JOURNAL OF MATHEMATICAL PSYCHOLOGY 39, 40–56 (1995)

Decomposition of Recurrent Choices into Stochastically Independent Counts

EHTIBAR N. DZHAFAROV

University of Illinois at Urbana-Champaign and The Beckman Institute for Advanced Science and Technology

AND

ULF BÖCKENHOLT*

University of Illinois at Urbana-Champaign

Consider a fixed set of alternatives $\{1, ..., k\}$ available at each of a random number $\mathbf{N}$ of choice opportunities, exactly one alternative from $\{1, ..., k\}$ being selected at each such choice opportunity. Let the distribution of the conditional random vector $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = \mathbf{N}\}$ be known, $\mathbf{X}_i$ being the number of times the ith alternative is chosen. What is the class of all possible $(k+1)$-vectors of probability mass functions $\{R(n), R_1(x_1), ..., R_k(x_k)\}$ such that if $\mathbf{N}$ is distributed according to $R(n)$, the components of the unconditional random vector $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ are mutually independent random variables distributed according to $R_1(x_1), ..., R_k(x_k)$, respectively? This paper presents a complete and constructive solution of this problem for a broad class of conditional random vectors $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = \mathbf{N}\}$. In particular, the solution applies to all situations where the sequence of potentially observable values of $\mathbf{X}_i$ (for any $i = 1, ..., k$) forms an interval of consecutive integers, finite or infinite. When, for some $i = 1, ..., k$, this sequence contains finite gaps, the solution may or may not apply in its entirety. It is suggested, however, that in many, if not all, such situations the representation of recurrent choices by conditional vectors $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = \mathbf{N}\}$ may not be optimal in the first place. A more natural representation, to which the solution proposed applies universally, is provided by $\{\mathbf{M}_1, ..., \mathbf{M}_k \mid \sum \mathbf{M}_i = \mathbf{M}\}$, where $\mathbf{M}_i$ is the ordinal position of an observable value of $\mathbf{X}_i$ in the sequence of all such values. © 1995 Academic Press, Inc.

INTRODUCTION: PROBLEM AND TERMINOLOGY

The subject of this paper is related to the following problem considered in Böckenholt (1993). Let there be k fixed alternatives one of which is to be selected (e.g., one of k widely spaced targets in a visual scene to fixate, or one of k brands of a certain product to purchase), and let such a selection be made *recurrently* whithin a certain observation period. The notion of recurrent choices implies that each choice consists in a selection of one and only one of the k alternatives, all of these alternatives are available at each choice opportunity, and the choices follow each other in a chronological, or quasi-chronological order. Different realizations of the choice sequences are observed repeatedly and represented by a k-vector of counts $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$, with $\mathbf{X}_i$ denoting the number of times the ith option is chosen (within the observation period). The overall number of choices made within the observation period is a random variable, $\mathbf{N} = \sum \mathbf{X}_i$, and conditioned on its value n, the partition $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = n\}$ is a random vector. As an example, suppose that the conditional partition vector is distributed multinomially, this is,

$$\text{Prob}\left\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \,\middle|\, \sum \mathbf{X}_i = n\right\} = \binom{n}{x_1, ..., x_k} p_1^{x_1} \cdots p_k^{x_k},$$

and let $\mathbf{N}$ be a Poisson count,

$$\text{Prob}\{\mathbf{N} = n\} = e^{-\xi}(\xi^n/n!).$$

The product of these two probability mass functions (p.m.f.'s) is an unconditional p.m.f. for the vector of counts $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$. By simple algebra one verifies that

$$\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} = e^{-\xi}\frac{\xi^n}{n!}\binom{n}{x_1, ..., x_k} p_1^{x_1} \cdots p_k^{x_k} = \prod_{i=1}^{k}\left[e^{-\xi p_i}\frac{(\xi p_i)^{x_i}}{x_i!}\right];$$

* The authors are indebted to A. A. J. Marley for helpful discussions and constructive criticisms.

Correspondence and reprint requests should be addressed to Ehtibar N. Dzhafarov, Department of Psychology, University of Illinois at Urbana-Champaign, 603 East Daniel Street, Champaign, IL 61820.

0022-2496/95 $6.00

that is, the unconditional p.m.f. for $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ is a product of k Poisson p.m.f.'s selectively associated with the respective k alternatives. Equivalently put, $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ consists of k independent Poisson counts.

This constitutes an example of the *decomposition of recurrent choices into stochastically independent counts* (or "independent decomposition," for short). For a given conditional distribution of $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = n\}$ one finds a distribution of $\mathbf{N}$ (the *overall count*), such that the induced unconditional vector $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ consists of k independent random variables (*component counts*). The conditional p.m.f.

$$C(x_1, ..., x_k) = \text{Prob}\left\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \,\middle|\, \sum \mathbf{X}_i = \sum x_i\right\},$$
$$k > 1,$$

is called the *choice function*. Note that n does not explicitly enter in the list of arguments of $C(x_1, ..., x_k)$.

The example just given shows that any multinomial choice function can be independently decomposed into Poisson component p.m.f.'s, with the overall count p.m.f. being Poisson as well. Moreover, as shown by Moran (1952) for the binomial case ($k = 2$), and by Bol'shev (1965) for arbitrary k, this decomposition is unique: multinomial choice functions cannot be independently decomposed into p.m.f.'s other than Poisson (this result is proved below, by different means, as an illustrative example). A general formulation of the problem addressed in this paper is as follows: given a k-variate choice function $C(x_1, ..., x_k)$, what is the class of *all possible* $(k+1)$-vectors of p.m.f.'s $\{R(x), R_1(x), ..., R_k(x)\}$ that yield independent decompositions of this choice function? This paper presents a complete and constructive solution of this problem for a broad class of choice functions: it establishes necessary and sufficient conditions under which these choice functions are independently decomposable, and it provides a general algorithm for computing all possible $(k+1)$-vectors of decomposing p.m.f.'s. If one restricts one's attention to sufficient conditions only, that allow one to compute *some* of the $(k+1)$-vectors of decomposing p.m.f.'s when these conditions are satisfied, then the solution proposed applies to all choice functions, with no restrictions. As suggested in the concluding section, the choice functions to which the necessity part of this solution does not apply are associated with situations where conditioning of choice probabilities on $\sum \mathbf{X}_i$ may be artificial to begin with.

From a mathematical point of view, the theory of independent decomposability falls within the scope of the problem of "characterization" of marginal distributions by conditional distributions (under the assumption of stochastic independence). The research in this field was originated by Patil and Seshadri (1964), whose formulations later were corrected and simplified by Menon (1966). Extensions and generalizations of these formulations that are relevant in the present context can be found in Mathai (1967), Kabe (1969), Janardan (1974, 1975), and Gerber (1980). The exposition below, however, is completely self-contained and follows a somewhat different logic.

To avoid dealing with cumbersome technicalities at the outset and to keep the core of the theory of independent decomposability intuitive, the theory will be presented in two stages. At first, the consideration will be confined to *everywhere-positive* choice functions only, that is, to choice functions $C(x_1, ..., x_k)$ such that

$$C(x_1, ..., x_k) > 0 \qquad \text{for all} \quad \{x_1, ..., x_k\};$$
$$x_i = 0, 1, ...; \qquad i = 1, ..., k.$$

Obviously, $C(x_1, ..., x_k)$ sums to 1 across all natural partitions of any given $\sum x_i$; in particular, $C(0, ..., 0) = 1$. At the second stage of analysis, the assumption of everywhere-positivity will be removed, and the results will be generalized to a broad class of choice functions that may attain zero values or be undefined for some k-vectors $\{x_1, ..., x_k\}$.

MULTIPLICATIVELY SEPARABLE CHOICE FUNCTIONS

For everywhere-positive choice functions, the formal definition of independent decomposability is as follows.

DEFINITION 1. A $(k+1)$-vector of p.m.f.'s $\{R(x), R_1(x), ..., R_k(x)\}$, $k > 1$, each of which is positive on the entire set of natural numbers, is said to *independently decompose* the random partition k-vector $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = \mathbf{N}\}$, described by an everywhere-positive choice function $C(x_1, ..., x_k)$, if

$$C(x_1, ..., x_k) = \frac{\prod_{i=1}^k R_i(x_i)}{R(\sum_{i=1}^k x_i)}. \tag{1}$$

That is, if $\mathbf{N}$ is distributed according to $R(x)$, then the components of the induced unconditional k-vector $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ are stochastically independent random variables distributed according to $\{R_1(x), ..., R_k(x)\}$, respectively. The p.m.f.'s $\{R_1(x), ..., R_k(x)\}$ are referred to as the *component count p.m.f.'s*; $R(x)$ is referred to as the *overall count p.m.f.* By abuse of language it will also be said in such cases that the choice function itself, $C(x_1, ..., x_k)$, is "independently decomposed" by $\{R(x), R_1(x), ..., R_k(x)\}$.

The concept introduced next serves as a bridge between the concept of a choice function in general (for now, under the assumption of everywhere-positivity) and that of an independently decomposable choice function.

DEFINITION 2. An everywhere-positive choice function $C(x_1, ..., x_k)$ is called *multiplicatively separable* (m-separable, for short) if there is a vector of real-valued functions

$\{q_1(x), ..., q_k(x)\}$, each of which is positive on the entire set of natural numbers, such that for any $\{x_1, ..., x_k\}$,

$$C(x_1, ..., x_k) = \frac{\prod_{i=1}^{k} q_i(x_i)}{q_{1\ldots k}(\sum_{i=1}^{k} x_i)}, \tag{2}$$

where

$$q_{1\ldots k}(x) = \sum_{\sum u_i = x} \prod_{i=1}^{k} q_i(u_i)$$

(i.e., the summation is over all possible natural partitions of x). The functions $\{q_1(x), ..., q_k(x)\}$ are said to *m-generate* the choice function $C(x_1, ..., x_k)$.

