SORITES WITHOUT VAGUENESS II: COMPARATIVE SORITES

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Abstract. We develop a mathematical theory for *comparative* sorites, considered in terms of a system mapping pairs of stimuli into a binary response characteristic whose values supervene on stimulus pairs and are interpretable as the complementary relations 'are the same' and 'are not the same' (overall or in some respect). Comparative sorites is about hypothetical sequences of stimuli in which every two successive elements are mapped into the relation 'are the same', while the pair comprised of the first and the last elements of the sequence is mapped into 'are not the same'. Although soritical sequences of this kind are logically possible, we argue that their existence is grounded in no empirical evidence and show that it is excluded by a certain psychophysical principle proposed for human comparative judgments in a context unrelated to soritical issues. We generalize this principle to encompass all conceivable situations for which comparative sorites can be formulated.

1. Introduction

In the companion paper (Dzhafarov and Dzhafarov, 2010), we introduced a behavioral approach to sorites, with 'behavior' understood in the broadest possible sense: as the relationship between stimuli acting upon some system[1] and that system's responses to these stimuli. The *classificatory sorites* analyzed in the companion paper can be presented as a conjunction of three assumptions which imply the existence of *classificatory soritical sequences*, or finite sequences $x_1, \ldots, x_n$ with the property that $\pi(x_i) = \pi(x_{i+1})$ for all $i = 1, \ldots, n-1$, intuitively because each x_{i+1} is only 'microscopically different' from x_i, yet $\pi(x_1) \neq \pi(x_n)$. Here, π is a *stimulus-effect function* mapping stimuli into *stimulus effects*, by which we understand any property of response that supervenes on stimuli, e.g., response identity, response time, or probability distribution over responses. Classificatory soritical sequences are clearly a logical impossibility, implying that in any situation where they are considered at least one of the three underlying assumptions identified in the companion paper is not satisfied. This means that:

(1) either the function π is not a well-defined function, i.e., the stimulus effects are not uniquely determined by stimuli, as happens, e.g., when a 'vague predicate' is applied to a physical object;

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[1]Not necessarily sentient or biological: it can, e.g., be a technical gadget or a set of normative rules.

(2) or the function π is not 'tolerant to small changes', i.e., there are stimulus values in every vicinity of which, however small, the stimulus-effect function is not constant;

(3) or else the stimulus set is not properly connected, meaning that no two stimulus values x, y with $\pi(x) \neq \pi(y)$ can be connected by a chain of stimuli each of which is 'only microscopically' different from its predecessor.

The reader is referred to the companion paper for motivation behind the behavioral (stimulus-effect) approach, and for a full formal treatment of classificatory sorites, including, in particular, a rigorous and general definition of 'connectedness by microscopic steps'.

In the present paper we undertake a similar analysis of a different variety of soritical problem, which we call the *comparative sorites*. This pertains to situations where a system responds to *pairs of stimuli* (x, y), and these responses have a binary property (stimulus effect upon the system) which supervenes on the stimulus pairs and whose values are interpretable as two complementary relations, 'x is matched by y' and 'x is not matched by y'. A common example involves a set of color patches or line segments visually presented pairwise and a human observer indicating in response to every pair whether the two stimuli were identical or not. But to determine whether a response has a property which can be interpreted in match/not match terms one cannot rely on the semantics of the words, and indeed the responding system need not be linguistic to begin with (e.g., it can be a two-pan balance 'comparing' pairs of weights). Although human comparative judgments do provide prominent guidance for one's intuitions, the interpretation in question, in accordance with our behavioral approach, should only rely on certain relations between matching and not matching pairs of stimuli. This will be discussed in detail below.

An initial, vague description of the comparative sorites can be given as follows.

Comparative Sorites 'Paradox'. A set of stimuli S acting upon a system and presentable in pairs (say, line segments visually presented in pairs to a human observer) may contain a finite sequence of stimuli, which we call a *comparative soritical sequence*, $x_1, \ldots, x_n$, such that 'from the system's point of view' x_i is matched by x_{i+1} for $i = 1, \ldots, n-1$, but x_1 is not matched by x_n.[2]

[2]It should be mentioned at the outset that the supervenience of the matching relation on stimulus pairs is critical for our analysis: we deal with a function $\pi(x, y)$ which attains two values, 'match' and 'not match'. This leaves no room for the possibility that the predicate 'x is matched by y' may be 'vague', in the sense of being inconsistent (the truth value of the predicate is not determined by x and y), multivalued (the predicate has more than two truth values), or 'indeterminate' (the truth value of the predicate for given x and y cannot be ascertained). The comparative sorites 'paradox' cannot be formulated with such predicates, as this formulation requires that we know definitively that x_i is matched by x_{i+1} for $i = 1, \ldots, n-1$, and that x_1 is not matched by x_n. One can start with inconsistent or indeterminate relations as one's empirical basis, but one should come up with a computation yielding the matching relation as a well-defined binary function of stimulus pairs. We will discuss in Section 2 how such a computation can be performed for (generally inconsistent) human judgments of the form 'x and y are different' or 'y is greater than x'. As an example involving indeterminacy or multiple truth values, let 'x_i is matched by x_{i+1}' have an intermediate truth value or no definite truth value for $i = 1, \ldots, n-1$, but let 'x_1 is matched by x_n' be definitely false. It would suffice in this case to redefine the predicate so that 'true' be equated to 'not definitely false'.

Comparative soritical sequences are easily constructed in abstract or idealized physical settings (see Example 2.1 in the next section or its improved version in the discussion related to Figure 2.3). Thus, unlike with the classificatory sorites, we cannot hope to uncover an inconsistent set of assumptions behind the comparative sorites 'paradox'. But in the case of systems resembling human comparative judgments in their responses we can argue that there is no empirical support for the existence of comparative soritical sequences. Furthermore, empirical evidence supports a certain psychophysical principle (Regular Mediality/Minimality) which we can generalize and use to show the impossibility of comparative soritical sequences if matching stimulus pairs are 'appropriately defined'. An informal outline of this argument appears in Section 2, and the formal presentation in Section 3.

The difference between the classificatory and comparative sorites is deeper than that between classifying single stimuli and pairs of stimuli. What one calls a single stimulus depends on one's, to a large extent arbitrary, conceptual partitioning of what acts upon the system being studied. Nothing prevents one from redefining a pair of stimuli (x_i, x_j) into a single 'bipartite' stimulus x_{ij}, and treating 'match' and 'not match' as classificatory responses to x_{ij}. This, however, would not make the comparative sorites a special case of the classificatory one, as one can easily see by forming a sequence

$$x_{12}, x_{23}, \ldots, x_{n-1,n}, x_{1n}$$

and applying to it the formulations of the two forms of sorites. The rationale for classificatory sorites is simply inapplicable here: while $x_{i+1,i+2}$ may very well be treated as being only 'microscopically' different from $x_{i,i+1}$ for $i = 1, \ldots, n-2$ (possibly because x_i is very close to x_{i+1}, which in turn is very close to x_{i+2}), the difference between $x_{n-1,n}$ and x_{1n} need not be small in the same sense.

Nor can one consider the classificatory sorites a special case of the comparative one. Indeed, given a sequence $x_1, \ldots, x_n$ with x_{i+1} 'microscopically' different from x_i for $i = 1, \ldots, n-1$, one can, with some ingenuity, recast any stimulus-effect function $\pi(x)$ into a function of two arguments interpretable as their 'comparison'. Thus, $\pi(x)$ can be presented as $f(x, x_0)$ for some fixed x_0, or even as $f(x, g(x))$ where $g(x)$ is a function mapping x into a stimulus 'microscopically' different from x, so that x_{i+1} in the sequence $x_1, \ldots, x_{n-1}$ equals $g(x_i)$. There is nothing in the classificatory sorites, however, that would necessitate a comparison of x_1 and x_n, which is the crux of the comparative sorites.

There is, however, a simple sense in which the comparative sorites can be obtained as a 'logical consequence' of the classificatory one: by postulating the existence of some stimulus-effect function $\pi(x)$ such that the relation 'x is matched by y' holds if and only if $\pi(x) = \pi(y)$. Then, very clearly, the logical impossibility of the classificatory sorites established in the companion paper forces one to reject the possibility of (this form of) the comparative one. The remainder of the paper would not be necessary if it were obvious that for every matching relation such a stimulus-effect function can be found.

2. Comparative Sorites: Informal Considerations

The conceptual vagueness in our formulation of the comparative sorites is greater than in the informal descriptions of the classificatory sorites, and it requires a great deal more conceptual machinery to be 'sharpened'. Moreover, the different pieces of this conceptual machinery must be all in place simultaneously to support each other, and this will have to wait until the formal analysis is presented in Section 3. The present section is aimed at building a context within which these formal constructs will be justified.

The main difficulty lies, of course, in the fact that unlike in the classificatory sorites, where one deals with any, arbitrarily chosen responses of a system to stimuli, here we are dealing with a specific stimulus effect, a binary variable with values 'y matches x' and 'y does not match x'; and the choice of definition for this stimulus effect immediately determines the possibility or impossibility of the comparative soritical sequences. Here are two simple examples.

Example 2.1. Let the stimulus set S be the set of positive reals, say, representing weights. Then, if the relation 'matches' is defined to mean 'is approximately equal to', e.g., for some $\varepsilon > 0$,

$$\text{`}y \text{ matches } x\text{'} \iff |\log y - \log x| \leq \varepsilon,$$

then the (x, y)-pairs satisfying 'y matches x' form a reflexive and symmetric relation which is not transitive.[3] It is easy to see that such a relation allows for comparative soritical sequences: e.g., any sequence $1, e^{\delta}, e^{2\delta}$ with $\frac{\varepsilon}{2} < \delta \leq \varepsilon$.

This example confirms our observation that the comparative sorites with freely definable 'y matches x' is logically independent of the classificatory sorites. Indeed, given any non-constant stimulus-effect function π, it is readily seen that we can find an x such that $\pi(x) \neq \pi(y)$ for some y with $x < y \leq e^{\varepsilon}x$, even though then $|\log y - \log x| \leq |\log e^{\varepsilon}x - \log x| = \varepsilon$ and so 'y matches x'.

