feedback as to whether their response for each of the two circles was correct or not. No feedback

		Right			
		Center (-c)	Up (-u)	Down (-d)	
Left	Center (c-)	$1/4$ (cc)	$1/8$ (cu)	$1/8$ (cd)	$1/2$
Left	Up (u-)	$1/8$ (uc)	$1/16$ (uu)	$1/16$ (ud)	$1/4$
Left	Down (d-)	$1/8$ (dc)	$1/16$ (du)	$1/16$ (dd)	$1/4$
		$1/2$	$1/4$	$1/4$	

FIG. 5: Probabilities with which a trial was allocated to one of the 9 conteXts, with the notation used for the conteXts and the conteNts: c, u, and d denote that the dot is, respectively, in the center, shifted up, or shifted down. The 9 conteXts are denoted cc, cu, du, etc., the left (right) symbol indicating the location of the dot in the left (respectively, right) circle. To denote conteNts, the location of a dot is shown on the left or on the right with a dash filling the other side: thus, c- denotes the dot in the center of the left circle, -d denotes the dot shifted down in the right circle, etc.

was given in the non-practice trials. The experimental sessions were preceded by one to three training sessions, excluded from the analysis.

3.2. Experimental ConteXts and ConteNts

In each of two circles the dot presented could be located either at its center, or 4 px above the center, or else 4 px under the center. These pairs of locations produce a total of nine conteXts. During each session, excepting the practice trials, the dot was presented at the center in a half of the trials, above the center in a quarter of them, and below the center in the remaining quarter, for each of the circles. Figure 5 presents the proportions of allocations of trials to each of the 9 conditions.

For each session, each trial was randomly assigned to one of the conditions in accordance with Table 5. The number of experimental sessions was chosen so that the expected number of (non-practice) trials in the conditions with lowest probabilities was at least 300. This number of observations was chosen based on Cepeda Cuervo et al. (2008), whose results show that coverage errors with respect to nominal values are below 1% for almost all confidence intervals for proportions with $n > 300$.

The system of random variables representing the data is shown in Figure 6.

3.3. Results

The results are shown in Figs. 7, 8, and 9, one for each of the three participants. Each row, together with its margins, specifies an empirical estimate of the joint distribution of the two random variables sharing the corresponding conteXt. This distribution is shown in the format

$$\Pr[X=1], \Pr[Y=1], \Pr[X=Y=1],$$

where X and Y are the two variables in the same row. Each column, together with its margins, shows an empirical estimate of the multimaximal coupling of the three random variables sharing the corresponding conteNt. The distribution of the coupling is shown in the format

$$\begin{array}{c} \Pr[A=1] \\ \Pr[B=1] \\ \Pr[C=1] \\ \Pr[A=B=1] \\ \Pr[B=C=1] \\ \Pr[A=C=1] \end{array},$$

where A, B, C are the three random variables in the same column listed from top down. The analysis of contextuality consists in considering a set of jointly distributed 18 binary random variables (corresponding to the star symbols in Fig. 6), and determining whether the 2^{18} values of this set can be assigned probabilities that sum to the probabilities whose empirical estimates are shown in the data matrices (Figs. 7, 8, and 9). This is a standard linear programing task,

$$\underset{46\times 2^{18}}{\mathbf{M}} \overset{2^{18}\times 1}{\mathbf{Q}} = \underset{46\times 1}{\mathbf{P}}, \ \mathbf{Q} > 0 \text{ (componentwise)}.$$

The number of the rows in $\mathbf{M}$ and $\mathbf{P}$ (i.e., the number of linear constraints imposed on $\mathbf{Q}$) is the number of the probability estimates shown in each of the data matrices (45) plus the constraint that ensures that all the 2^{18} probabilities in $\mathbf{Q}$ sum to 1. (The number of the probability estimates could be reduced from 45 to 39, because one of the three marginal probabilities for each column could be eliminated. We did not, however, make use of this small reduction in our computations.) The linear programing was performed by using the GLPK (GNU Linear Programming Kit) package (version 4.6; Makhorin, 2012) and the R interface to the package (Rglpk, version 0.6-1; Theussl & Hornik, 2015).

	c-	-c	u-	-u	d-	-d
cc	⋆	⋆				
uc		⋆	⋆			
uu			⋆	⋆		
du				⋆	⋆	
dd					⋆	⋆
cu	⋆			⋆		
ud			⋆			⋆
dc		⋆			⋆	
cd	⋆					⋆

FIG. 6: The conteNt-conteXt system of measurements for the double detection experiment. The cell corresponding to context xy and content z (with z being x- or -y), if it contains a star, represents the random variable R_z^{xy}; the absence of a star means that content z was not measured in context xy. For instance, $xy = cc$ and $z = c$- define a random variable $R_{c\text{-}}^{cc}$. The random variables within a given row (in the same conteXt) are jointly distributed. In our design there are two random variables, $R_{x\text{-}}^{xy}$ and $R_{\text{-}y}^{xy}$ in each conteXt xy, and their joint distribution is uniquely defined by three probabilities: $\Pr[R_{x\text{-}}^{xy}=1]$, $\Pr[R_{\text{-}y}^{xy}=1]$, and $\Pr[R_{x\text{-}}^{xy}=R_{\text{-}y}^{xy}=1]$.