Comments on Definition 2. It is trivial to verify that an m-separable choice function is a legitimate everywhere-positive p.m.f. (i.e., it is defined and positive for all k-vectors of natural numbers, and it sums to 1 across all k-partitions of any natural number). The resemblance between Definitions 1 and 2 is obvious, but it is essential to see the difference: the m-generating functions $q_i(x)$ in Definition 2 *are not generally p.m.f.'s*; they do not necessarily sum to 1 across their domain. Representation (2), therefore, generally does not constitute an independent decomposition in the sense of Definition 1. Observe, however, that if $\{q_1(x), ..., q_k(x)\}$ are p.m.f.'s, and if (2) holds, then $q_{1\ldots k}(x)$ is a p.m.f., too. Indeed,

$$\begin{aligned}\sum_{x=0}^{\infty} q_{1\ldots k}(x) &= \sum_{x=0}^{\infty} \sum_{\sum u_i = x} \prod_{i=1}^{k} q_i(u_i)\\ &= \sum_{u_1, ..., u_k} \prod_{i=1}^{k} q_i(u_i)\\ &= \prod_{i=1}^{k} \sum_{u_i=0}^{\infty} q_i(u_i) = 1.\end{aligned}$$

The usefulness of the concept of m-separability in dealing with the problem of independent decomposability stems from two facts. First, it immediately follows from Definitions 1 and 2 that an independently decomposable choice function is m-separable: it is m-generated by the count p.m.f.'s $\{q_1(x) = R_1(x), ..., q_k(x) = R_k(x)\}$, with $R(x) = q_{1\ldots k}(x)$ (see Comments above). As a result, insofar as independent decomposability is concerned, one can restrict one's attention to m-separable functions only. (As shown in the next section, however, not all m-separable choice functions can be independently decomposed.) Second, whereas it is typically non-obvious whether a choice function is independently decomposable, its m-separability is usually apparent from merely contemplating its mathematical form.

Examples. (E1) A multinomial choice function

$$\binom{n}{x_1, ..., x_k} p_1^{x_1} \cdots p_k^{x_k}, \qquad \sum_{i=1}^{k} p_i = 1, \qquad n = \sum_{i=1}^{k} x_i,$$

is m-separable; it is m-generated by

$$q_i(x) = \frac{p_i^x}{x!}, \qquad i = 1, ..., k.$$

The function $q_{1\ldots k}(n)$ has the same form as the m-generating functions

$$q_{1\ldots k}(n) = \frac{1}{n!} = \frac{(\sum p_i)^n}{n!}.$$

In relation to Comments on Definition 2, note that $q_i(x)$ and $q_{1\ldots k}(n)$ are not p.m.f.'s, because for $p \neq 0$,

$$\sum_{x=0}^{\infty} \frac{p^x}{x!} = e^p \neq 1.$$

(E2) A Dirichlet-compound multinomial (i.e., a multinomial p.m.f. integrated over Dirichlet-distributed vectors of probabilities; Johnson & Kotz, 1969)

$$\frac{n!\, \Gamma(\theta) \prod_1^k \Gamma(x_i + \theta_i)}{\Gamma(n+\theta) \prod_1^k \Gamma(\theta_i)\, x_i!}, \qquad \theta = \sum_{i=1}^{k} \theta_i, \qquad n = \sum_{i=1}^{k} x_i,$$

is m-separable: it is m-generated by

$$q_i(x) = \frac{\Gamma(x + \theta_i)}{\Gamma(\theta_i)\, x!}, \qquad i = 1, ..., k.$$

As in the previous example, the function $q_{1\ldots k}(n)$ has the same form as the m-generating functions:

$$q_{1\ldots k}(n) = \frac{\Gamma(n+\theta)}{\Gamma(\theta)\, n!}.$$

(E3) Definition 2 can be used to construct "new" choice functions by using arbitrary m-generating functions $\{q_1(x), ..., q_k(x)\}$. Let $k = 2$, and $\{q_1(x), q_2(x)\} = \{\exp(\beta_1 x^3), \exp(\beta_2 x^3)\}$, where β_1 and β_2 are positive real constants. Then the m-generated choice function is

$$C(x_1, x_2) = \frac{\exp(\beta_1 x_1^3 + \beta_2 x_2^3)}{\sum_{u_1 + u_2 = x_1 + x_2} \exp(\beta_1 u_1^3 + \beta_2 u_2^3)}.$$

INDEPENDENT DECOMPOSABILITY OF *MULTIPLICATIVELY SEPARABLE CHOICE FUNCTIONS*

To establish the necessary and sufficient conditions under which an m-separable everywhere-positive choice function is independently decomposable, some preliminary work is needed. It begins with establishing the uniqueness properties of the $q_i(x)$-functions m-generating a given m-separable function.

THEOREM 1. *For an* m-*separable everywhere-positive choice function* $C(x_1, ..., x_k)$ m-*generated by* $\{q_1(x), ..., q_k(x)\}$, *a k-vector of functions* $\{Q_1(x), ..., Q_k(x)\}$ m-*generates this choice function if, and only if,*

$$Q_i(x) = \alpha_i q_i(x)\, e^{-\lambda x}, \qquad i = 1, ..., k.$$

where α_i *are positive reals and* λ *is a real number (common to all* $q_i(x)$*-functions).*

Comments on Theorem 1. The proof is omitted, for two reasons. First, the mathematical structure of the proof (although not the formulation of the theorem as given) is only a trivial generalization of a technique repeatedly used in the statistical characterization literature (Patil & Seshadri, 1964; Menon, 1966; Janardan, 1974): it consists in reducing the problem to a Cauchy–Pexider functional equation (Aczél, 1966, 1975) with respect to logarithms of m-generating functions. Second, Theorem 1 will later be derived by different means, as part of a constructive corollary to Theorem 3. One advantage of that derivation is that it more readily lends itself to generalizations beyond everywhere-positive choice functions.

Since all independently decomposable choice functions are m-separable, one comes to the conclusion that a choice function m-generated by $\{q_1(x), ..., q_k(x)\}$ is independently decomposable if, and only if, at least one of the admissible transformations of $q_i(x)$-functions given by Theorem 1 results in a k-vector of p.m.f.'s; $q_{1\ldots k}(x)$ then must be a p.m.f., too (see Comments on Definition 2). For a closer look at the problem, the following definition is needed.

DEFINITION 3. A function $F(x)$, defined (but not necessarily positive) on the entire set of natural numbers, is said to be of Laplace order σ (a real number) if its Laplace transformation $\tilde{F}(\lambda)$, given by

$$\tilde{F}(\lambda) = \sum_{x=0}^{\infty} e^{-\lambda x} F(x),$$

exists (the sum converges) at $\lambda > \sigma$ but does not exist (the sum diverges) at $\lambda < \sigma$. The Laplace order of $F(x)$ is denoted by ord $F(x)$.

Comments on Definition 3. The term "Laplace order" corresponds to the "abscissa of absolute convergence of the Laplace transform" in the general theory of integral transformations (Hameister, 1946). The definition is well constructed, because if the sum above converges (diverges) for some value of λ, it also converges (diverges) for all greater (smaller) values. The existence of the sum at $\lambda = \sigma$ is immaterial. The Laplace order of $-\infty$ means that the Laplace transform $\tilde{F}(\lambda)$ exists everywhere (i.e., for all λ); the Laplace order of $+\infty$ means that $\tilde{F}(\lambda)$ exists nowhere (i.e., for no λ).

THEOREM 2. *An m-separable everywhere-positive choice function* $C(x_1, ..., x_k)$, *m-generated by* $\{q_1(x), ..., q_k(x)\}$, *can be independently decomposed if, and only if,*

$$\max\{\text{ord}\, q_1(x), ..., \text{ord}\, q_k(x)\} = \sigma < \infty. \tag{3}$$

The $(k+1)$*-vector of count p.m.f.'s* $\{R(x), R_1(x), ..., R_k(x)\}$ *is then computed as*

$$R_i(x) = \frac{q_i(x)\, e^{-\lambda x}}{\tilde{q}_i(\lambda)}, \qquad i = 1, ..., k, \tag{4}$$

$$R(x) = \frac{q_{1\ldots k}(x)\, e^{-\lambda x}}{\prod_{i=1}^{k} \tilde{q}_i(\lambda)} = \frac{q_{1\ldots k}(x)\, e^{-\lambda x}}{\widetilde{q_{1\ldots k}}(\lambda)}, \tag{5}$$

where λ *is any real number exceeding* σ.

Proof. (i) *Necessity.* Let $C(x_1, ..., x_k)$ be independently decomposed by some p.m.f.'s $\{R(x), R_1(x), ..., R_k(x)\}$. Then $C(x_1, ..., x_k)$ is m-generated by $\{R_1(x), ..., R_k(x)\}$. Since it is also m-generated by $\{q_1(x), ..., q_k(x)\}$, from Theorem 1 it follows that there should exist a real λ and a set of positive reals $\{\alpha_1, ..., \alpha_k\}$ such that

$$R_i(x) = \alpha_i q_i(x)\, e^{-\lambda x}, \qquad i = 1, ..., k.$$

Because $R_i(x)$ are p.m.f.'s,

$$\sum_{x=0}^{\infty} q_i(x)\, e^{-\lambda x} = \alpha_i^{-1}, \qquad i = 1, ..., k.$$

By Definition 3, this means that (since α_i are all strictly positive) the Laplace order of all $q_i(x)$-functions is at most λ. Hence

$$\max\{\text{ord}\, q_1(x), ..., \text{ord}\, q_k(x)\} \leqslant \lambda < \infty,$$

and the necessity is proven.

(ii) *Sufficiency.* Let now (3) hold. Choose an arbitrary $\lambda > \sigma$. By Definition 3,

$$\sum_{x=0}^{\infty} q_i(x)\, e^{-\lambda x} = \tilde{q}_i(\lambda) < \infty, \qquad i = 1, ..., k.$$

Hence

$$R_i(x) = \frac{q_i(x)\, e^{-\lambda x}}{\tilde{q}_i(\lambda)}, \qquad i = 1, ..., k,$$

are legitimate (and positive) p.m.f.'s. Due to Theorem 1, the $R_i(x)$-functions m-generate $C(x_1, ..., x_k)$. The sufficiency is proven, and (4) is established. Equation (5) now follows by straightforward algebra,

$$R(x) = \sum_{\sum u_i = x} \prod_{i=1}^{k} \frac{q_i(u_i)\, e^{-\lambda u_i}}{\tilde{q}_i(\lambda)}$$
$$= \frac{e^{-\lambda x}}{\prod_{i=1}^{k} \tilde{q}_i(\lambda)} \left(\sum_{\sum u_i = x} \prod_{i=1}^{k} q_i(u_i) \right)$$
$$= \frac{q_{1\ldots k}(x)\, e^{-\lambda x}}{\prod_{i=1}^{k} \tilde{q}_i(\lambda)},$$

and, by the well-known property of the Laplace transforms for convolutions,

$$\prod_{i=1}^{k} \tilde{q}_i(\lambda) = \widetilde{q_{1\ldots k}}(\lambda). \quad \blacksquare$$

Comments on Theorem 2. Observe that the only role played by the everywhere-positivity constraint in this theorem is in ensuring that admissible transformations of m-generating functions are restricted to those given by Theorem 1. If the admissibility of these, and *only these*, transformations is established for a broader class of choice functions, Theorem 2 will be applicable with no modifications. This issue is further clarified below, after the everywhere-positivity constraint is removed.