Example 2.2. To consider a case where the two varieties of sorites are interrelated, let the relation 'matches' be defined to mean 'identical in some (crude) property', e.g., for some $\lambda > 0$,

$$\text{`}y \text{ matches } x\text{'} \iff [\lambda x] = [\lambda y],$$

where $[a]$ is the floor of a, i.e., the greatest integer not exceeding a (thus, e.g., $[3.8] = 3$). Then the relation in question is reflexive, symmetric, and transitive, whence no comparative soritical sequences involving this relation are possible. This agrees with the obvious fact that $[\lambda x]$ can be viewed as a stimulus-effect function defined on individual number-stimuli, so any comparative soritical sequence thereby would imply the existence of a (logically impossible) classificatory one.

We see that the issue of the comparative sorites is to a large extent definitional. The matching relation can be so understood (Example 2.2) that comparative soritical sequences are as logically impossible as the classificatory ones. It can also be so understood, however, that comparative soritical sequences are possible, in a rather trivial sense too (Example 2.1). With some conceptual emendations, both our examples can represent behaviors of physically realizable systems (see, e.g., Figs. 2.3 and 2.3 in Section 2.6 below). What we argue for in this paper can be viewed, with caveats, as a position that the definition of matching in Example 2.2 (matching means identity in some, possibly crude, property) is a better choice than that in Example 2.1 (matching means approximate equality in some property).

We will present two justifications for this preference. The first and the main one is that human (more generally, biological) perception, usually presented as providing 'incontrovertible' evidence in favor of matching as approximate equality, provides

[3] This is an example of an algebraic structure known as a semi-order (see Luce 1956).

no such evidence; and that, moreover, a certain psychophysical principle proposed in a context unrelated to soritical considerations excludes the possibility of comparative soritical sequences. Our second justification is more general, although less compelling as it relates to theoretical desiderata rather than empirical generalizations: the definition of matching underlying Example 2.1 allows for the possibility that two stimuli matching each other 'from the point of view' of a system will differ from each other is some respect which, from the same 'point of view', is relevant for comparing these stimuli. The definition of matching underlying Example 2.2 does not allow for this possibility.

2.1. **A naive notion of differential threshold.** In the philosophical literature the consideration of comparative sorites is usually confined to human observers asked to indicate whether two given stimuli are 'the same' or 'different'.[4] Our approach is broader, but we do consider human comparative judgments a prominent prototype for a reasonable definition of matching. A brief overview of what is known of these judgments therefore should provide us with indispensable guidance.

It is often considered both common sense and a well established empirical fact that the relation described by the judgment 'y is the same as x' is reflexive, symmetric, but not transitive (Goodman 1951, Armstrong 1968, Dummett 1975, Wright 1975).[5] Essentially, this means the choice of Example 2.1 over Example 2.2. The 'common sense' argument is based on the general idea of the 'tolerance' of macroscopic stimulus effects to microscopic changes in stimuli: if y is judged to be the same as x, then y' must also be judged to be the same as x provided y' differs from y sufficiently little. We know from the companion paper that this argument is faulty for connected stimulus spaces as it leads to classificatory soritical sequences. Many believe, however, that in the case of pairwise comparisons a restricted form of tolerance known as *differential thresholds* or *just-noticeable differences* is all that the comparative sorites requires, and some believe that this restricted form of tolerance is a well-known empirical fact:

Differential Threshold Property. Any stimulus x always looks exactly the same as itself, i.e., (x, x) is a pair of indistinguishable stimuli; and there is always a small vicinity V_x of x, known in psychophysics as the *differential threshold at* x, any element of which looks the same as x. To construct a comparative soritical sequence all one needs is to find a sequence $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$ such that $x_{i+1} \in V_{x_i}$ for $i = 1, 2, \ldots n-1$, but x_n is outside V_{x_1}.

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[4]The literature on the comparative (or 'observational', as it is sometimes termed) sorites is not nearly as rich as that on the classificatory one. One interesting context in which comparative sorites has been discussed, pointed out to us by Gustav Arrhenius, is the situation when the choice between two 'indistinguishable' stimuli is associated with rewards and punishments (e.g., in the 'self-torturer' version discussed by Warren Quinn 1990, a person chooses between two very close intensities of electric shock, and is rewarded for choosing the higher of the two). This area is outside the scope of this paper as it focuses on the rationality or axiology of the person's decisions (Parfit 1984) rather than the rationale for the (non-)existence of comparative soritical sequences. Thus, Wlodek Rabinowicz's (1989, p. 44) analysis of whether in the situations in questions the rationality of the decision making is well-defined would be equally applicable if the pairwise differences (say, between the levels of pain in Quinn's example) were merely very small rather than unnoticeable, while the rewards for choosing the more painful option were sufficiently large.

[5]The transitivity of 'y is the same as x', together with its 'incontrovertible' reflexivity and symmetry, also has its proponents (see, e.g., Jackson and Pinkerton 1973, Graff 2001).

This 'empirical fact' is in reality an unfounded belief. Almost everything in it contradicts or oversimplifies what we know from modern psychophysics. A stimulus x does not necessarily look the same as 'itself' if one takes into account the fact that two stimuli being compared necessarily belong to distinct 'stimulus areas', as explained below. There is generally no special stimulus set containing x that can be designated as *the* differential threshold at x: a standard definition would be something like 'the set of stimuli whose probability of being judged greater than x is between $\frac{1}{2} - p$ and $\frac{1}{2} + p$', where p is chosen arbitrarily in the interval $]0, 1/2[$. Judgments like 'x looks the same as y' or 'x is greater than y' given by human observers in response to stimulus pairs cannot generally be considered predicates on the set of stimulus pairs, as these responses do not supervene on them: an indirect approach is needed to derive a matching relation from these (inconsistent) judgments.[6] Whether this (generally) derived matching relation is reflexive and symmetric is not a trivial question, and its transitivity or intransitivity may not even be formulable in a conventional way.

2.2. **Stimulus values and stimulus areas.** A correct formulation of the matching relation and its properties requires the notion of distinct 'stimulus areas'. It is a simple but fundamental fact that in order to speak of two stimuli appearing the same or different, one does have to deal with *two distinct stimuli*. In particular, to say meaningfully that two physically identical stimuli, x and x, are judged as being the same or different, overall or in some specified respect, the two x's have to designate identical properties of two otherwise different stimuli. Thus, one of them can be presented on the left and another on the right from a certain point, or one presented chronologically first and the other second (with a sufficient separation in-between to prevent perceptual interference). Otherwise 'x and x' would mean a single stimulus x, and instead of asking an observer whether two given stimuli are the same or different, one would be asking whether the single stimulus being presented exhibits some specified property, such as flicker or motion in the case of visual stimuli. In psychophysics, tasks of the latter variety are classified as *detection tasks*, as opposed to perceptual *discrimination*, or *pairwise comparison* tasks. Only pairwise comparison tasks pertain to comparative sorites as defined in this paper, in terms of a system responding to pairs of stimuli. Even if one allows for experimental paradigms intermediate between detection of change and comparison, it would be safe and wise to confine one's attention to clear-cut cases of comparison only, such as the one shown in Figure 2.1, where two stimuli occupy fixed, well separated spatial positions in the visual field (left-right) and are clearly so perceived. Everything else in the two stimuli is the same and fixed, except for their levels of luminance, x and y. The stimuli themselves, therefore, should be referred to by both their *values* (levels of luminance x, y) and their *stimulus areas* (left and right).[7] The complete reference therefore is $(x, left)$ and $(y, right)$, or

[6] To prevent misunderstanding, our view of matching is not critically based on the fact of statistical variability of human responses, however basic this fact may be for psychophysical theory. Thus, our formal treatment of matching in Section 3 does not utilize probabilistic notions. The point being made is that the statement presented as the Differential Threshold Property above is not an account of factual knowledge, as it is usually taken to be, but rather a simplistic theoretical belief.

[7] The term used in Dzhafarov (2002) where the concept was introduced in a systematic fashion was 'observation area', but 'stimulus area' seems preferable in view of the intended generalizations of the present analysis to non-perceptual responses.

more briefly,

$$x^{(l)} \text{ and } y^{(r)}.$$

Of course, stimulus values may be much more complex than levels of luminance (consider comparing two motions, two faces, two pieces of melody), and stimulus areas need not be defined only by spatiotemporal positions of stimuli. Thus, the two stimuli in Figure 2.1 while being compared in their brightness may be of two different fixed colors, two line segments compared in their length may be of two different fixed orientations, and of two faces compared in terms of depicting the same or different persons one can be a still photograph and the other a short movie clip.

FIGURE 2.1. A prototypical example of a stimulus pair presented for a comparative judgment such as "are they the same in brightness?" or "which of them is brighter?"

2.3. **Psychometric functions and definition of matching.** Let us see now how the matching relation is defined in modern psychophysics (following Luce and Galanter 1963, and Dzhafarov 2002, 2003). Figure 2.2 provides an illustration for the case where stimulus values are represented by an interval of real numbers, as, e.g., the luminance values in Figure 2.1. The right-hand panels show two cross-sections of a typical 'probability of being judged to be different' function,

$$\psi(x^{(l)}, y^{(r)}) = \Pr[x^{(l)} \text{ and } y^{(r)} \text{ are judged to be different}],$$

for an observer asked to say whether two stimuli are the same or different, either with respect to a specified subjective property or overall, but ignoring the conspicuous difference in the stimulus area. If, as is the case with the stimuli in Figure 2.1, the percepts of the stimuli contain subjective components which are linearly ordered (such as brightness, heaviness, loudness, etc.), then the question to an observer can also be formulated in terms of which of the two stimuli has a greater value of this subjective component. The left-hand panels in Figure 2.2 show two cross-sections of a typical 'probability of being judged to be greater' ('brighter', 'heavier', etc.) function,

$$\gamma(x^{(l)}, y^{(r)}) = \Pr[y^{(r)} \text{ is judged to be greater than } x^{(l)}].$$

One conspicuous feature of both these probability functions is that they attain values between 0 and 1: the responses 'same/different' and 'greater/less' per se, taken as binary variables, do not supervene on stimulus pairs. The best one can hope for here, and what a psychophysicist would routinely assume, is that the probabilities $\psi(x^{(l)}, y^{(r)})$ or $\gamma(x^{(l)}, y^{(r)})$ of these responses are stimulus-effect functions,