P1	c-	-c	u-	-u	d-	-d	$\Pr[X=Y=1]$	# of trials
cc	.7175	.6365					.5476	1260
uc		.5587	.2476				.2095	630
uu			.5238	.4857			.3746	315
du				.0444	.7810		.0286	315
dd					.7556	.6508	.5714	315
cu	.8095			.2302			.2175	630
ud			.0762			.4571	.0571	315
dc		.3032			.7937		.2778	630
cd	.4063					.6349	.3730	630
$\Pr[A=B=1]$	.7175	.5587	.2476	.0444	.7556	.4571		
$\Pr[B=C=1]$	.4063	.3032	.0762	.0444	.7556	.4571		
$\Pr[A=C=1]$	.4063	.3032	.0762	.2302	.7810	.6349		

FIG. 7: Empirical data (relative frequencies) for the conteNt-conteXt system in Fig. 6 for participant P1. For every conteXt xy and every conteNt z measured in xy (either x- or -y), the cell for R_z^{xy} contains the frequency estimate of $\Pr[R_z^{xy}=1]$; the right margins of the row for xy shows the frequency estimate of $\Pr\left[R_{x\text{-}}^{xy}=R_{\text{-}y}^{xy}=1\right]$ and the total number of measurements in this conteXt. Since xy and z vary, the column for joint probabilities denotes the two random variables by $X=R_{x\text{-}}^{xy}$ and $Y=R_{\text{-}y}^{xy}$. The bottom margins in the column for conteNt z show the three frequency estimates of the maximal values of $\Pr[R_z^{xy}=R_z^{uv}=1]$, $\Pr\left[R_z^{uv}=R_z^{st}=1\right]$, and $\Pr\left[R_z^{xy}=R_z^{st}=1\right]$ (where xy, uv, st are three conteXts in which z was measured). To make notation compact, the three random variables in each column are labeled A, B, C (from top down), and the three probabilities are shown as $\Pr[A=B=1]$, $\Pr[B=C=1]$, and $\Pr[A=C=1]$ (one of which is always redundant but shown for completeness).

P2	c-	-c	u-	-u	d-	-d	$\Pr[X=Y=1]$	# of trials
cc	.8659	.7841					.6746	1260
uc		.7619	.3968				.1968	630
uu			.5556	.5841			.3746	315
du				.6317	.1333		.0254	315
dd					.2413	.2032	.1175	315
cu	.8508			.4587			.3444	630
ud			.6127			.1111	.0063	315
dc		.8905			.1667		.1476	630
cd	.9429					.0683	.0571	630
$\Pr[A=B=1]$	.8508	.7619	.3968	.5556	.1333	.1111		
$\Pr[B=C=1]$	.8508	.7619	.5556	.4587	.1667	.0683		
$\Pr[A=C=1]$	.8659	.7841	.3968	.4587	.1333	.0683		

FIG. 8: Empirical data (frequencies) for the conteNt-conteXt system in Fig. 6 for participant P2. The rest is as in Fig. 7.

P3	c-	-c	u-	-u	d-	-d	$\Pr[X=Y=1]$	# of trials
cc	.6791	.5973					.3654	1259
uc		.8302	.1349				.0905	630
uu			.2548	.1688			.0732	314
du				.1460	.3746		.0127	315
dd					.3460	.4127	.1397	315
cu	.8381			.0746			.0524	630
ud			.1178			.3917	.0159	314
dc		.6714			.2921		.1127	630
cd	.6968					.3238	.1746	630
$\Pr[A=B=1]$	.6791	.5973	.1349	.1460	.3460	.3917		
$\Pr[B=C=1]$	.6968	.6714	.1178	.0746	.2921	.3238		
$\Pr[A=C=1]$	.6791	.5973	.1178	.0746	.2921	.3238		

FIG. 9: Empirical data (frequencies) for the conteNt-conteXt system in Fig. 6 for participant P3. The rest is as in Fig. 7.

The outcome of the analysis is that, for all three participants, the system of linear equations has a solution in nonnegative values —that is, the data matrices in Figs. 7, 8, and 9 describe noncontextual systems of random variables. Note that in this case the empirical estimates were fit by the solution precisely, eliminating the need for statistical analysis.

4. CONCLUSION

The experiment presented in this paper illustrates the use of the double factorial paradigm in the search of contextuality in behavioral systems, namely in the responses of human observers in a double-detection task. This paradigm provides the closest analogue in psychophysical research to the Alice-Bob EPR/Bohm paradigm (Bell, 1964; Clauser et al., 1969; Fine, 1982). We have found that for the participants in the study there was no evidence of contextuality in their responses. These results add to the existing evidence that points towards lack of contextuality in behavioral data (Cervantes & Dzhafarov, 2017; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017). The present result is in fact stronger than the previous ones, as it uses a more stringent than before criterion of noncontextuality. This criterion is based on multimaximality rather than on the simple maximality of the couplings in cyclic systems. However, we should emphasize that in the absence of a predictive theory on a par with quantum mechanics, no failure to find contextuality in even a large number of experiments can be safely generalized: contextuality may very well be found under as yet unexplored modifications of experimental conditions. Consider, e.g., the Alice-Bob EPR/Bohm paradigm, and imagine that we have no theory that could guide us in choosing the specific axes along which Alice and Bob are to measure the spins in their respective particles. It would be rather unlikely to hit at a "right" combination of the angles by pure chance, and after numerous failures one could very well conclude, in this case wrongly, that contextuality is absent in this paradigm. More work is needed.

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