Theorem 2 is now illustrated using the examples considered in the previous section.

EXAMPLES. (E1, continued) Applying (4) to the multinomial choice function, the component count p.m.f.'s, if they exist, should have the form

$$R_i(x) = \frac{q_i(x)\, e^{-\lambda x}}{\tilde{q}_i(\lambda)}$$
$$= \frac{e^{-\lambda x}(p_i^x/x!)}{\sum_{u=0}^{\infty} e^{-\lambda u}(p_i^u/u!)}$$
$$= \frac{(\xi p_i)^x/x!}{\sum_{u=0}^{\infty} (\xi p_i)^u/u!},$$

where $e^{-\lambda}$ is denoted by ξ. The denominator is known to converge to $e^{\xi p_i}$ for all ξ (i.e., for all λ), which means that the Laplace order of all these $q_i(x)$-functions is $-\infty$, and hence the component count p.m.f.'s exist for all values of ξ (or λ). One comes to a set of Poisson count p.m.f.'s,

$$R_i(x) = e^{-\xi p_i} \frac{(\xi p_i)^x}{x!} = \mathscr{P}_{(\xi p_i)}(x), \qquad i = 1, ..., k.$$

Since $q_{1\ldots k}(x)$ here has the same form as $q_i(x)$, with 1 replacing p_i, the overall count p.m.f. (5) is,

$$R(x) = e^{-\xi} \frac{\xi^x}{x!} = \mathscr{P}_{(\xi)}(x),$$

again a Poisson p.m.f. By Theorem 2, an independent decomposition of a multinomial choice function can be obtained with any such, and only such, $(k+1)$-vectors of Poisson functions. Schematically,

$$\mathscr{M}_{(p_1 \cdots p_k)} \leftrightarrow \{\mathscr{P}_{(\xi)}, \mathscr{P}_{(\xi p_1)}, ..., \mathscr{P}_{(\xi p_k)}\}, \qquad \text{for any} \quad \xi > 0.$$

As mentioned in the introduction, this result was obtained by Bol'shev (1965), by different means.

(E2, continued) For a Dirichlet-compound multinomial choice function the decomposing component count p.m.f.'s, if they exist, should have the form, due to (4),

$$R_i(x) = \frac{q_i(x)\, e^{-\lambda x}}{\tilde{q}_i(\lambda)}$$
$$= \frac{[\Gamma(x+\theta_i)/\Gamma(\theta_i)\, x!]\, e^{-\lambda x}}{\sum_{u=0}^{\infty} [\Gamma(u+\theta_i)/\Gamma(\theta_i)\, u!]\, e^{-\lambda u}}$$
$$= \frac{[\Gamma(x+\theta_i)/\Gamma(\theta_i)\, x!]\, \zeta^x}{\sum_{u=0}^{\infty} [\Gamma(u+\theta_i)/\Gamma(\theta_i)\, u!]\, \zeta^u},$$

where ζ replaces $e^{-\lambda}$. The denominator can be shown to converge to $(1-\zeta)^{-\theta_i}$ if $\zeta < 1$ (i.e., $\lambda > 0$) and to diverge otherwise. For $\zeta < 1$, the obtained function,

$$R_i(x) = (1-\zeta)^{\theta_i} \frac{\Gamma(x+\theta_i)}{\Gamma(\theta_i)\, x!} \zeta^x = \mathscr{NB}_{(\zeta, \theta_i)},$$

is a negative binomial p.m.f. with parameters (ζ, θ_i). Since $q_{1\ldots k}(x)$ here has the same form as $q_i(x)$, with $\theta = \sum \theta_i$ replacing θ_i, the overall count p.m.f., due to (5), is

$$R(x) = (1-\zeta)^{\theta} \frac{\Gamma(x+\theta)}{\Gamma(\theta)\, x!} \zeta^x = \mathscr{NB}_{(\zeta, \sum \theta_i)},$$

a negative binomial p.m.f. with parameters (ζ, θ). Based on Theorem 2, one concludes that a Dirichlet-compound multinomial choice function can be decomposed by any such, and only such, $(k+1)$-vectors of negative binomial functions. Schematically,

$$\mathscr{DM}_{(\theta_1 \cdots \theta_k)} \leftrightarrow \{\mathscr{NB}_{(\zeta, \sum \theta)}, \mathscr{NB}_{(\zeta, \theta_1)}, ..., \mathscr{NB}_{(\zeta, \theta_k)}\}$$
$$\text{for any} \quad \zeta < 1.$$

(E3, continued) Here, no independent decomposition can be constructed because β_1 and β_2 are assumed to be positive. Indeed, the Laplace order of both $\exp(\beta_1 x^3)$ and $\exp(\beta_2 x^3)$ is $+\infty$. Even though Theorem 1 says that one can form an infinity of alternative m-separable representations, m-generated by

$$\{Q_1(x), Q_2(x)\} = \{\alpha_1 \exp(\beta_1 x^3 - \lambda x), \alpha_2 \exp(\beta_2 x^3 - \lambda x)\},$$

none of these, due to Theorem 2, would form a pair of legitimate p.m.f.'s.

A CRITERION FOR MULTIPLICATIVE SEPARABILITY

As shown in the preceding section, both existence and uniqueness of independent decompositions for an everywhere-positive choice function are established in a simple and constructive way, assuming that a single vector of functions m-generating this choice function has been found. Analogous and special cases of this assumption (although not the general concept of m-separability) are common in the literature on statistical characterization: the closest examples are Menon's (1966) reformulation of Patil and Seshadri's (1964) Theorem 1 for univariate distributions ($k=2$), and Janardan's (1974) extension of this theorem to the relationship between two random vectors (less direct analogous can be found in Mathai, 1967, and Kabe, 1969). *De facto*, m-separability of many choice functions is indeed apparent from merely contemplating their mathematical form. Still, a complete theory of independent decomposability should include a criterion (i.e., a necessary and sufficient condition) for m-separability based on the *values* of choice functions, rather than their mathematical *form*. The closest result in this area is Theorem 3 by Patil and Seshadri (1964; a corrected formulation is in Panaretos, 1982), but its relevance is rather indirect, and it only applies to univariate distributions on a finite support.

In the criterion theorem below, the following notation is used. For a given choice function $C(x_1, ..., x_k)$,

$$C_a(x) = C(x_1 = 0, ..., x_a = x, ..., x_k = 0), \qquad a = 1, ..., k$$

(all arguments but x_a are zero). Analogously,

$$C_{ab}(x, y) = C(x_1 = 0, ..., x_a = x, ..., x_b = y, ..., x_k = 0),$$
$$a, b \in \{1, ..., k\}, \qquad a \neq b$$

(all arguments but x_a and x_b are zero).

THEOREM 3. *An everywhere-positive choice function $C(x_1, ..., x_k)$ is m-separable if, and only if, the following identity holds across all k-vectors $\{x_1, ..., x_k\}$:*

$$C(x_1, ..., x_k) = \frac{\prod_{i=1}^k C_i(x_i)\, P(x_i)}{P(\sum_{i=1}^k x_i)}, \qquad (6)$$

where the function $P(x)$ (mapping natural numbers into positive reals) is defined as

$$P(x) = \begin{cases} \prod_{i=0}^{x-1} \dfrac{C_a(i)\, C_b(1)}{C_{ab}(i, 1)} & \text{if} \quad x > 0, \\ 1 & \text{if} \quad x = 0, \end{cases} \qquad (7)$$

for some $a, b \in \{1, ..., k\}$, $a \neq b$.

Proof. (i) *Necessity* is proved by using Definition 2 and expressing all choice functions in (6) and (7) through m-generating functions $\{q_1(x), ..., q_k(x)\}$ and $q_{1\ldots k}(x)$. From (7) one derives, for $x > 0$,

$$P(x) = \prod_{i=0}^{x-1} \left(\frac{\dfrac{q_a(i)}{q_{1\ldots k}(i)} \dfrac{q_b(1)}{q_{1\ldots k}(1)} \dfrac{\prod_{j=1}^k q_j(0)}{q_a(0)} \dfrac{\prod_{j=1}^k q_j(0)}{q_b(0)}}{\dfrac{q_a(i)\, q_b(1)}{q_{1\ldots k}(i+1)} \dfrac{\prod_{j=1}^k q_j(0)}{q_a(0)\, q_b(0)}} \right)$$
$$= \left(\frac{\prod_{i=1}^k q_i(0)}{q_{1\ldots k}(1)} \right)^x \prod_{i=0}^{x-1} \frac{q_{1\ldots k}(i+1)}{q_{1\ldots k}(i)},$$

which simplifies to

$$P(x) = q_{1\ldots k}(x) \frac{A^x}{q_{1\ldots k}(0)}, \qquad (*)$$

where

$$A = \frac{\prod_{i=1}^k q_i(0)}{q_{1\ldots k}(1)}. \qquad (**)$$

Equation (*) also holds for $x = 0$, yielding $P(0) = 1$, as required in (7). After substituting for $P(x)$ in the right-hand side of (6) and observing that $q_{1\ldots k}(0) = \prod q_i(0)$, the expression transforms into

$$\frac{\prod_{i=1}^k \dfrac{q_i(x_i)[\prod_{j=1}^k q_j(0)/q_i(0)]}{q_{1\ldots k}(x_i)}\, q_{1\ldots k}(x_i) \dfrac{A^{x_i}}{q_{1\ldots k}(0)}}{q_{1\ldots k}(\sum_{i=1}^k x_i) \dfrac{A^{\sum x_i}}{q_{1\ldots k}(0)}}$$
$$= \frac{\prod_{i=1}^k q_i(x_i)}{q_{1\ldots k}(\sum_{i=1}^k x_i)},$$

which, by Definition 2, proves that (6) holds identically.