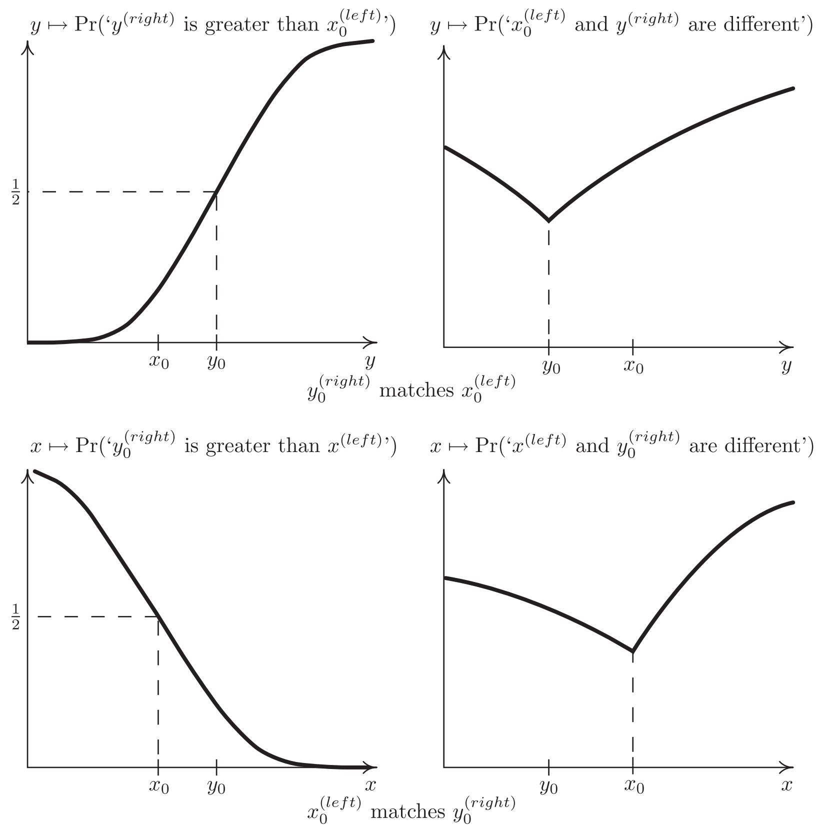

FIGURE 2.2. Graphs of psychometric functions $\psi(x^{(l)}, y^{(r)}) = \Pr[x^{(l)}$ and $y^{(r)}$ are judged to be different] (right-hand panels) and $\gamma(x^{(l)}, y^{(r)}) = \Pr[y^{(r)}$ is judged to be greater than $x^{(l)}]$ (left-hand panels) cross-sectioned at $x = x_0$ (top panels) and $y = y_0$ (bottom panels). For same-different judgments, a point y_0 is a matching point for x_0 if $\psi(x_0^{(l)}, y_0^{(r)})$ is the minimum value for $\psi(x_0^{(l)}, y^{(r)})$ across all stimuli $y^{(r)}$; and a point x_0 is a matching point for y_0 if $\psi(x_0^{(l)}, y_0^{(r)})$ is the minimum value for $\psi(x^{(l)}, y_0^{(r)})$ across all stimuli $x^{(l)}$. Note that in this picture, for one and the same pair (x_0, y_0), y_0 matches x_0 and x_0 matches y_0. For greater-less judgments, a point y_0 is a matching point for x_0 (and then also x_0 is a matching point for y_0) if $\gamma(x_0^{(l)}, y_0^{(r)}) = \frac{1}{2}$.

i.e. are uniquely determined by stimulus pairs.[8] The probability function ψ can be

[8]In the philosophical literature the probabilistic nature of matching was emphasized by C.L. Hardin (1988). As stated in the companion paper, one may object to treating probabilities as stimulus effects on the grounds that probabilities do not characterize individual responses. We do not share this concern, which is at odds with the established views in physics and behavioral sciences, but a particular ontology of probabilities is not critical for our approach (see footnote 6). If one denies probabilities the status of something 'occurring in response to', one should

interpreted as a 'difference' measure, and the same interpretation can be afforded to $|\gamma - \frac{1}{2}|$.

Let us generically denote the relation '$x^{(l)}$ is matched by $y^{(r)}$' by $x^{(l)}\mathrm{M}y^{(r)}$, using the variants M_ψ and M_γ to indicate whether this relation is defined from the function ψ or the function γ. The relation M_ψ is naturally defined by

$$\begin{array}{c} x^{(l)}\mathrm{M}_\psi y^{(r)} \text{ iff } \psi(x^{(l)}, y^{(r)}) = \min_z \psi(x^{(l)}, z^{(r)}) \\ y^{(r)}\mathrm{M}_\psi x^{(l)} \text{ iff } \psi(x^{(l)}, y^{(r)}) = \min_z \psi(z^{(l)}, y^{(r)}) \end{array}, \tag{2.1}$$

whereas M_γ is defined by

$$\begin{array}{c} x^{(l)}\mathrm{M}_\gamma y^{(r)} \text{ iff } \gamma(x^{(l)}, y^{(r)}) = \frac{1}{2} \\ y^{(r)}\mathrm{M}_\gamma x^{(l)} \text{ iff } \gamma(x^{(l)}, y^{(r)}) = \frac{1}{2} \end{array}. \tag{2.2}$$

No claim is being made that $\mathrm{M}_\psi \equiv \mathrm{M}_\gamma$. In fact, there are many procedures and procedural variants by which one can obtain matching pairs of stimuli,[9] and no two of them would generally define one and the same matching relation. Whichever definition is applied, however, it leaves no grounds for the belief formulated above as the Differential Threshold Property. The notion of a differential threshold is merely a crude characterization of the rate of increase of a psychometric function near its median (if using γ) or minimum (for ψ). When using $y \mapsto \gamma(x_0^{(l)}, y^{(r)})$ it may be defined (in the 'right' stimulus area) as the interval between $y_{\gamma^*}^{(r)}$ and $y_{1/2}^{(r)}$, or between $y_{\gamma^*}^{(r)}$ and $y_{1-\gamma^*}^{(r)}$, where γ^* is an arbitrarily chosen probability between $\frac{1}{2}$ and 1, and $y_{\gamma^*}^{(r)}$ is defined so that

$$\gamma(x^{(l)}, y_{\gamma^*}^{(r)}) = \gamma^*.$$

Following a psychophysical tradition, the interval $]y_{1/2}^{(r)}, y_{\gamma^*}^{(r)}[$, or $]y_{1-\gamma^*}^{(r)}, y_{\gamma^*}^{(r)}[$, can be called a γ^*-threshold (because different choices of γ^* define different intervals). The definition of a threshold in the 'left' stimulus area, for a fixed $y_0^{(r)}$, is analogous.

There is no tradition of computing thresholds from $y \mapsto \psi(x_0^{(l)}, y^{(r)})$ and $x \mapsto \psi(x^{(l)}, y_0^{(r)})$, but if needed they may be defined by the intervals on which these functions do not exceed some elevation $\Delta\psi$ above their minimum values (and then they can be termed $\Delta\psi$-thresholds).

Whatever the case, a differential threshold is not a stimulus subset whose elements all match a fixed stimulus in another stimulus area — unless one simply defines 'y matches x' by x falling within a γ^* or $\Delta\psi$ differential threshold for x, with arbitrarily chosen γ^* or $\Delta\psi$. Such a definition would allow one to form comparative soritical sequences, of the kind considered in Example 2.1, but this would hardly be of much theoretical interest. It seems more interesting to us to take for prototypes of matching the relations in Eqs. 2.1 and 2.2.

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either give up an attempt to base one's analysis of comparative sorites on human comparative judgments, or else seek a redefinition of $x^{(l)}$ and $y^{(r)}$ to include 'hidden' components determining these judgments.

[9]The traditional psychophysical term for a stimulus $b^{(\beta)}$ matching stimulus $a^{(\alpha)}$ is a *point of subjective equality* for $a^{(\alpha)}$ (whatever the physical meaning of α and β, left-right, first-second, or anything else.)

2.4. **Regular Mediality and Regular Minimality.** We will now stipulate two properties of the matching relation M, as defined in Eqs. 2.1 and 2.2 and illustrated in Figure 2.2, which are critical for our treatment of comparative sorites. These properties constitute a principle which is called *Regular Mediality* when applied to γ and M_γ and *Regular Minimality* when applied to ψ and M_ψ (Dzhafarov 2002, 2003; Dzhafarov and Colonius 2006). We present this principle in a form modified to better suit our purposes. Here is its first statement.

Regular Mediality/Minimality, part 1 (RM1). For every stimulus in either of the two stimulus areas (in our example, right or left) one can find a stimulus in the other stimulus area (respectively, left or right) such that if $x^{(l)}$ and $y^{(r)}$ are the stimuli in question then

$$x^{(l)}\mathrm{M}y^{(r)} \text{ and } y^{(r)}\mathrm{M}x^{(l)}.$$

With regards to γ this assumption is very nonrestrictive. It only requires that, for every $x_0^{(l)}$, the function $y \mapsto \gamma(x_0^{(l)}, y^{(r)})$ does reach the median level $\gamma = \frac{1}{2}$ at some point $y_0^{(r)}$. Then the function $x \mapsto \gamma(x^{(l)}, y_0^{(r)})$ should reach the median level at the point $x_0^{(l)}$: the relations $x_0^{(l)}\mathrm{M}_\gamma y_0^{(r)}$ and $y_0^{(r)}\mathrm{M}_\gamma x_0^{(l)}$ mean one and the same thing, $\gamma(x_0^{(l)}, y_0^{(r)}) = \frac{1}{2}$.

With regards to ψ the assumption is more restrictive. Requiring that the functions $y \mapsto \psi(x_0^{(l)}, y^{(r)})$ and $x \mapsto \psi(x^{(l)}, y_0^{(r)})$ do reach their minima at some points is innocuous enough, so that every $x_0^{(l)}$ has a $y_0^{(r)}$ matching it, and every $y_0^{(r)}$ has an $x_0^{(l)}$ matching it. But it does not follow that, for every x_0, if y_0 minimizes the function $y \mapsto \psi(x_0^{(l)}, y^{(r)})$ then x_0 minimizes the function $x \mapsto \psi(x^{(l)}, y_0^{(r)})$. In other words, the relations $x_0^{(l)}\mathrm{M}_\psi y_0^{(r)}$ and $y_0^{(r)}\mathrm{M}_\psi x_0^{(l)}$ are not mathematically equivalent, so their conjunction in RM1 is an independent assumption.