(ii) *Sufficiency* is obtained immediately by putting

$$q_i(x) = C_i(x)\, P(x), \qquad i = 1, ..., k,$$

and observing that (6) satisfies Definition 2. That

$$P\left(\sum_{i=1}^{k} x_i\right) = q_{1\ldots k}\left(\sum_{i=1}^{k} x_i\right)$$

follows from the fact that $C(x_1, ..., x_k)$ sums to 1 across all partitions of $\sum x_i$. ∎

Comments on Theorem 3. To simplify derivations in the necessity proof, one could use the admissible transformations of Theorem 1 and select the coefficients in $\alpha_i q_i(x) e^{-\lambda x}$ so that

$$\alpha_i = \frac{1}{q_i(0)}, \qquad e^{\lambda} = q_{1\ldots k}(1).$$

By this one achieves

$$q_i(0) = 1, \qquad i = 1, ..., k; \qquad \text{hence} \quad q_{1\ldots k}(0) = 1,$$

$$q_{1\ldots k}(1) = \sum_{i=1}^{k} q_i(1) = 1,$$

and $P(x)$ becomes simply $q_{1\ldots k}(x)$.

The simplification just mentioned has not been used in the formal proof to emphasize that Theorem 3 is independent of Theorem 1. Moreover, a remarkable fact is that Theorem 1 can now be obtained as a corollary to Theorem 3. This corollary provides a constructive algorithm for computing the m-generating functions $\{q_1(x), ..., q_k(x)\}$ from the (everywhere-positive) values of $C(x_1, ..., x_k)$.

COROLLARY TO THEOREM 3. *For an m-separable everywhere-positive choice function $C(x_1, ..., x_k)$, the m-generating functions $\{q_1(x), ..., q_k(x)\}$ are determined as follows*:

(i) *compute the auxiliary function $P(x)$ according to* (7), $x = 0, 1, ...$; *the values of $P(x)$ are determined uniquely, that is, they do not depend on the choice of a and b in* (7);

(ii) *set $\{q_1(0), ..., q_k(0)\}$ equal to arbitrary positive reals, $\{\alpha_1, ..., \alpha_k\}$*;

(iii) *choose an arbitrary real λ, and compute $q_{1\ldots k}(1)$ from*

$$\lambda = \log \frac{\prod_{i=1}^{k} \alpha_i}{q_{1\ldots k}(1)}.$$

(iv) *for any $x = 0, 1, ...$, compute $q_i(x)$ and $q_{1\ldots k}(x)$ as*

$$q_i(x) = \alpha_i C_i(x) P(x) e^{-\lambda x}, \qquad i = 1, ..., k, \tag{8}$$

$$q_{1\ldots k}(x) = P(x) e^{-\lambda x} \prod_{i=1}^{k} \alpha_i. \tag{9}$$

$C(x_1, ..., x_k)$ can only be m-generated by k-vectors of functions that satisfy (8). *Hence different k-vectors of m-generating functions $\{q_1(x), ..., q_k(x)\}$ are interrelated by the transformations given in Theorem* 1.

Proof. That $P(x)$ does not depend on the choice of a and b in (7) follows from the fact that, for any x and any $a, b, c, d \in \{1, ..., k\}$, $a \neq b$, $c \neq d$,

$$\frac{C_a(x)\, C_b(1)}{C_{ab}(x, 1)} = \frac{C_c(x)\, C_d(1)}{C_{cd}(x, 1)}.$$

Indeed, if $C(x_1, ..., x_k)$ is m-generated by some $\{q_1(x), ..., q_k(x)\}$, then, by Definition 2, both ratios algebraically transform into

$$\frac{q_{1\ldots k}(x+1)}{q_{1\ldots k}(x)} \left(\frac{\prod_{i=1}^{k} q_i(0)}{q_{1\ldots k}(1)}\right).$$

Equation (9) immediately follows from (*) of Theorem 3; observe that the product of α_i's is $q_{1\ldots k}(0)$ and e^{λ} equals A, as defined in (**) of Theorem 3. By Definition 2,

$$C_i(x) = \frac{q_i(x)[\prod_{j=1}^{k} q_j(0)/q_i(0)]}{q_{1\ldots k}(x)}.$$

Equation (8) is obtained from this and (9) by simple algebra.

Suppose now that $C(x_1, ..., x_k)$ is m-generated by a k-vector $\{Q_1(x), ..., Q_k(x)\}$, such that $q_i(0) = Q_i(0)$, $i = 1, ..., k$, and $q_{1\ldots k}(1) = Q_{1\ldots k}(1)$. By construction, this means that $\{q_1(x), ..., q_k(x)\}$ and $\{Q_1(x), ..., Q_k(x)\}$ are associated with the same choice of $\{\alpha_1, ..., \alpha_k\}$ and λ. Since $P(x)$ is determined uniquely, it follows from (8) that $Q_i(x) = q_i(x)$, $i = 1, ..., k$, for any x. Since (8) is satisfied for arbitrary $\{\alpha_1, ..., \alpha_k\}$ and λ, the transformations of Theorem 1 are admissible. Since, for given $\{\alpha_1, ..., \alpha_k\}$ and λ, the m-generating functions $\{q_1(x), ..., q_k(x)\}$ are determined uniquely, no other transformations are admissible. The proof is complete. ∎

EXAMPLES. (E1, continued) The m-separability of a multinomial choice function is apparent from its mathematical form. For illustration purposes, however, it will now be proven by using the criterion of m-separability provided by Theorem 3;

$$P(x) = \prod_{i=0}^{x-1} \frac{p_a^i p_b}{((i+1)!/i!)\, p_a^i p_b} = \prod_{i=0}^{x-1} \frac{i!}{(i+1)!} = \frac{1}{x!},$$

which satisfies the requirement that $P(0)=1$. Identity (6) is satisfied because

$$\binom{n}{x_1, ..., x_k} p_1^{x_1} \cdots p_k^{x_k} = \frac{(p_1^{x_1}(1/x_1!)) \cdots (p_k^{x_k}(1/x_k!))}{1/n!} = \frac{\prod_{i=1}^{k} C_i(x)\, P(x_i)}{P(n)}.$$

This proves that the choice function is m-separable. To formally reconstruct m-generating functions, set $\alpha_i = 1$ and $\lambda = 0$ in (8) and (9):

$$q_i(x) = C_i(x)\, P(x) = \frac{p_i^x}{x!}, \qquad i = 1, ..., k;$$

$$q_{1 \cdots k}(x) = P(x) = \frac{1}{x!}.$$

These expressions coincide with those previously found by direct inspection.

(E4) m-separability is by far not a general property of all choice functions. For instance, an additive mixture of two binomial distributions,

$$C(x_1, x_2) = \pi \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2} + (1-\pi) \binom{n}{x_1, x_2} q_1^{x_1} q_2^{x_2},$$

$$\sum_{i=1}^{2} p_i = \sum_{i=1}^{2} q_i = 1, \qquad \sum_{i=1}^{2} x_i = n,$$

is not m-separable unless it is trivial, that is, unless $\pi(1-\pi)(p_1 - q_1) = 0$. As a straightforward, although rather tedious exercise in algebra, one can show that this function does not satisfy the identity

$$\frac{C(1,0)\, C(0,2)}{C(0,1)\, C(2,0)} = \frac{C(1,2)}{C(2,1)},$$

derived by applying (6) and (7) with $x_1 = 1$, $x_2 = 2$. In general, a non-trivial additive mixture $\sum \pi_j C_j(x_1, ..., x_k)$ of m-separable choice functions $C_j(x_1, ..., x_k)$, $\sum \pi_j = 1$, is not m-separable.

REMOVING ASSUMPTION OF EVERYWHERE-POSITIVITY

Without loss of generality, one can assume that all unconditional p.m.f.'s $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\}$ are defined for all possible $\{x_1, ..., x_k\}$-vectors: indeed, if some $\{x_1, ..., x_k\}$-vectors were excluded from the domain of an unconditional p.m.f., the latter can always be redefined as attaining zero values on these, initially excluded, k-vectors. Obviously, if $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} = 0$ at some k-vector $\{x_1, ..., x_k\}$, the conditional p.m.f. $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \sum \mathbf{X}_i = \sum x_i\}$ cannot be positive at this k-vector. A difficulty is, however, that this value need not be zero, it may also be *undefined* (indeterminate). Indeed, by definition,

$$\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \textstyle\sum \mathbf{X}_i = \sum x_i\} = \frac{\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\}}{\sum_{\sum u_i = \sum x_i} \text{Prob}\{\mathbf{X}_1 = u_1, ..., \mathbf{X}_k = u_k\}}.$$

Obviously, this expression is defined if, and only if, its denominator is positive. If $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} = 0$ for *all* $\{x_1, ..., x_k\}$-vectors partitioning a given total $\sum x_i$, then the conditional p.m.f. becomes an indeterminate form 0/0, and all such $\{x_1, ..., x_k\}$-vectors must be excluded from its domain. This possibility turns out to be a nuisance for multiplicative decompositions of conditional p.m.f.'s.

To circumvent this difficulty, it is convenient to modify the definition of a choice function $C(x_1, ..., x_k)$ in the following way.

Definition 4. For a given partition k-vector $\{\mathbf{X}_1, ..., \mathbf{X}_k \mid \sum \mathbf{X}_i = \sum x_i\}$,

(i) $C(x_1, ..., x_k) = \text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \sum \mathbf{X}_i = \sum x_i\}$ for all $\sum x_i$ at which this probability is defined;

(ii) $C(x_1, ..., x_k) = 0$ for all $\sum x_i$ at which $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \sum \mathbf{X}_i = \sum x_i\}$ is undefined (indeterminate).

Comments on Definition 4. The definition is well constructed, because $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \sum \mathbf{X}_i = \sum x_i\}$ is either defined for all or is not defined for any of $\{x_1, ..., x_k\}$-vectors partitioning a given total. Indeed, to be a legitimate p.m.f., $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \sum \mathbf{X}_i = \sum x_i\}$ has a definite value (positive or zero) at $\{x_1, ..., x_k\}$ if, and only if, it sums to 1 across all partitions of $\sum x_i$. The goal achieved by this definition is that the choice function $C(x_1, ..., x_k)$ is defined for all $\{x_1, ..., x_k\}$, but the value of $\sum C(x_1, ..., x_k)$, across all partitions of a given $\sum x_i$, is either 1 or 0. In the former case, the choice functions is a legitimate p.m.f., satisfying the identity

$$\frac{C(x_1, ..., x_k)}{\sum_{\sum u_i = \sum x_i} C(u_1, ..., u_k)} = C(x_1, ..., x_k).$$

When $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k \mid \sum \mathbf{X}_i = \sum x_i\}$ is undefined, the denominator equals 0, and the left-hand ratio becomes an indeterminate form 0/0. It will be assumed throughout the remainder of this paper that all choice functions are non-degenerate, that is, they are positive for at least two different k-vectors $\{x_1, ..., x_k\}$.

The following definition combines and generalizes Definitions 1 and 2.

DEFINITION 5. A choice function $C(x_1, ..., x_k)$ is m-separable if there is a vector of (m-generating) real-valued functions $\{q_1(x), ..., q_k(x)\}$, each of which is non-negative on the entire set of natural numbers, such that for any $\{x_1, ..., x_k\}$ and x,

$$\frac{C(x_1, ..., x_k)}{\sum_{\sum u_i = \sum x_i} C(u_1, ..., u_k)} = \frac{\prod_{i=1}^k q_i(x_i)}{q_{1...k}(\sum_{i=1}^k x_i)}, \qquad q_{1...k}(x) = \sum_{\sum u_i = x} \prod_{i=1}^k q_i(u_i).$$

(Indeterminate forms 0/0 are treated as equal, 0/0 = 0/0.)

The choice function $C(x_1, ..., x_k)$ is independently decomposed by a $(k+1)$-vector of p.m.f.'s $\{R(x), R_1(x), ..., R_k(x)\}$ if it is m-generated by $\{q_1(x), ..., q_k(x)\} = \{R_1(x), ..., R_k(x)\}$ and $q_{1...k}(x) = R(x)$.

Comments on Definition 5. The difference between this definition and Definitions 1 and 2 is not only in replacing all references to positive functions with those to non-negative functions. In addition, the decompositions into m-generating functions (or count p.m.f.'s) are allowed now to assume the form of an "equality" of two indeterminate forms (0/0 = 0/0), rather than a numerical equality between a choice function and a product of functions.

EXAMPLE E5. Let function $\mathscr{D}^{a_1, ..., a_s}_{p_1, ..., p_s}(x)$ be defined as

$$\mathscr{D}^{a_1, ..., a_s}_{p_1, ..., p_s}(x) = \prod_{i=1}^s (p_i^{x-a_i} + p_i^{a_i - x} - 2),$$

where $s = 1, 2, ..., \{p_1, ..., p_s\}$ are positive reals (different from 1), and $\{a_1, ..., a_s\}$ are non-negative integers. This function is positive everywhere except at $x = a_i$, $i = 1, ..., s$, where it is zero. As this function is real-valued and non-negative on the entire set of natural numbers, it can be used to construct choice functions in accordance with Definition 5. For instance, the choice function $C(x_1, x_2)$ defined by

$$\frac{C(x_1, x_2)}{\sum_{u_1+u_2 = x_1+x_2} C(u_1, u_2)} = \frac{\mathscr{D}^{1,2}_{p_1,p_2}(x_1)\, \mathscr{D}^{1,4}_{q_1,q_2}(x_2)}{\sum_{u_1+u_2=x_1+x_2} \mathscr{D}^{1,2}_{p_1,p_2}(u_1)\, \mathscr{D}^{1,4}_{q_1,q_2}(u_2)}$$

satisfies Definition 5. It is positive whenever x_1 is not 1 or 2 and x_2 is not 1 or 4. The corresponding conditional p.m.f., $\text{Prob}\{\mathbf{X}_1 = x_1, \mathbf{X}_2 = x_2 \mid \mathbf{X}_1 + \mathbf{X}_2 = x_1 + x_2\}$ is defined at all values of $x_1 + x_2$ except at $x_1 + x_2 = 1$, because for both partitions of 1, $\{0, 1\}$ and $\{1, 0\}$,

$$\mathscr{D}^{1,2}_{p_1,p_2}(0)\, \mathscr{D}^{1,4}_{q_1,q_2}(1) = \mathscr{D}^{1,2}_{p_1,p_2}(1)\, \mathscr{D}^{1,4}_{q_1,q_2}(0) = 0.$$

It is easy to establish now that the following weakened version of Theorems 1 and 2 holds for arbitrary m-separable functions. Recall that Definition 3 of Laplace order applies to all functions that are defined on the entire set of natural numbers (which is true for m-generating functions of Definition 5). For convenience, the equations that are identical to those of Theorems 1 and 2 are replicated explicitly, rather than referenced.

THEOREM 4. *Let $C(x_1, ..., x_k)$ be a choice function m-generated by a k-vector of functions $\{q_1(x), ..., q_k(x)\}$. Then the following propositions hold*:

(i) *$C(x_1, ..., x_k)$ is m-generated by any k-vector of functions $\{Q_1(x), ..., Q_k(x)\}$ such that, for some positive reals $\{\alpha_1, ..., \alpha_k\}$ and a real λ,*

$$Q_i(x) = \alpha_i q_i(x)\, e^{-\lambda x}, \qquad i = 1, ..., k. \tag{10}$$

(ii) *If $\max\{\text{ord}\, q_1(x), ..., \text{ord}\, q_k(x)\} = \sigma < \infty$, then $C(x_1, ..., x_k)$ is independently decomposable. In particular, it is independently decomposed by any $(k+1)$-vector of count p.m.f.'s $\{R(x), R_1(x), ..., R_k(x)\}$ such that, for some real $\lambda > \sigma$,*

$$R_i(x) = \frac{q_i(x)\, e^{-\lambda x}}{\sum_{x=0}^\infty q_i(x)\, e^{-\lambda x}} = \frac{q_i(x)\, e^{-\lambda x}}{\tilde{q}_i(\lambda)}, \qquad i = 1, ..., k, \tag{11}$$

$$R(x) = \frac{q_{1...k}(x)\, e^{-\lambda x}}{\prod_{i=1}^k \sum_{x=0}^\infty q_i(x)\, e^{-\lambda x}} = \frac{q_{1...k}(x)\, e^{-\lambda x}}{\prod_{i=1}^k \tilde{q}_i(\lambda)} = \frac{q_{1...k}(x)\, e^{-\lambda x}}{\widetilde{q_{1...k}}(\lambda)}. \tag{12}$$

(iii) *If all k-vectors of functions $\{Q_1(x), ..., Q_k(x)\}$ m-generating $C(x_1, ..., x_k)$ are related to $\{q_1(x), ..., q_k(x)\}$ by transformations* (10), *then $C(x_1, ..., x_k)$ is independently decomposable only if $\max\{\text{ord}\, q_1(x), ..., \text{ord}\, q_k(x)\} = \sigma < \infty$, and then all possible $(k+1)$-vectors of count p.m.f.'s $\{R(x), R_1(x), ..., R_k(x)\}$ are given by* (11) *and* (12).

Proof. (i) By simple algebra, if $\{q_1(x), ..., q_k(x)\}$ satisfy Definition 5, then the same is true for $\{Q_1(x), ..., Q_k(x)\}$.

(ii) Recall (see Comments on Definition 4) that $C(x_1, ..., x_k)$ is assumed to be non-degenerate (hence not identically equal to zero). As a result, every $q_i(x)$-function has at least one positive value and $\tilde{q}_i(\lambda) > 0$ for any λ.

Obviously, $\widetilde{q_{1\dots k}}(\lambda) > 0$, too. With this addition, the proof is identical to that of the sufficiency part of Theorem 2.

(iii) The proof is identical to that of the necessity part of Theorem 2, except that here the inadmissibility of transformations other than (10) is assumed, rather than derived from Theorem 1. ∎

It must be clear now (see also Comments on Theorem 2) that the principal role of the everywhere-positivity constraint adopted in the previous sections was to ensure that transformations (10) are *the only* admissible transformations for $q_i(x)$-functions m-generating an m-separable $C(x_1, \dots, x_k)$. That transformations (10) are admissible is true for *any* m-separable choice function (Theorem 4(i)), and among the k-vectors $\{Q_1(x), \dots, Q_k(x)\}$ related to a given k-vector $\{q_1(x), \dots, q_k(x)\}$ by transformations (10) one can always find all k-vectors of p.m.f.'s, or establish that such k-vectors do not exist (Theorem 4(ii)). If no other transformations of $\{q_1(x), \dots, q_k(x)\}$ are admissible, then *all* independent decompositions of a given choice function are known (Theorem 4(iii)). As the following two examples show, this assumption is not true for *arbitrary* m-separable choice functions: the class of admissible transformations may very well include those beyond transformations (10). As a result, a choice function may be independently decomposed by $(k+1)$-vectors of p.m.f.'s other than those given by (11) and (12).

Examples. (E6) Let $C(x_1, x_2)$ be m-generated by a pair of functions $\{q_1(x), q_2(x)\}$ such that

$$\begin{cases} q_1(x) > 0 & \text{if} \quad x \leqslant h, \\ q_1(x) = 0 & \text{if} \quad x > h, \end{cases}$$

$$\begin{cases} q_2(x) > 0 & \text{if} \quad x \leqslant h \quad \text{or} \quad x \geqslant h+g, \\ q_2(x) = 0 & \text{if} \quad h < x < h+g, \end{cases}$$

where $h-1$ and g are positive integers. For $g > h$, one can verify by simple algebra that $C(x_1, x_2)$ is m-generated by any pair of functions

$$Q_1(x) = \alpha_1 q_1(x)\, e^{-\lambda x}, \qquad Q_2(x) = \alpha_2(x)\, q_2(x)\, e^{-\lambda x},$$

where

$$\begin{cases} \alpha_2(x) = \alpha_2^* > 0 & \text{if} \quad x \leqslant h, \\ \alpha_2(x) = \alpha_2^{**} > 0 & \text{if} \quad x > h. \end{cases}$$

Obviously, these transformations reduce to (10) only when the two α-values are set equal to each other.