The next property of M which is apparent in Figure 2.2 is that the matches are determined uniquely. There is only one $y^{(r)}$ that matches a given $x^{(l)}$ (satisfies $x^{(l)}\mathrm{M}y^{(r)}$), and inversely, there is only one $x^{(l)}$ that matches a given $y^{(r)}$ (satisfies $y^{(r)}\mathrm{M}x^{(l)}$). In view of RM1 one can equivalently say that there is only one $x^{(l)}$ satisfying $x^{(l)}\mathrm{M}y^{(r)}$ for a given $y^{(r)}$, and for this $x^{(l)}$, $z^{(r)}\mathrm{M}x^{(l)}$ holds only when $z^{(r)} = y^{(r)}$. These statements, however, are too restrictive to be generalized to arbitrary sets of stimulus values. Consider, e.g., the possibility that the two circles in Figure 2.1 can in fact be presented in different sizes, varying from trial to trial, but that the observer's task remains the same: to compare the two stimuli in their brightness. It is well known that a stimulus of luminance level l_1 and size s_1 can have the same (subjective) brightness as a stimulus of some other luminance l_2 and size s_2, regardless of whether the two stimuli belong to the 'left' or 'right' stimulus area. One would expect then that $(l_2, s_2)^{(l)}\mathrm{M}(l, s)^{(r)}$ would be true if and only if $(l_1, s_1)^{(l)}\mathrm{M}(l, s)^{(r)}$ is also true even though $(l_1, s_1) \neq (l_2, s_2)$. That is, a given right stimulus would match more than one left stimulus (and, of course, vice versa). A more familiar example: think of all possible radiometric spectra which produce a color with a given color appearance, as measured, e.g., by conventional CIE coordinates. Such examples suggest that the uniqueness of stimuli matching and being matched by a given stimulus should be more generally replaced with their equivalence, in the following sense.

Let two stimuli in a given stimulus area be called *equivalent* if the sets of stimuli they match in the other stimulus area are identical. That is, $x_1^{(l)}$ and $x_2^{(l)}$ are equivalent, in symbols $x_1^{(l)}\mathrm{E}x_2^{(l)}$, if, for every $y^{(r)}$,

$$y^{(r)}\mathrm{M}x_1^{(l)} \iff y^{(r)}\mathrm{M}x_2^{(l)},$$

and $y_1^{(r)}$ and $y_2^{(r)}$ are equivalent, $y_1^{(r)}\mathrm{E}y_2^{(r)}$, if, for every $x^{(l)}$,

$$x^{(l)}\mathrm{M}y_1^{(r)} \iff x^{(l)}\mathrm{M}y_2^{(r)}.$$

Except for the use of stimulus areas (which allows for the possibility that equivalent values may be different in different stimulus areas), this is essentially Nelson Goodman's (1951) definition of perceptual indistinguishability.[10]

The second part of the Regular Minimality/Mediality principle now can be formulated as follows. In essence it says that matches are determined uniquely up to the equivalence relation E.

Regular Mediality/Minimality, part 2 (RM2). Any two stimuli in one stimulus area are equivalent if they are matched by one and the same stimulus in the other stimulus area, i.e. (continuing to use our example with left and right),

$$\begin{array}{l} \text{if } x_1^{(l)}\mathrm{M}y^{(r)} \text{ and } x_2^{(l)}\mathrm{M}y^{(r)} \text{ then } x_1^{(l)}\mathrm{E}x_2^{(l)} \\ \text{if } y_1^{(r)}\mathrm{M}x^{(l)} \text{ and } y_2^{(r)}\mathrm{M}x^{(l)} \text{ then } y_1^{(r)}\mathrm{E}y_2^{(r)} \end{array}.$$

2.5. **Properties of matching following from Regular Mediality/Minimality.**

It is easy to see that RM1-RM2 imply the symmetry of the matching relation M in the following form:

$$\mathrm{Sym}: \ x^{(l)}\mathrm{M}y^{(r)} \Longleftrightarrow y^{(r)}\mathrm{M}x^{(l)},$$

for all $x^{(l)}, y^{(r)}$.

Note that this property has nothing to do with invariance with respect to an exchange of values between the two stimulus areas, i.e. the properties whose formulations are: for all x, y,

$$\overrightarrow{\mathrm{Exch}}: \ x^{(l)}\mathrm{M}y^{(r)} \Longleftrightarrow y^{(l)}\mathrm{M}x^{(r)}$$

and

$$\overleftarrow{\mathrm{Exch}}: \ y^{(r)}\mathrm{M}x^{(l)} \Longleftrightarrow x^{(r)}\mathrm{M}y^{(l)}.$$

The symmetry property Sym says: if for a given $x^{(l)}$ one finds a $y^{(r)}$ matching it, then this $x^{(l)}$ will also match this $y^{(r)}$, and vice versa. The values x and y remain in their respective stimulus areas (left and right, respectively) in both parts of this statement. By contrast, in $\overrightarrow{\mathrm{Exch}}$ and $\overleftarrow{\mathrm{Exch}}$ the values x and y exchange their stimulus areas: the pairs $(x^{(l)}, y^{(r)})$ and $(y^{(l)}, x^{(r)})$ are not the same two stimuli differently ordered: if $x \neq y$, then together they contain four distinct stimuli. There is therefore no compelling reason to expect that the properties $\overrightarrow{\mathrm{Exch}}$ and $\overleftarrow{\mathrm{Exch}}$ hold empirically, and in fact they generally do not. On the other hand, the 'true' symmetry property Sym seems to be supported by all available empirical evidence (Dzhafarov 2002; Dzhafarov and Colonius 2006) and underlies all psychophysical models dealing with matching-type relations, where traditionally one speaks of 'subjectively equal' or 'matching' stimuli without specifying which of them matches which.

[10] In Dzhafarov and Colonius (2006) the equivalence is defined in a stronger way: $x_1^{(l)}\mathrm{E}x_2^{(l)}$ if $\psi(x_1^{(l)}, y^{(r)}) = \psi(x_2^{(l)}, y^{(r)})$ for all $y^{(r)}$, and $y_1^{(r)}\mathrm{E}y_2^{(r)}$ if $\psi(x^{(l)}, y_1^{(r)}) = \psi(x^{(l)}, y_2^{(r)})$, for all $x^{(l)}$. The weaker definition adopted in this paper is sufficient for our purposes and is more easily generalizable beyond perception, to arbitrary systems with match/not match-type responses.

The relation M may only hold between stimuli belonging to different stimulus areas, never one and the same area. One cannot compare $x^{(l)}$ to $y^{(l)}$ without redefining the operational meaning of the stimulus areas, e.g., by switching from the left-right scheme to a first-second (in time) one: but then $(x^{(l)}, y^{(l)})$ would have to be properly labeled as $(x^{(first)}, y^{(second)})$. This consideration implies

$$\neg x^{(l)}\mathrm{M}y^{(l)} \text{ and } \neg x^{(r)}\mathrm{M}y^{(r)},$$

for any x and y.[11] In particular, M is irreflexive,

$$\text{Irrefl}: \;\; \neg x^{(l)}\mathrm{M}x^{(l)} \text{ and } \neg x^{(r)}\mathrm{M}x^{(r)},$$

for any x.

Again, this statement should not be confused with questions about whether two stimuli with the same value but belonging to different stimulus areas always match each other, i.e. whether it is true that, for all x,

$$\overrightarrow{\text{Refl}}: \;\; x^{(l)}\mathrm{M}x^{(r)}$$

and

$$\overleftarrow{\text{Refl}}: \;\; x^{(r)}\mathrm{M}x^{(l)}.$$

These questions are generally answered in the negative (see Figure 2.2). In any case, although legitimate, they are of little interest to us, as they critically depend on which of the (potential infinity of) equivalent representations of stimulus values one chooses to use.[12]

Comparative sorites is usually discussed in terms of the transitivity of the matching relation M, with its reflexivity and symmetry being taken for granted. Upon the introduction of stimulus areas and careful formulation of what psychophysics tells us about comparative judgments, it is clear that M is irreflexive but (if one accepts RM1-RM2) symmetric in the meaning of Sym. The transitivity of M is easily seen to be false (or unformulable) in the traditional, triadic way: if $a^{(l)}\mathrm{M}b^{(r)}$ and $b^{(r)}\mathrm{M}c^{(l)}$, we know that $\neg a^{(l)}\mathrm{M}c^{(l)}$, because no two stimuli within the same stimulus area can be compared. In this sense one can say that M is intransitive.

It is more constructive, however, to look at the following suitably modified, tetradic formulation of transitivity:

$$\text{Trans}: \begin{cases} \text{if } a^{(l)}\mathrm{M}b^{(r)} \text{ and } b^{(r)}\mathrm{M}c^{(l)} \text{ and } c^{(l)}\mathrm{M}d^{(r)} \text{ then } a^{(l)}\mathrm{M}d^{(r)} \\ \text{if } a^{(r)}\mathrm{M}b^{(l)} \text{ and } b^{(l)}\mathrm{M}c^{(r)} \text{ and } c^{(r)}\mathrm{M}d^{(l)} \text{ then } a^{(r)}\mathrm{M}d^{(l)} \end{cases}.$$

It can be shown that this notion of transitivity does follow from RM1-RM2. So, if one is guided by what is known, or at least what does not contradict what is known about human comparative judgments (as opposed to conceptually and empirically erroneous beliefs like the Differential Threshold Property), then the relation M satisfies the (tetradic form of) transitivity. The transitivity does not allow for a

[11] One could also say that M is undefined for two stimuli from the same stimulus area, rather than that the statements in question are false. The present approach is adopted more or less arbitrarily, as it appears to be more convenient for formalization.

[12] Consider a simple example. Let the two patches in Figure 2.1 be of different but fixed sizes, say, the right one a smaller one. The stimulus areas then can be referred to as 'left and large' and 'right and small'. The validity of $\overrightarrow{\text{Refl}}$ and $\overleftarrow{\text{Refl}}$ will be determined by whether the values x of the stimuli are defined as luminance, luminance×area, or any other (perhaps unconventional, from a physicist's view) combination of luminance and size (say, luminance×diameter).

comparative soritical sequence. The latter should now be redefined to indicate a sequence of $2n$ stimuli ($n \geq 2$)

$$a_1^{(\alpha)}, b_1^{(\beta)}, a_2^{(\alpha)}, b_2^{(\beta)}, \ldots, a_n^{(\alpha)}, b_n^{(\beta)},$$

in which (α, β) is either (left,right) or (right,left), each stimulus is matched by the next one, but $\neg a_1^{(\alpha)} \mathrm{M} b_n^{(\beta)}$.