(E7) Let $C(x_1, x_2)$ be m-generated by a pair of functions $\{q_1(x), q_2(x)\}$ such that $q_1(x) > 0$ for all x, and $q_2(x) > 0$ if, and only if, $x = mh$, h being a positive integer, $m = 0, 1, 2, \dots$. If $h = 1$, then $C(x_1, x_2)$ is everywhere-positive, hence transformations (10) are indeed the only admissible transformations. If $h > 1$, however, then, presenting natural numbers x as $mh + d$, $d = 0, 1, \dots, h-1$, one can easily verify that $C(x_1, x_2)$ is m-generated by

$$Q_1(x) = Q_1(mh+d) = \alpha_1^{(d)} e^{-\lambda(mh+d)} q_1(mh+d),$$

$$Q_2(x) = \begin{cases} Q_2(mh) = \alpha_2 e^{-\lambda mh} q_2(mh) & \text{if} \quad d = 0, \\ Q_2(mh+d) = 0 & \text{if} \quad d \neq 0, \end{cases}$$

where $\{\alpha_1^{(0)}, \alpha_1^{(1)}, \dots, \alpha_1^{(h-1)}\}$ and α_2 are arbitrary positive reals, and λ is, as usual, an arbitrary real. Obviously, transformations (10) obtain only if $\alpha_1^{(0)} = \alpha_1^{(1)} = \cdots = \alpha_1^{(h-1)}$.

(E8) One might be tempted to conjecture, based on the previous examples, that admissible transformations of m-generating functions are always of the form $\alpha_i(x)\, q_i(x)\, e^{-\lambda x}$, where $\alpha_i(x)$ assumes at most a finite number of different values. Even this generalization is not true, however. Let $C(x_1, x_2)$ be m-generated by a pair of functions $\{q_1(x), q_2(x)\}$ such that

$$q_1(x) > 0 \quad \text{iff} \quad x = 2^m, \qquad m = 0, 1, 2, \dots,$$

$$q_2(x) > 0 \quad \text{iff} \quad x = 3^m, \qquad m = 0, 1, 2, \dots.$$

Since for any given $s = 0, 1, \dots$, there is at most one pair (m, m^*) such that $2^m + 3^{m^*} = s$, the choice function $C(x_1, x_2)$ only attains values 1, 0, and 0/0 (indeterminate). Obviously, $C(x_1, x_2)$ is m-generated by *any* two functions $\{Q_1(x), Q_2(x)\}$ whose positive domains coincide with those of $\{q_1(x), q_2(x)\}$, componentwise.

It turns out, however, that admissible transformations of m-generating functions are indeed confined to transformations (10) for a very broad class of choice functions, that includes everywhere-positive ones as a proper subclass. Crudely put, the essential feature of these choice functions (termed "*connected*" ones) is that the areas of $\{x_1, \dots, x_k\}$-vectors at which $C(x_1, \dots, x_k) > 0$ are not separated from each other by "excessively large" gaps of zero values; as a result, the mutual constraints imposed by the neighboring "islands" of positive values on each other (due to the inherent properties of conditional p.m.f.'s) are sufficiently restrictive to enable an analogue of the constructive algorithm for m-generating functions provided by Theorem 3 and its corollary. The vectors of m-generating functions, if they exist, are reconstructed uniquely up to admissible transformations (10), and Theorem 2 applies in its entirety to all connected m-separable choice functions. In relation to the examples given in this section, the choice function constructed in E5 is connected, whereas the choice functions described in E6, E7, and E8 are not.

Substantively, the theory of connected choice functions is very similar to the one developed for everywhere-positive choice functions, but the technicalities involved are considerably more demanding. In particular, one has to introduce a few additional definitions and some new notation whose intuitive meaning and utility may not be immediately transparent. For this reason, the entire theory of connected choice functions, together with illustrating examples, is relegated to the Appendix. One very simple special case, however, has to be introduced here, as it is required for the discussion in the concluding section of this paper.

DEFINITION 6. A choice function $C(x_1, ..., x_k)$ is called *simply-connected* if, for any $i = 1, ..., k$, there is an interval $X_i = \{0, ..., \max_i\}$ of *consecutive* natural numbers, from 0 through some (finite or infinite) $\max_i > 0$, such that $C(x_1, ..., x_k) > 0$, if, and only if, $\{x_1, ..., x_k\} \in X_1 \times \cdots \times X_k$.

Simply-connected functions are a straightforward generalization of everywhere-positive ones (obtained from Definition 6 by putting $\max_i = \infty$ for all i), and the theory of independent decomposability applies here in a most straightforward fashion, too. It is easy to verify that none of the results established for everywhere-positive choice functions utilizes the assumption that the positive domains X_i of variables x_i $(i = 1, ..., k)$ are infinite (as opposed to the assumption, essential for the proofs of Theorems 1 and 3, that the positive domains consist of consecutive natural numbers). As a result, all one has to do in order to generalize these results to simply-connected functions is to redefine m-generating functions $q_i(x)$ and component count p.m.f.'s $R_i(x)$ as being positive on the intervals $X_i = \{0, ..., \max_i\}$ and equal to zero above $\max_i$ $(i = 1, ..., k)$; the function $q_{1 \cdots k}(x)$ and the overall count p.m.f. $R_i(x)$ are then positive at $x \leqslant \sum \max_i$ and equal to zero above this value. There is no need to formalize these observations as separate theorems, because they trivially follow as a special case from the general theory of connected choice functions presented in the Appendix (see Comments on Definition A2).

ORDINAL CHOICE FUNCTIONS

A straightforward line of further mathematical development would consist in considering choice functions with progressively weaker constraints imposed on their positive domains. Moving along this line, one would construct progressively more general criteria of m-separability, delimit the corresponding classes of admissible transformations for m-generating functions, and investigate conditions under which these transformations yield p.m.f.'s. The mathematical complexity (or at least cumbrousness) of this task seems to be formidable. It is worthwhile, therefore, to discuss an approach according to which such an investigation, interesting as it might be from a purely mathematical point of view; may be unnecessary for probabilistic representations of recurrent choices.

As stated in the introduction, the notion of recurrent choices involves a chronological (or quasi-chronological) sequence of choice decisions *made one at a time*, each time with respect to the same k fixed alternatives $\{1, ..., k\}$. Different realizations of this sequence (within a certain period) are observed repeatedly and are represented by the k-vector of counts $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$, the choice function $C(x_1, ..., x_k)$ being the probability of observing $\{x_1, ..., x_k\}$ among all observable k-vectors with the same total. Intuitively, one expects that if an ith option $(i = 1, ..., k)$ is sometimes chosen x times, and sometimes, say, $x + 2$ times, then it should also be "possible" for this option to be chosen $x + 1$ times—unless there is a "structural rule" that "compels" the chooser, once the $(x + 1)$th choice of option i has been made, to always choose this option again, moreover, to do so before the end of the observation period. One possible interpretation of such a "structural rule" is that the two choices, $(x + 1)$th and $(x + 2)$th, are "based on a single decision act," the decision to choose option i *twice* (before a certain deadline). This contradicts the notion that choice decisions are made one at a time, one choice decision corresponding to one factual act of choice.

If one dismisses such situations as "impossible" or requiring a different kind of analysis, then all the remaining situations are representable by everywhere-positive or, at least, simply-connected choice functions. Based on empirical distributions of partition k-vectors it is never possible to distinguish between zero probabilities and sufficiently small probabilities: therefore, unless there are substantive reasons to believe that "structural rules" do intervene (so that a subject can decide, e.g., to select a certain option "three more times within the remaining 30 minutes"), one can always confine oneself to modelling recurrent choices by simply-connected choice functions only. It is trivial to show that for any m-separable choice function $C(x_1, ..., x_k)$ there exists a sequence of simply-connected m-separable functions $C_p(x_1, ..., x_k)$, $p = 1, 2, ...$, uniformly converging to $C(x_1, ..., x_k)$ as p increases indefinitely. It is sufficient, for example, to replace $\{q_1(x), ..., q_k(x)\}$ m-generating $C(x_1, ..., x_k)$ by $\{q_1(x) + p^{-1}, ..., q_k(x) + p^{-1}\}$ m-generating $C_p(x_1, ..., x_k)$ to obtain a sequence of everywhere-positive choice functions uniformly converging to $C(x_1, ..., x_k)$. In special cases this can be achieved by more elegant means: for instance, by replacing natural numbers $\{a_1, ..., a_s\}$ with $\{a_1 + \varepsilon, ..., a_s + \varepsilon\}$ in the definition of the m-generating functions $\mathscr{D}^{a_1, ..., a_s}_{p_1, ..., p_s}(x)$, introduced in E5 of the previous section, one obtains, as $\varepsilon \to 0$, arbitrarily close m-generating functions that are positive on the entire set of natural numbers.

Suppose, however, that substantive reasons for admitting "structural rules" in a sequence of recurrent choices do exists, so that the sets $\{X_1, ..., X_k\}$ of potentially observable counts do contain true "internal gaps." In other words, a certain number x can never be the count of times an ith option is chosen, not because $x > \max_i$ (as would be the case for simply-connected functions), but because the xth choice of the ith option is necessarily followed by several additional choices of the same option before the observation interval ends. A contemplation of the examples of such choice functions given in the previous section (E6–E8) suggests that the main question here is whether it is reasonable at all to use these functions to represent recurrent choices with "structural rules" involved. This is especially apparent for the choice functions introduced in E7 and E8. In E7, option 2 is always chosen in groups of h, whereas option 1 is chosen one at a time. It seems quite artificial then to treat, say, partition vectors $\{\mathbf{X}_1 = 1, \mathbf{X}_2 = h\}$ and $\{\mathbf{X}_1 = h+1, \mathbf{X}_2 = 0\}$ as partitioning the same total number of choice decisions, $h+1$; a more natural approach would be to view $\{\mathbf{X}_1 = 1, \mathbf{X}_2 = h\}$ as involving just two choice decisions, the same as in $\{\mathbf{X}_1 = 2, \mathbf{X}_2 = 0\}$ and $\{\mathbf{X}_1 = 0, \mathbf{X}_2 = 2h\}$. Put differently, it seems more natural to represent these recurrent choices by the choice function

$$\mathscr{C}(m_1, m_2) = \text{Prob}\{\mathbf{X}_1 = m_1, \mathbf{X}_2 = m_2 h \mid m_1 + m_2\},$$
$$m_i = 0, 1, 2, ..., \qquad i = 1, 2,$$

than by the choice function

$$C(x_1, x_2) = \text{Prob}\{\mathbf{X}_1 = x_1, \mathbf{X}_2 = x_2 \mid x_1 + x_2\},$$
$$x_1 = 0, 1, 2, ...; \qquad x_2 = 0, h, 2h, ...$$

Obviously, the domain of $C(x_1, x_2)$ contains "internal gaps," whereas $\mathscr{C}(m_1, m_2)$ is everywhere-positive. Analogously, in E8 a natural approach would be to represent the recurrent choices by

$$\mathscr{C}(m_1, m_2) = \text{Prob}\{\mathbf{X}_1 = 2^{m_1}, \mathbf{X}_2 = 3^{m_2} \mid m_1 + m_2\},$$
$$m_i = 0, 1, 2, ..., \qquad i = 1, 2,$$

again an everywhere-positive choice function.