2.6. **A plausible argument for (tetradic) transitivity of matching across two stimulus areas.** In reference to our two 'naive' opening Examples 2.1 and 2.2, Trans implies that the latter example is being upheld over the former. To deal with more general and more adequately constructed examples, with stimulus values properly associated with stimulus areas, consider the idealized physical systems depicted in Figs. 2.3 and 2.4. The system in Figure 2.3 generalizes Example 2.1: the (corrected version of the) latter is obtained if one sets $\alpha = 1-\beta$ and $\varepsilon = \log \frac{1-\alpha}{\alpha}$. The system in Figure 2.4 generalizes (and corrects) Example 2.2, which is obtained by setting $\kappa = \lambda$. To the extent these two examples provide a reasonable guide for one's intuition, our view of the matching relation M can be characterized by saying that M indicates precise equality of stimuli in some, possibly crude property (Figure 2.4) rather than their approximate equality (Figure 2.3). This view is supported, or at least not contradicted by available psychophysical evidence and theory. Its plausibility can also be justified by the following reasoning.[13]

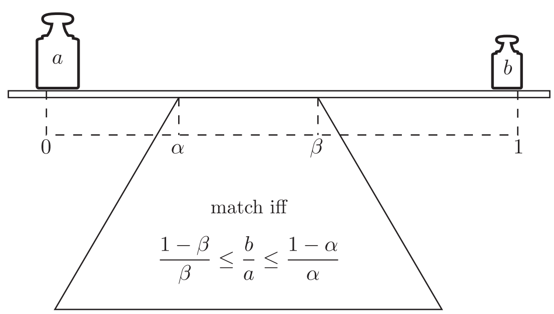

FIGURE 2.3. An unequal-arms balance beam placed on a flat-top fulcrum. This weight comparison system obeys RM1 but not RM2: consequently it is irreflexive, symmetric, and not transitive. The formula shows the equilibrium condition in accordance with Archimedes's law.

Assume that a system possesses three stimulus-effect functions: $\mu(x^{(l)}, y^{(r)})$, $\pi_l(x)$, and $\pi_r(y)$. The function $\mu(x^{(l)}, y^{(r)})$ has two values, 'm' and '¬m', interpreted as indicating the match and not match relations $x^{(l)} \mathrm{M} y^{(r)}$ and $\neg x^{(l)} \mathrm{M} y^{(r)}$, respectively. Let the effects of individual stimuli upon the system, $\pi_l(x)$, and $\pi_r(y)$, be interpreted as indicating certain properties of $x^{(l)}$ and $y^{(r)}$ which are 'relevant for comparison'. This proviso may appear too vague to be of use, but there is a simple way to ensure it without attempting to define what the relevant properties

[13]Which is essentially a streamlined version of what Delia Graff calls "the truisms that if two things look the same then the way they look is the same and that if two things look the same then if one looks red, so does the other" (Graff, 2001).

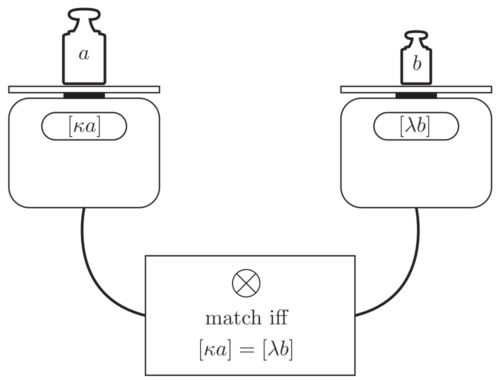

FIGURE 2.4. Two digital scales connected by a comparison gadget which indicates a 'match' if and only if the two digital scales show one and the same number. The latter, in response to weight x, equal to $[\kappa x]$ for the left scale and $[\lambda x]$ for the right one, where $[z]$ denotes the integer part of z. This weight comparison system satisfies both RM1 and RM2. Consequently it is irreflexive, symmetric, and (tetradic) transitive.

are. Let $(x_0^{(l)}, y_0^{(r)})$ be a pair such that $\mu(x_0^{(l)}, y_0^{(r)}) = \mathrm{m}$, and let $\pi_l(x)$ and $\pi_r(y)$ be defined by

$$\begin{cases} \pi_l(x) = \mu(x^{(l)}, y_0^{(r)}) \\ \pi_r(y) = \mu(x_0^{(l)}, y^{(r)}) \end{cases}.$$

Thus $\pi_l(x)$ and $\pi_r(y)$ each take on 'm' and '¬m' as their possible values, and, being specializations of μ, they clearly characterize $x^{(l)}$ and $y^{(r)}$ in terms relevant for their comparison. It is reasonable to expect now that $\mu(x^{(l)}, y^{(r)}) = \mathrm{m}$ should imply $\pi_l(x) = \pi_r(y)$, although not necessarily vice versa. One can easily verify that the system in Figure 2.4 indeed complies with this desideratum:

$$\underbrace{\mu(x^{(l)}, y^{(r)}) = \mathrm{m}}_{[\kappa x]=[\lambda y]} \text{ and } \underbrace{\mu(x_0^{(l)}, y_0^{(r)}) = \mathrm{m}}_{[\kappa x_0]=[\lambda y_0]} \implies \underbrace{\pi_l(x) = \mathrm{m}}_{[\kappa x]=[\lambda y_0]} \text{ iff } \underbrace{\pi_r(y) = \mathrm{m}}_{[\lambda y]=[\kappa x_0]}.$$

But the system in Figure 2.3 may violate it: choose, e.g., $\alpha = 1 - \beta = \frac{1}{e+1}$, $x_0 = y_0 = 1$, $x = e^{3/4}$, $y = e^{5/4}$, and observe that

$$\mu(x^{(l)}, y^{(r)}) = \mathrm{m}$$

but

$$\pi_l(x) = \mathrm{m} \neq \pi_r(y) = \neg\mathrm{m}.$$

2.7. **Multiple stimulus areas.** So far we have only discussed the comparisons of stimuli which belong to two fixed stimulus areas, like 'left' and 'right'. The generalization of this discussion to situations where the two stimuli being compared belong to any two of multiple stimulus areas (as in Figure 2.5) is far from trivial, which may be the main reason why the principle of Regular Minimality/Mediality in psychophysics has only been formulated for two stimulus areas. Our presentation of this principle in the form RM1-RM2 is chosen with the aim of such a generalization, to be presented formally in Section 3 . Note, in reference to Figure 2.5, that all four stimuli in the upper panel and all three in the lower could in principle be presented simultaneously, and the responses could then be recorded for all the pairwise relations involved (procedural variants are numerous).

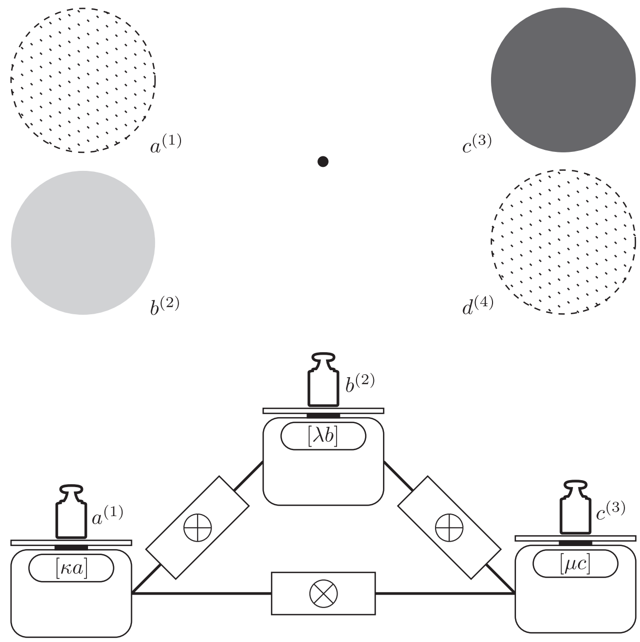

FIGURE 2.5. Examples of multiple stimulus areas for pairwise comparisons. Top panel: two stimuli being presented to a human observer for a comparative judgment (see Figure 2.1) can occupy any two of the four fixed locations. Bottom: the comparison of weights can be made by using any two of the three scales like the ones shown in Figure 2.4.

Here, we will briefly outline analogues of RM1-RM2 for three distinct stimulus areas, denoted $1, 2, 3$. This is not a straightforward generalization of the previously considered case, with two distinct stimulus areas. Rather, as shown in Section 3, these two cases, the 'bi-areal' and 'tri-areal' ones, can be viewed as forming two basic prototypes to which any other case can be reduced to.

Tri-areal Regular Mediality/Minimality, part 1 (RM*1). For every value x in any stimulus area one can find stimulus values y and z in the other two stimulus areas such that any two of the three stimuli match each other.

Thus, if x, y, z belong to the areas $1, 2, 3$, respectively, then

$$\begin{array}{ll} x^{(1)}\mathrm{M}y^{(2)} & y^{(2)}\mathrm{M}x^{(1)} \\ x^{(1)}\mathrm{M}z^{(3)} & z^{(3)}\mathrm{M}x^{(1)} \\ y^{(2)}\mathrm{M}z^{(3)} & z^{(3)}\mathrm{M}y^{(2)} \end{array}$$

In essence, this assumption generalizes (and corrects) the naive belief that one can always choose the same stimulus value x in all three stimulus areas, and any two of them will match each other.

Tri-areal Regular Mediality/Minimality, part 2 (RM*2). If two stimuli in a given stimulus area, say, $x_1^{(1)}$ and $x_2^{(1)}$, are matched by one and the same stimulus in another stimulus area, say, $y^{(2)}$ or $z^{(3)}$, then $x_1^{(1)}$ and $x_2^{(1)}$ are equivalent, in the sense that $x_1^{(1)}$ matches another stimulus if and only if so does $x_2^{(1)}$.

It can be shown now that M thus defined is irreflexive, symmetric, and transitive in the traditional triadic sense:

$$\text{Trans}^*: \text{ if } a^{(\alpha)}\text{M}b^{(\beta)} \text{ and } b^{(\beta)}\text{M}c^{(\gamma)} \text{ then } a^{(\alpha)}\text{M}c^{(\gamma)},$$

for all permutations (α, β, γ) of $(1, 2, 3)$. (One may find it surprising that the tri-areal case provides for a more conventional formulation of transitivity than the bi-areal case.)[14]

A comparative soritical sequence should here be defined as

$$a_1^{(\omega_1)}, a_2^{(\omega_2)}, \ldots, a_n^{(\omega_n)},$$

where each ω_i is $1, 2$, or 3, $a_i^{(\omega_i)}\text{M}a_{i+1}^{(\omega_{i+1})}$ for all $i = 1, 2, \ldots, n-1$ (which requires $\omega_i \neq \omega_{i+1}$), $\omega_1 \neq \omega_n$, and $\neg a_1^{(\omega_1)}\text{M}a_n^{(\omega_n)}$. As in the bi-areal case, the transitivity property here makes such sequences impossible. The inequality $\omega_1 \neq \omega_n$ is important, because if $\omega_1 = \omega_n$ (consider, e.g., a sequence $a^{(1)}, b^{(2)}, c^{(3)}, d^{(1)}$) then $\neg a_1^{(\omega_1)}\text{M}a_n^{(\omega_n)}$ 'automatically'.

3. A Formal Treatment of Comparative Sorites

We are ready now to embark on a systematic formal treatment of the general case. Stimuli will be assumed to belong to a set $S \times \Omega$, where S is a set of stimulus values and Ω a set of stimulus areas, both containing at least two elements. We will continue to use the more convenient $x^{(\omega)}$ in place of (x, ω) for stimuli, the elements of $S \times \Omega$.

We will never need to equate or compare values of two stimuli belonging to different stimulus areas. In other words, in relating $x^{(\alpha)}$ to $y^{(\beta)}$ ($\alpha, \beta \in \Omega$, $\alpha \neq \beta$), two stimuli which are different by virtue of belonging to different stimulus areas, we never need to compare their values x and y, e.g., to assert their equality. This is an important observation, leading to the following generalized interpretation of our conceptual set-up:

Generalized Interpretation. Different sets $S \times \{\omega\}$ and $S \times \{\omega'\}$ may simply be viewed as sets with different, further unanalyzable elements. In other words, instead of $S \times \Omega$ we can think of a collection of sets S_ω indexed by an arbitrary set Ω (whose elements, the indices, are called stimulus areas).

This 'automatic' generalization is indispensable in situations where one would want to speak of matching between entities of different nature, e.g., abilities of examinees and difficulties of the tests offered to them (as is routinely done in psychometric models). One need not keep this generalization in mind throughout the rest of this paper, but it determines the style of how we quantify our formal statements: e.g., we prefer to say

for any $\omega, \omega' \in \Omega$ and $a^{(\omega)}, b^{(\omega')} \in S \times \Omega$, the stimuli $a^{(\omega)}$ and $b^{(\omega')}$ are ...

[14]Of course, the tetradic transitivity is satisfied for any two of the three stimulus areas.

instead of the seemingly more economic

for any $\omega, \omega' \in \Omega$ and $a, b \in S$, the stimuli $a^{(\omega)}$ and $b^{(\omega')}$ are ...

3.1. **Matching relation and matching sequences.** The stimulus set $S \times \Omega$ is assumed to be endowed with a binary relation $x_1^{(\omega_1)} \mathrm{M} x_2^{(\omega_2)}$ (read as 'x_1 in ω_1 is matched by x_2 in ω_2' or 'x_2 in ω_2 matches x_1 in ω_1', see Figure 3.1) thus promoting the set $S \times \Omega$ to a space $(S \times \Omega, \mathrm{M})$. The most basic property of M is

$$x_1^{(\omega_1)} \mathrm{M} x_2^{(\omega_2)} \Longrightarrow \omega_1 \neq \omega_2. \tag{3.1}$$

This implies, in particular, that M is irreflexive: for all $x^{(\omega)}$,

$$\neg x^{(\omega)} \mathrm{M} x^{(\omega)}.$$

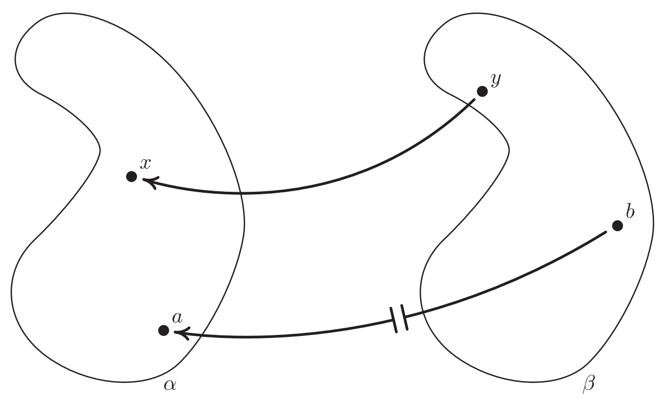

FIGURE 3.1. Diagrammatic representation of matches and non-matches used in the illustrations below. Stimulus values are shown within stimulus areas (e.g., $x^{(\alpha)}$ is shown as x in area α). The arrow from $y^{(\beta)}$ to $x^{(\alpha)}$ indicates that $x^{(\alpha)}$ is matched by $y^{(\beta)}$, $x^{(\alpha)} \mathrm{M} y^{(\beta)}$. The interrupted arrow from $b^{(\beta)}$ to $a^{(\alpha)}$ indicates that $a^{(\alpha)}$ is not matched by $b^{(\beta)}$, $\neg a^{(\alpha)} \mathrm{M} b^{(\beta)}$.

Definition 3.1. Given a space $(S \times \Omega, \mathrm{M})$, a sequence $x_1^{(\omega_1)}, \ldots, x_n^{(\omega_n)}$ of elements of $S \times \Omega$ is called *chain-matched* if $x_i^{(\omega_i)} \mathrm{M} x_{i+1}^{(\omega_{i+1})}$ for $i = 1, \ldots, n-1$. A sequence $x_1^{(\omega_1)}, \ldots, x_n^{(\omega_n)}$ is called *well-matched* if $\omega_i \neq \omega_j \implies x_i^{(\omega_i)} \mathrm{M} x_j^{(\omega_j)}$ for all $i, j \in \{1, \ldots n\}$.

The two forms of matchedness do not imply each other logically. Soritical sequences (defined below) are always chain-matched but never well-matched. A sequence $a^{(1)}, b^{(1)}, c^{(2)}$ may be well-matched (if $a^{(1)} \mathrm{M} c^{(2)}$, $b^{(1)} \mathrm{M} c^{(2)}$, $c^{(2)} \mathrm{M} a^{(1)}$, $c^{(2)} \mathrm{M} b^{(1)}$) but not chain-matched (because $\neg a^{(1)} \mathrm{M} b^{(1)}$). Any sequence $x_1^{(\omega)}, \ldots, x_n^{(\omega)}$ (with one and the same ω) is trivially well-matched but not chain-matched.

Definition 3.2. A chain-matched sequence $x_1^{(\omega_1)}, \ldots, x_n^{(\omega_n)}$ in a space $(S \times \Omega, \mathrm{M})$ is called *soritical* if

(1) $\omega_1 \neq \omega_n$
(2) $\neg x_1^{(\omega_1)} \mathrm{M} x_n^{(\omega_n)}$.

3.2. **Triadic and tetradic sequences.** It is immediate from the definition that there are no soritical sequence with just two elements, and that every soritical sequence consisting of three elements is of the form $a^{(\alpha)}, b^{(\beta)}, c^{(\gamma)}$ with $\{\alpha, \beta, \gamma\}$ pairwise distinct. Longer soritical sequences, as it turns out, can always be reduced to one of two types: three-element sequences like the one just mentioned, and four-element sequences with two alternating stimulus areas, $a^{(\alpha)}, b^{(\beta)}, c^{(\alpha)}, d^{(\beta)}$. (See Figure 3.2 for an illustration.)

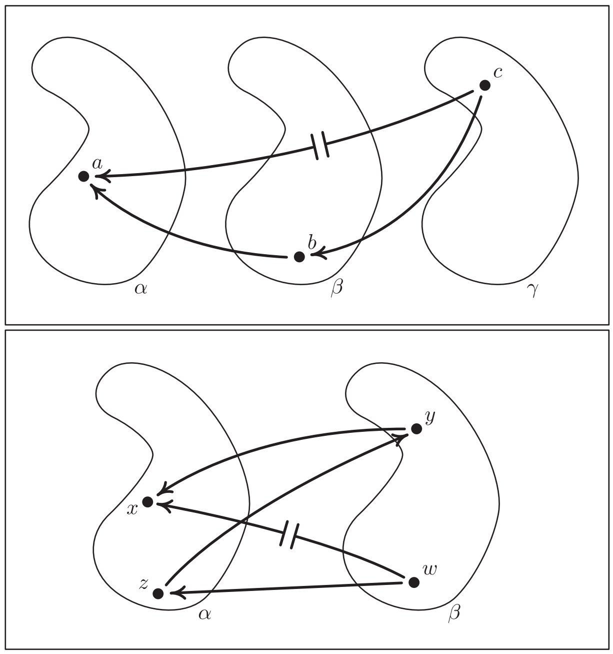

FIGURE 3.2. An illustration of Lemma 3.3: every soritical sequence contains a triadic soritical subsequence (top) or a tetradic soritical subsequence (bottom). Thus, in the top panel, $b^{(\beta)}$ matches $a^{(\alpha)}$, $c^{(\gamma)}$ matches $b^{(\beta)}$, but $c^{(\gamma)}$ does not match $a^{(\alpha)}$. In the bottom panel, $y^{(\beta)}$ matches $x^{(\alpha)}$, $z^{(\alpha)}$ matches $y^{(\beta)}$, $w^{(\beta)}$ matches $z^{(\alpha)}$, but $w^{(\beta)}$ does not match $x^{(\alpha)}$.