A generalization suggests itself immediately. As shown in the Appendix (Definition A1, Lemma A1), a choice function $C(x_1, ..., x_k)$ may be m-separable only if the unconditional random k-vector of counts $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ is "domain-separable," that is, if there are subsets X_i $(i = 1, ..., k)$ of (not necessarily consecutive) natural numbers, such that $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} > 0$ if, and only if, $\{x_1, ..., x_k\} \in X_1 \times \cdots \times X_k$. The subsets X_i $(i = 1, ..., k)$ are referred to as *sets of potentially observable values* (for counts $\mathbf{X}_i$), and they may be finite or infinite. Let these sets of potentially observable values be ordered as

$$X_i = x_0^{(i)} < \cdots < x_m^{(i)} < \cdots < x_{\max_i}^{(i)}, \qquad x_{\max_i}^{(i)} > x_0^{(i)},$$
$$m = 0, ..., \max_i, \qquad i = 1, ..., k,$$

$\max_i$ being finite or infinite. Let $\mathbf{M}_i$ be defined as the *ordinal position* of $\mathbf{X}_i$ in the ordered set X_i, $i = 1, ..., k$; that is, $\mathbf{M}_i = m$ if, and only if, $\mathbf{X}_i = x_m^{(i)}$. Finally, let $\mathscr{F}(m_1, ..., m_k)$ denote the unconditional p.m.f. defined as

$$\mathscr{F}(m_1, ..., m_k) = \text{Prob}\{\mathbf{M}_1 = m_1, ..., \mathbf{M}_k = m_k\}$$
$$= \text{Prob}\{\mathbf{X}_1 = x_{m_1}^{(1)}, ..., \mathbf{X}_k = x_{m_k}^{(k)}\}.$$

The point being made is that in many, if not all, situations where $\{X_1, ..., X_k\}$ contain internal gaps, the k-vector $\{\mathbf{M}_1, ..., \mathbf{M}_k\}$ may provide more adequate representation for recurrent choices than $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$. If so, then the corresponding conditional partition vectors are $\{\mathbf{M}_1, ..., \mathbf{M}_k \mid \sum \mathbf{M}_i = \sum m_i\}$, and the choice functions $\mathscr{C}(m_1, ..., m_k)$ associated with them (*ordinal choice functions*) are computed as

$$\frac{\mathscr{C}(m_1, ..., m_k)}{\sum_{\sum u_i = \sum m_i} \mathscr{C}(u_1, ..., u_k)} = \frac{\mathscr{F}(m_1, ..., m_k)}{\sum_{\sum u_i = \sum m_i} \mathscr{F}(u_1, ..., u_k)}$$

(compare with Definition 4). Obviously, $\mathscr{C}(m_1, ..., m_k)$ thus defined is positive on $\{0, ..., \max_1\} \times \cdots \times \{0, ..., \max_k\}$, and it is simply-connected in the sense of Definition 6. As a result (see the conclusion of the previous section), the theory of independent decomposability applies here in its simplest form. Observe that for everywhere-positive choice functions the vectors $\{\mathbf{M}_1, ..., \mathbf{M}_k\}$ and $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ coincide, and $\mathscr{C}(m_1, ..., m_k) = C(m_1, ..., m_k)$, for all k-vectors $\{m_1, ..., m_k\}$ of natural numbers. For simply-connected functions, $\mathscr{C}(m_1, ..., m_k) = C(m_1, ..., m_k)$ for all $\{m_1, ..., m_k\}$ such that $0 \leqslant m_i \leqslant \max_i$ $(i = 1, ..., k)$. Therefore, the use of conditional partition vectors $\{\mathbf{M}_1, ..., \mathbf{M}_k \mid \sum \mathbf{M}_i = \sum m_i\}$ to represent recurrent choices can be viewed as a general rule, whether or not the sets $\{X_1, ..., X_k\}$ of potentially observable counts contain internal gaps.

From this "ordinal-position" point of view, the results established in this paper can be summarized as follows. Let choices be made recurrently from a fixed set of k options within a certain observation period, so that at the end of this period the ith option is chosen a random number $\mathbf{X}_i$ times $(i = 1, ..., k)$. For every i, let the set X_i of possible (i.e., potentially observable) values of $\mathbf{X}_i$ be known; $X_i = x_0^{(i)} < \cdots < x_m^{(i)} < \cdots < x_{\max_i}^{(i)}$, where $\max_i > 0$ is finite or infinite $(i = 1, ..., k)$. It is assumed that a *single* decision to "choose the ith option again," given that it has already been chosen $x_m^{(i)}$ times, is the decision to select this option $x_{m+1}^{(i)} - x_m^{(i)}$ times $(m = 0, ..., x_{\max_i - 1}^{(i)})$. Once decided upon, the way that the factual selections are arranged in time should exclude the possibility that the observation period may end before all the $x_{m+1}^{(i)} - x_m^{(i)}$ selections have been made; otherwise, some intermediate values between $x_m^{(i)}$ and $x_{m+1}^{(i)}$ would have to be included in X_i. Under this assumption, the numerical value of $x_m^{(i)}$ in X_i becomes immaterial, and one should treat $x_m^{(i)}$ as simply the mth possible value (counting from zero) for the number of times that the ith option is chosen. As a result, the recurrent choices can be represented by random vectors $\{\mathbf{M}_1, ..., \mathbf{M}_k\}$, where $\mathbf{M}_i$ is the ordinal position of $\mathbf{X}_i$ in the set X_i. The problem addressed in this paper can now be presented as that of finding the necessary and sufficient conditions for the independent decomposability of $\mathscr{C}(m_1, ..., m_k)$, the ordinal choice function determining the distribution of a conditional partition vector $\{\mathbf{M}_1, ..., \mathbf{M}_k \mid \sum \mathbf{M}_i = \sum m_i\}$. According to the solution proposed, one should ask first whether $\mathscr{C}(m_1, ..., m_k)$ is m-separable, and the answer is provided by Theorem 3 (more precisely, by Theorem A1 specialized to simply-connected choice functions). If the answer is affirmative, all k-vectors of m-generating functions $\{q_1(x), ..., q_k(x)\}$ are related to each other by the transformations given in (10). Then, according to Theorem 2, the ordinal choice function $\mathscr{C}(m_1, ..., m_k)$ is independently decomposable if, and only if, the Laplace orders $\{\text{ord } q_1(x), ..., \text{ord } q_k(x)\}$ are all finite, in which case the class of all possible $(k+1)$-vectors of count p.m.f.'s independently decomposing this choice function is computed according to (11)–(12). This solution is complete, constructive, and applicable to all possible ordinal choice functions.

APPENDIX: CONNECTED CHOICE FUNCTIONS AS A GENERALIZATION OF EVERYWHERE-POSITIVE CHOICE FUNCTIONS

Definition A1. An unconditional random k-vector of counts $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ is called *domain-separable* if there are non-empty subsets $\{X_1, ..., X_k\}$ of natural numbers ("sets of potentially observable values"), such that $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} > 0$ for all $\{x_1, ..., x_k\} \in X_1 \times \cdots \times X_k$, and $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} = 0$ for all other $\{x_1, ..., x_k\}$. A choice function $C(x_1, ..., x_k)$ defined by

$$\frac{C(x_1, ..., x_k)}{\sum_{\sum u_i = \sum x_i} C(u_1, ..., u_k)} = \frac{\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\}}{\sum_{\sum u_i = \sum x_i} \text{Prob}\{\mathbf{X}_1 = u_1, ..., \mathbf{X}_k = u_k\}}$$

is called domain-separable if $\{\mathbf{X}_1, ..., \mathbf{X}_k\}$ is domain-separable.

Comments on Definition A1. Observe, first, that a domain-separable $C(x_1, ..., x_k)$ is in complete compliance with Definition 4. Second, to exclude trivial cases, a set X_i of potentially observable values of $\mathbf{X}_i$ will always be assumed to contain at least two different elements, for any $i = 1, ..., k$. Indeed, if $\mathbf{X}_i$ always equals a fixed number x, then the ith option can be excluded from the set $\{1, ..., k\}$, and $C(x_1, ..., x_k)$ can be redefined as a $(k-1)$-variate choice function.

The motivation for this definition is provided by the following lemma, whose proof is omitted as trivial.

Lemma A1. (i) *A domain-separable choice function* $C(x_1, ..., x_k)$ *is positive if, and only if,* $\text{Prob}\{\mathbf{X}_1 = x_1, ..., \mathbf{X}_k = x_k\} > 0$; *that is,* $C(x_1, ..., x_k) > 0$ *if, and only if,* $\{x_1, ..., x_k\} \in X_1 \times \cdots \times X_k$.

(ii) *Any m-separable choice function* $C(x_1, ..., x_k)$ *is domain-separable, with*

$$X_i = \{x \mid q_i(x) > 0\}, \qquad i = 1, ..., k.$$

By counterposition, a choice function that is not domain-separable is not m-separable; hence it cannot be independently decomposed.

Examples. The m-separable choice functions $C(x_1, x_2)$ constructed in E5–E8 are domain-separable, with the following sets X_1 and X_2 of potentially observable values:

(E5, continued) $X_1 = \{x \mid \mathscr{D}^{1,2}_{p_1, p_2}(x) > 0\} = \{0, 3, 4, 5, 6, 7, ...\}$; $X_2 = \{x \mid \mathscr{D}^{1,4}_{q_1, q_2}(x) > 0\} = \{0, 2, 3, 5, 6, 7, ...\}$.

(E6, continued) $X_1 = \{0, 1, ..., h\}$; $X_2 = \{0, 1, ..., h, h+g, h+g+1, h+g+2, ...\}$.

(E7, continued) $X_1 = \{0, 1, 2, ...\}$; $X_2 = \{0, h, 2h, ...\}$.

(E8, continued) $X_1 = \{1, 2, 2^2, ...\}$, and $X_2 = \{1, 3, 3^2, ...\}$.