Lemma 3.3. *If $x_1^{(\omega_1)}, \ldots, x_n^{(\omega_n)}$ in a space $(S \times \Omega, \mathrm{M})$ is a soritical sequence, then it contains either a triadic soritical subsequence $a^{(\alpha)}, b^{(\beta)}, c^{(\gamma)}$ or a tetradic soritical subsequence $a^{(\alpha)}, b^{(\beta)}, c^{(\alpha)}, d^{(\beta)}$.*

Proof. Let $x_{i_1}^{(\omega_{i_1})}, \ldots, x_{i_m}^{(\omega_{i_m})}$ be a soritical subsequence of our sequence of the shortest possible length. If there exists an ℓ such that $1 < \ell < m$ and $\omega_{i_1} \neq \omega_{i_\ell} \neq \omega_{i_m}$ then it must be that $x_{i_1}^{(\omega_{i_1})} \mathrm{M} x_{i_\ell}^{(\omega_{i_\ell})}$: otherwise $x_{i_1}^{(\omega_{i_1})}, \ldots, x_{i_\ell}^{(\omega_{i_\ell})}$ would be yet a shorter soritical subsequence of the original sequence. Similarly, it must be that $x_{i_\ell}^{(\omega_{i_\ell})} \mathrm{M} x_{i_m}^{(\omega_{i_m})}$. Hence,

$$(a^{(\alpha)}, b^{(\beta)}, c^{(\gamma)}) = (x_{i_1}^{(\omega_{i_1})}, x_{i_\ell}^{(\omega_{i_\ell})}, x_{i_m}^{(\omega_{i_m})})$$

is a triadic subsequence of the kind desired. If no such ℓ exists, then it must be that $m \geq 4$ and that $\omega_{i_1} = \omega_{i_3} \neq \omega_{i_2} = \omega_{i_m}$. Again, the choice of $x_{i_1}^{(\omega_{i_1})}, \dots, x_{i_m}^{(\omega_{i_m})}$ as a shortest soritical subsequence ensures that $x_{i_3}^{(\omega_{i_3})} \mathrm{M} x_{i_m}^{(\omega_{i_m})}$, so in this case

$$(a^{(\alpha)}, b^{(\beta)}, c^{(\alpha)}, d^{(\beta)}) = (x_{i_1}^{(\omega_{i_1})}, x_{i_2}^{(\omega_{i_2})}, x_{i_3}^{(\omega_{i_3})}, x_{i_m}^{(\omega_{i_m})})$$

is a tetradic soritical subsequence of our sequence. □

3.3. **Well-matched spaces.** This concept is a generalization of the properties $\mathsf{RM}1$ and RM^*1 formulated above. Refer to Figure 3.3 for an illustration.

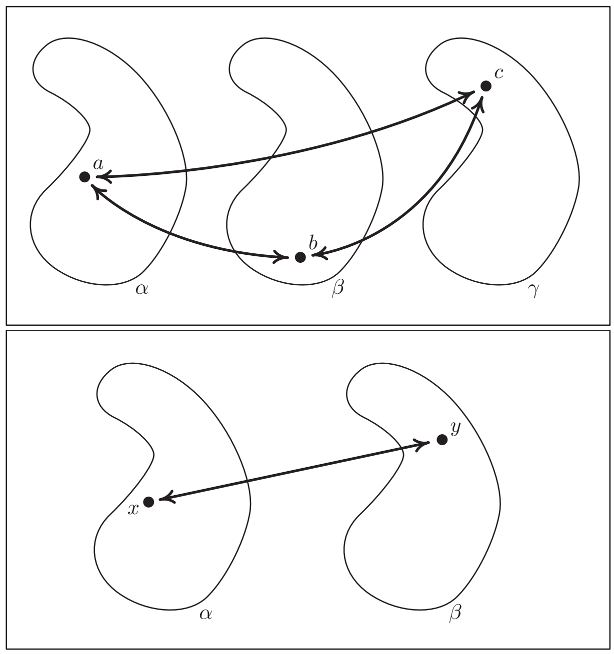

FIGURE 3.3. An illustration of a well-matched space. Top: if the space contains three distinct stimulus areas α, β, γ, then for every value a in α one can find a value b in β and a value c in γ such that any two of the stimuli $a^{(\alpha)}, b^{(\beta)}, c^{(\gamma)}$ match each other. Bottom: if the space only contains two distinct stimulus areas α, β, then for every value x in α one can find a value y in β such that $x^{(\alpha)}$ and $y^{(\beta)}$ match each other.

Definition 3.4. $(S \times \Omega, \mathrm{M})$ is a *well-matched space* if, for any sequence $\alpha, \beta, \gamma \in \Omega$ and any $a^{(\alpha)} \in S \times \Omega$, there is a well-matched sequence $a^{(\alpha)}, b^{(\beta)}, c^{(\gamma)}$.

Note that α, β, γ in this definition need not be pairwise distinct. In particular, the following observation involving just two stimulus areas deserves to be stated separately.

Lemma 3.5. *If $(S \times \Omega, \mathrm{M})$ is a well-matched space, then for any $\alpha, \beta \in \Omega$ and any $a^{(\alpha)} \in S \times \Omega$ one can find a $b^{(\beta)} \in S \times \Omega$ such that $a^{(\alpha)} \mathrm{M} b^{(\beta)}$ and $b^{(\beta)} \mathrm{M} a^{(\alpha)}$.*

Proof. Consider a sequence α, β, β and apply Definition 3.4. □

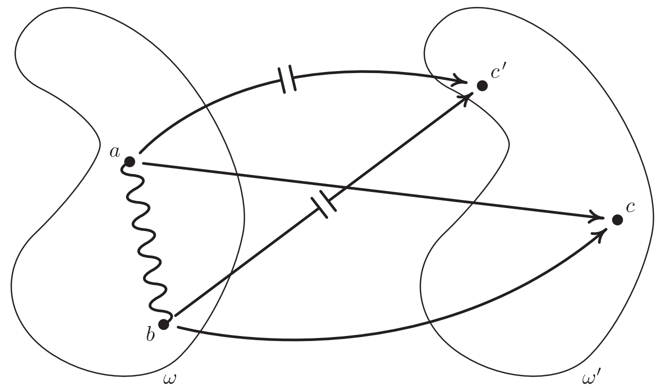

FIGURE 3.4. An illustration of the equivalence of stimuli $a^{(\omega)}$ and $b^{(\omega)}$ (indicated by the wiggly line connecting them) belonging to the same stimulus area ω: for any stimulus in another stimulus area (ω'), $a^{(\omega)}$ and $b^{(\omega)}$ either both match it (arrows from $a^{(\omega)}$ and $b^{(\omega)}$ to $c^{(\omega')}$) or both do not match it (interrupted arrows from $a^{(\omega)}$ and $b^{(\omega)}$ to $c'^{(\omega')}$).

3.4. **Equivalence relation and regular spaces.**

Definition 3.6. Given a space $(S \times \Omega, M)$, we call two elements $a^{(\omega)}, b^{(\omega')}$ of this space *equivalent*, and write $a^{(\omega)}\mathrm{E}b^{(\omega')}$, if for any $c^{(\iota)} \in S \times \Omega$,

$$c^{(\iota)}\mathrm{M}a^{(\omega)} \iff c^{(\iota)}\mathrm{M}b^{(\omega')}.$$

The lemma presented next justifies our calling E an equivalence, and, for well-matched spaces, restricting the equivalence relation to stimuli from one and the same stimulus area (see Figure 3.4 for an illustration).

Lemma 3.7. *In Definition 4.8,* E *is an equivalence relation on* $S \times \Omega$*. If the space is well-matched then* $a^{(\omega)}\mathrm{E}b^{(\omega')}$ *holds only if* $\omega = \omega'$*.*

Proof. The first claim is obvious. For the second, notice that if it were the case that $a^{(\omega)}\mathrm{E}b^{(\omega')}$, $\omega \neq \omega'$, and $S \times \Omega$ is well-matched, then we could find some $c^{(\omega)}$ with $c^{(\omega)}\mathrm{M}b^{(\omega')}$ by Lemma 3.5. But this would imply that $c^{(\omega)}\mathrm{M}a^{(\omega)}$ since $a^{(\omega)}$ and $b^{(\omega')}$ are equivalent, in contradiction to Eq. 3.1 above. □

The concept of equivalence is used to define the notion of a *regular space*. The latter generalizes the properties RM2 and RM*2 from above. See Figure 3.5 for an illustration.

Definition 3.8. $(S \times \Omega, \mathrm{M})$ is a *regular space* if, for any $a^{(\omega)}, b^{(\omega)}, c^{(\omega')} \in S \times \Omega$ with $\omega \neq \omega'$,

$$a^{(\omega)}\mathrm{M}c^{(\omega')} \wedge b^{(\omega)}\mathrm{M}c^{(\omega')} \Longrightarrow a^{(\omega)}\mathrm{E}b^{(\omega)}.$$

3.5. **Regular well-matched spaces.** Figure 3.6 shows that well-matchedness and regularity are independent properties. Our primary interest is in the spaces which are both regular and well-matched.

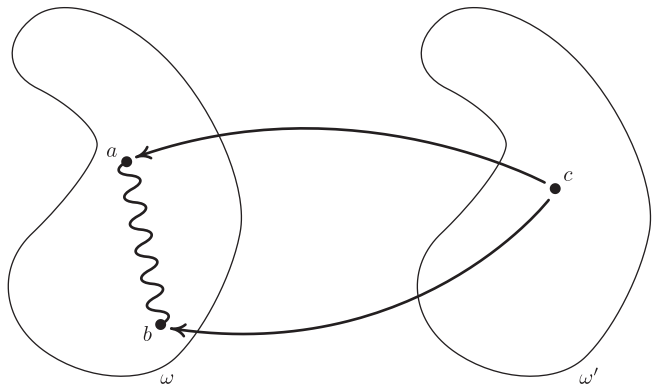

FIGURE 3.5. An illustration of a regular space. If a stimulus $c^{(\omega')}$ matches two stimuli $a^{(\omega)}$ and $b^{(\omega)}$, then these two stimuli are equivalent.

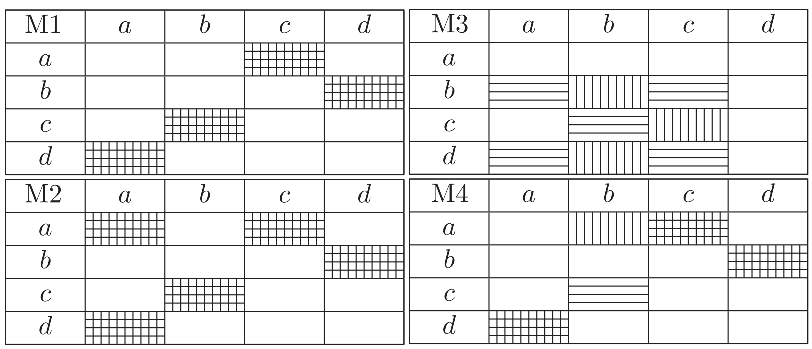

FIGURE 3.6. Toy examples of stimulus spaces with $\Omega = \{1, 2\}$ (rows and columns, resp.) and $S = \{a, b, c, d\}$. For $x, y \in S$, if a cell (x, y) is filled with horizontal lines then $x^{(1)}\mathrm{M}y^{(2)}$; if it is filled with vertical lines, then $y^{(2)}\mathrm{M}x^{(1)}$; a checkered pattern thus indicates both $x^{(1)}\mathrm{M}y^{(2)}$ and $y^{(2)}\mathrm{M}x^{(1)}$. Notice that $x^{(1)}$ and $y^{(1)}$ are equivalent if and only if the entries in row x filled with vertical lines are the same as those in row y, while $x^{(2)}$ and $y^{(2)}$ are equivalent if and only if the entries in column x filled with horizontal lines are the same as those in column y. Matrix M1 represents a regular and well-matched space; matrix M2 represents a well-matched but not regular space (e.g., $a^{(1)}$ matches $a^{(2)}$ and $c^{(2)}$ which are not equivalent); matrix M3 represents a regular but not well-matched space (e.g., $a^{(1)}$ does not have a matching column stimulus); and matrix M4 represents a space which is neither regular ($a^{(1)}$ matches $b^{(2)}$ and $c^{(2)}$ which are not equivalent) nor well-matched ($b^{(2)}$ does not have a row stimulus which matches and is matched by it).

Lemma 3.9. *If $(S\times\Omega, \mathrm{M})$ is a regular well-matched space, then, for any $a^{(\omega)}, b^{(\omega')} \in S \times \Omega$,*

$$a^{(\omega)}\mathrm{M}b^{(\omega')} \iff b^{(\omega')}\mathrm{M}a^{(\omega)}.$$

Proof. Due to the symmetry of the formulation we only have to prove $a^{(\omega)}\mathrm{M}b^{(\omega')} \Longrightarrow b^{(\omega')}\mathrm{M}a^{(\omega)}$. Suppose $a^{(\omega)}\mathrm{M}b^{(\omega')}$ holds, and let $b^{(\omega')}, x^{(\omega)}$ be a well-matched sequence guaranteed by Lemma 3.5. Then $a^{(\omega)}\mathrm{M}b^{(\omega')} \wedge x^{(\omega)}\mathrm{M}b^{(\omega')}$, whence, by the regularity of the space, $a^{(\omega)}\mathrm{E}x^{(\omega)}$. But we also have $b^{(\omega')}\mathrm{M}x^{(\omega)}$, which together with $a^{(\omega)}\mathrm{E}x^{(\omega)}$ implies $b^{(\omega')}\mathrm{M}a^{(\omega)}$. □

Lemma 3.10. *If* $(S \times \Omega, \mathrm{M})$ *is a regular well-matched space, then, for any* $a^{(\omega)}$, $b^{(\omega)}$,$c^{(\omega')} \in S \times \Omega$, *each of the statements*

$$a^{(\omega)}\mathrm{M}c^{(\omega')} \wedge b^{(\omega)}\mathrm{M}c^{(\omega')}$$

and

$$c^{(\omega')}\mathrm{M}a^{(\omega)} \wedge c^{(\omega')}\mathrm{M}b^{(\omega)}$$

implies the statement

$$a^{(\omega)}\mathrm{E}b^{(\omega)}.$$

Proof. Immediately follows from Definition 3.8 and the symmetry of M (Lemma 3.9). □

Lemma 3.11. *If* $(S\times\Omega, \mathrm{M})$ *is a regular well-matched space, then, for any* $a^{(\omega)}$,$x^{(\omega)}$, $b^{(\omega')}$,$y^{(\omega')} \in S \times \Omega$,

$$a^{(\omega)}\mathrm{E}x^{(\omega)} \wedge b^{(\omega')}\mathrm{E}y^{(\omega')} \Longrightarrow \{a^{(\omega)}\mathrm{M}b^{(\omega')} \Longleftrightarrow x^{(\omega)}\mathrm{M}y^{(\omega')}\}.$$

In particular,

$$a^{(\omega)}\mathrm{E}x^{(\omega)} \Longrightarrow \left\{ \begin{array}{c} b^{(\omega')}\mathrm{M}a^{(\omega)} \Longleftrightarrow b^{(\omega')}\mathrm{M}x^{(\omega)} \\ \textit{and} \\ a^{(\omega)}\mathrm{M}b^{(\omega')} \Longleftrightarrow x^{(\omega)}\mathrm{M}b^{(\omega')} \end{array} \right. .$$

Proof. Obvious. □

3.6. **No sorites theorems.** We are now ready to prove the impossibility of comparative soritical sequences. In accordance with Lemma 3.3, we can confine our attention to two types of soritical sequences:

$$a^{(\alpha)}, b^{(\beta)}, c^{(\gamma)}$$

and

$$a^{(\alpha)}, b^{(\beta)}, c^{(\alpha)}, d^{(\beta)}.$$

In the former case we can consequently assume that $\Omega = \{1, 2, 3\}$, and in the latter that $\Omega = \{1, 2\}$.

Theorem 3.12. *Let* $(S \times \{1, 2, 3\}, \mathrm{M})$ *be a regular well-matched space. Then any chain-matched sequence* $a^{(1)}, b^{(2)}, c^{(3)}$ *in this space is well-matched.*

Proof. The chain-matchedness of $a^{(1)}, b^{(2)}, c^{(3)}$ means $a^{(1)}\mathrm{M}b^{(2)} \wedge b^{(2)}\mathrm{M}c^{(3)}$. All we have to prove is that then $a^{(1)}\mathrm{M}c^{(3)}$, as the rest of the matches in $a^{(1)}, b^{(2)}, c^{(3)}$ then obtain by the symmetry of M. By Definition 3.4, there exists a well-matched sequence $x^{(1)}, b^{(2)}, y^{(3)}$. Since

$$a^{(1)}\mathrm{M}b^{(2)} \wedge x^{(1)}\mathrm{M}b^{(2)},$$

$a^{(1)}$ and $x^{(1)}$ are equivalent by Lemma 3.10,

$$a^{(1)}\mathrm{E}x^{(1)}.$$

Since

$$b^{(2)}\mathrm{M}c^{(3)} \wedge b^{(2)}\mathrm{M}y^{(3)},$$

$c^{(3)}$ and $y^{(3)}$ are equivalent by the same lemma,

$$c^{(3)}\mathrm{E}y^{(3)}.$$

Since $x^{(1)}, b^{(2)}, y^{(3)}$ is well-matched, we have $x^{(1)}\mathrm{M}y^{(3)}$, and, by Lemma 3.11,

$$a^{(1)}\mathrm{E}x^{(1)} \wedge c^{(3)}\mathrm{E}y^{(3)} \wedge x^{(1)}\mathrm{M}y^{(3)} \Longrightarrow a^{(1)}\mathrm{M}c^{(3)}.$$

□

Theorem 3.13. *Let* $(S \times \{1, 2\}, \mathrm{M})$ *be a regular well-matched space. Then any chain-matched sequence* $a^{(1)}, b^{(2)}, c^{(1)}, d^{(2)}$ *in this space is well-matched.*

Proof. The chain-matchedness of $a^{(1)}, b^{(2)}, c^{(1)}, d^{(2)}$ means $a^{(1)}\mathrm{M}b^{(2)} \wedge b^{(2)}\mathrm{M}c^{(1)} \wedge c^{(1)}\mathrm{M}d^{(2)}$. We have to show that $a^{(1)}\mathrm{M}d^{(2)}$. Consider a well-matched sequence $x^{(1)}, b^{(2)}, y^{(1)}, z^{(2)}$ (which exists by Definition 3.4). Since

$$b^{(2)}\mathrm{M}c^{(1)} \wedge b^{(2)}\mathrm{M}y^{(1)},$$

$c^{(1)}$ and $y^{(1)}$ are equivalent by Lemma 3.10,

$$c^{(1)}\mathrm{E}y^{(1)}.$$

Then we should have $y^{(1)}\mathrm{M}d^{(2)}$ (by Lemma 3.11, because $c^{(1)}\mathrm{M}d^{(2)}$). Since

$$y^{(1)}\mathrm{M}d^{(2)} \wedge y^{(1)}\mathrm{M}z^{(2)}$$

$d^{(2)}$ and $z^{(2)}$ are equivalent by Lemma 3.10,

$$d^{(2)}\mathrm{E}z^{(2)}.$$

By the same lemma, we also have

$$a^{(1)}\mathrm{E}x^{(1)},$$

because

$$a^{(1)}\mathrm{M}b^{(2)} \wedge x^{(1)}\mathrm{M}b^{(2)}.$$

But now $a^{(1)}\mathrm{E}x^{(1)}$ and $d^{(2)}\mathrm{E}z^{(2)}$, so from the fact that $x^{(1)}\mathrm{M}z^{(2)}$ it follows that

$$a^{(1)}\mathrm{M}d^{(2)},$$

by Lemma 3.11. □

Corollary 3.14. *One cannot form a soritical sequence in a regular well-matched space: any chain-matched sequence in such a space is well-matched.*

This completes the formal account of the comparative sorites.

We add without elaborating that the property of regular well-matchedness allows one, by an appropriate bijective (re)labeling of stimuli in all stimulus areas, to merge the irreflexive relation M and the equivalence relation E into a single identity relation EM. The idea of this (re)labeling (called 'canonical' in Dzhafarov, 2003, and Dzhafarov & Colonius, 2006) is very simple. Given a regular well-matched space $S \times \Omega$, any two equivalent stimuli $a^{(\omega)}$ and $b^{(\omega)}$ in any stimulus area ω can be assigned one and the same label (say, $\mathbf{x}$). Then every new label in any one stimulus area will match and be matched by one and only one label in any other stimulus area—and then it is possible to assign the same label $\mathbf{x}$ to all stimuli in all stimulus areas which match (and are matched by) $\mathbf{x}^{(\omega)}$. The resulting simplicity is the reward: for any two stimulus areas ω and ω', any 'relabeled stimulus' $\mathbf{x}^{(\omega)}$

matches the 'relabeled stimulus' $\mathbf{x}^{(\omega')}$ and none other; and in any given stimulus area any $\mathbf{x}^{(\omega)}$ is only equivalent to itself:

$$\mathbf{x}^{(\omega)}\mathrm{EM}\mathbf{y}^{(\omega')} \Longleftrightarrow \mathbf{x} = \mathbf{y},$$

where ω and ω' need not be distinct.

4. Conclusion

Comparative sorites is very different from classificatory sorites. The naive idea that two sufficiently close stimuli must be indistinguishable and that one can therefore construct a comparative soritical sequence of stimuli is far from being a well-known, let alone obvious, empirical fact. This idea overlooks the fundamental notion of a stimulus area and the probabilistic nature of comparative judgments in humans. Psychophysical analysis of comparative judgments is consistent with the notion that an idealized matching relation between stimuli should be defined so that it prevents the existence of comparative soritical sequences. In this paper this definition was developed into a general mathematical theory of what we call regular well-matched stimulus spaces: the matching relation in such spaces is irreflexive, symmetric, and transitive in the triadic or tetradic sense.

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