The following notation is used in the rest of this Appendix. Let $C(x_1, ..., x_k)$ be domain-separable, positive on $X_1 \times \cdots \times X_k$, and let $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$ be a fixed k-vector. Then the sets $\{X_1^{o_1}, ..., X_k^{o_k}\}$ are defined as

$$X_i^{o_i} = \{x - o_i \mid x \in X_i\}, \qquad i = 1, ..., k$$

(observe that $X_i^{o_i}$-sets generally contain both non-negative and negative integers.) Let $\tilde{C}(x_1, ..., x_k)$ denote $C(o_1 + x_1, ..., o_k + x_k)$. Obviously, $\tilde{C}(x_1, ..., x_k) > 0$ if, and only if, $\{x_1, ..., x_k\} \in X_1^{o_1} \times \cdots \times X_k^{o_k}$. The one-argument and two-argument substitutions are defined by analogy with the notation used in Theorem 3. For a given k-vector $\{o_1, ..., o_k\}$,

$$\begin{aligned}\tilde{C}_a(x) &= \tilde{C}(x_1 = 0, ..., x_a = x, ..., x_k = 0)\\ &= C(x_1 = o_1, ..., x_a = o_a + x, ..., x_k = o_k),\\ &\qquad a = 1, ..., k;\\ \tilde{C}_{ab}(x, y) &= \tilde{C}(x_1 = 0, ..., x_a = x, ..., x_b = y, ..., x_k = 0)\\ &= C(x_1 = o_1, ..., x_a = o_a + x, ...,\\ &\qquad x_b = o_b + y, ..., x_k = o_k),\\ &\qquad a, b = 1, ..., k, \quad a \neq b.\end{aligned}$$

Lemma A2. *For a domain-separable choice function* $C(x_1, ..., x_k)$, *positive on* $X_1 \times \cdots \times X_k$, *and for a given* k-*vector* $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$,

(i) *$x \in U = \bigcup_{i=1}^{k} X_i^{o_i}$ if, and only if, $\tilde{C}_a(x) > 0$ for some $a = 1, ..., k$;*

(ii) *if $\tilde{C}_a(x) > 0$ and $\tilde{C}_b(y) > 0$, for some $a, b = 1, ..., k$, $a \neq b$, then $\widetilde{C_{ab}}(x, y) > 0$.*

Proof. (i) $x \in U$ if, and only if, for some $a = 1, ..., k$, $x \in X_a^{o_a}$. The latter is equivalent to $\{x_1 = 0, ..., x_a = x, ..., x_k = 0\} \in X_1^{o_1} \times \cdots \times X_k^{o_k}$, which, in turn, is equivalent to $\tilde{C}_a(x) > 0$.

(ii) Due to the previous result, $\{x_1 = 0, ..., x_a = x, ..., x_b = y, ..., x_k = 0\} \in X_1^{o_1} \times \cdots \times X_k^{o_k}$; hence $\widetilde{C_{ab}}(x, y) > 0$. ∎

The concept introduced next plays the key role in the further development.

Definition A2. A domain-separable choice function $C(x_1, ..., x_k)$, positive on $X_1 \times \cdots \times X_k$, is called *connected to* a k-vector $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$ if:

(i) the union set $U = \bigcup_{i=1}^{k} X_i^{o_i}$ forms an interval (finite or infinite) of consecutive integers: $u, u+1, ...$;

(ii) there are at least two different positions $b, b^* = 1, ..., k$, at which $\tilde{C}_b(1)\, \tilde{C}_{b^*}(1) > 0$.

$C(x_1, ..., x_k)$ is called *connected* if it is connected to some k-vector $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$.

Comments on Definition A2. A *simply-connected* choice function (see Definition 6) is connected to any k-vector $\{o_1, ..., o_k\}$ in its positive domain $\{0, ..., \max_1\} \times \cdots \times \{0, ..., \max_k\}$, except for those that contain upper boundaries $\max_i$. In particular, and most naturally, it is connected to $\{0, ..., 0\}$. With respect to $\{0, ..., 0\}$, $X_i = X_i^0$ is a set of consecutive natural numbers, for any $i = 1, ..., k$. Then the set U is also a set of consecutive natural numbers. Condition (i), therefore, is satisfied. Condition (ii) is satisfied because the function is positive on the entire $\{0, ..., \max_1\} \times \cdots \times \{0, ..., \max_k\}$, and $\max_i > 0$ for any $i = 1, ..., k$. As a special case, an everywhere-positive choice function is connected to any k-vector $\{o_1, ..., o_k\}$. It is shown below that the possibility of choosing an "origin point" $\{o_1, ..., o_k\}$ for which (i) and (ii) hold is sufficient for the theory of independent decomposability of everywhere-positive choice functions to be generalized with no essential modifications.

Examples. (E5, continued) The m-separable choice function $C(x_1, x_2)$ is connected. For instance, it is connected to $\{o_1, o_2\} = \{5, 6\}$. With respect to this pair, $X_1^5 = \{-5, -2, -1, 0, 1, 2, ...\}$, $X_2^6 = \{-6, -4, -3, -1, 0, 1, ...\}$, and U is the (infinite) set of consecutive integers beginning at -6. Since $k = 2$, Definition A2(ii) requires that both $\tilde{C}_1(1)$ and $\tilde{C}_2(1)$ be positive. This is the case, indeed, $\tilde{C}_1(1) = C(5+1, 6) > 0$ and $\tilde{C}_2(1) = C(5, 6+1) > 0$.

(E6, continued) The m-separable choice function $C(x_1, x_2)$ is connected if, and only if, $g \leqslant h$. For $g \leqslant h$, $C(x_1, x_2)$ is connected to $\{o_1, o_2\} = \{h-1, h+g\}$. Indeed: $C(h-1, h+g) > 0$; Definition A2(i) is satisfied because $U = X_1^{h-1} \cup X_2^{h+g} = \{-h-g, -h-g+1, ...\}$; Definition A2(ii) is satisfied because $C(h, h+g)\, C(h-1, h+g+1) > 0$. For $g > h$, however, it is easy to see that conditions (i) and (ii) of Definition A2 cannot be satisfied simultaneously by any pair $\{o_1, o_2\}$.

(E7, continued) The m-separable choice function $C(x_1, x_2)$ is connected if, and only if, $h = 1$. For $h = 1$, $C(x_1, x_2)$ is everywhere-positive, hence connected to any $\{x_1, x_2\}$. If $h > 1$, however, $C(x_1, x_2)$ is not connected, because $C(x_1, x_2) > 0$ only if $x_2 = mh$, but then $C(x_1, x_2 + 1) = 0$, contrary to condition (ii) of Definition A2.

(E8, continued) The m-separable choice function $C(x_1, x_2)$ is not connected, because $C(x_1, x_2) > 0$ only if $x_1 = 2^{m}$, $x_2 = 3^{m^*}$, but then $C(x_1, x_2 + 1) = C(x_1 + 1, x_2) = 0$, contrary to condition (ii) of Definition A2.

Lemma A3. *Let $C(x_1, ..., x_k)$, positive on $X_1 \times \cdots \times X_k$, be connected to $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$. Let U be the union set of Definition* A2. *Then*

(i) *$0 \in U$, and it is the highest possible value for* min U;

(ii) *$1 \in U$, and it is the lowest possible value for* max U;

(iii) *for any $x \in U$, one can find two different positions $a_x, b_x = 1, ..., k$, such that*

$$0 < \frac{\tilde{C}_{a_x}(x)\, \tilde{C}_{b_x}(1)}{\widetilde{C_{a_x b_x}}(x, 1)} < \infty.$$

Proof. (i) By definition of $X_i^{o_i}$-sets, $0 \in X_i^{o_i}$ because $o_i \in X_i$, for all $i = 1, ..., k$. If $o_i = \min X_i$ for all $i = 1, ..., k$, then $0 = \min U$; otherwise, $\min U < 0$.

(ii) By condition (ii) of Definition A2, $\tilde{C}_i(1) > 0$ for some $i = 1, ..., k$. Hence $1 \in X_i^{o_i} \subset U$. If for all $i = 1, ..., k$, $\max X_i \leqslant o_i + 1$, then $1 = \max U$; otherwise, $\max U > 1$ (in particular, it may be ∞).

(iii) By Lemma A2(i), one can find a position $a = 1, ..., k$, at which $\tilde{C}_a(x) > 0$. By condition (ii) of Definition A2, one can find a different position $b = 1, ..., k$, at which $\tilde{C}_b(1) > 0$. Then $\widetilde{C_{ab}}(x, 1) > 0$ by Lemma A2(ii). The subscript x at a and b reflects the fact that these values are generally different for different x. ∎

This completes the preparatory work for the key theorem on connected choice functions that is presented next. It generalizes Theorem 3 from everywhere-positive choice functions to all connected choice functions. Both the formulation and the proof of this theorem are structured to maximally resemble those of Theorem 3.

Theorem A1. *A choice function $C(x_1, ..., x_k)$ connected to $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$ is m-separable if, and only if,* *the following identity holds across all k-vectors* $\{x_1, ..., x_k\} \in X_1^{o_1} \times \cdots \times X_k^{o_k}$,

$$\tilde{C}(x_1, ..., x_k) = \frac{\prod_{i=1}^k \tilde{C}_i(x_i)\, P(x_i)}{\sum_{\sum u_i = \sum x_i} \prod_{i=1}^k \tilde{C}_i(u_i)\, P(u_i)} \tag{A1}$$

(*summation is across all partitions of* $\sum x_i$ *belonging to* $X_1^{o_1} \times \cdots \times X_k^{o_k}$), *where, for all* $x \in U = \bigcup_{i=1}^k X_i^{o_i}$,

$$P(x) = \begin{cases} \prod_{i=0}^{x-1} \dfrac{\tilde{C}_{a_i}(i)\, \tilde{C}_{b_i}(1)}{\widetilde{C_{a_ib_i}}(i, 1)}, & \text{if} \quad x > 0, \\ 1, & \text{if} \quad x = 0, \\ \prod_{i=1}^{|x|} \dfrac{\widetilde{C_{a_{-i}b_{-i}}}(-i, 1)}{\widetilde{C_{a_{-i}}}(-i)\, \widetilde{C_{b_{-i}}}(1)}, & \text{if} \quad x < 0, \end{cases} \tag{A2}$$

for some sequence of position pairs $a_x, b_x = 1, ..., k$, $a_x \neq b_x$.

Proof. Observe, first, that since $\{x_1, ..., x_k\} \in X_1^{o_1} \times \cdots \times X_k^{o_k}$, the left-hand expression in (A1) is finite and positive. Hence $P(x)$ must be finite and positive for all $x \in U$, which, by (A2), means that

$$0 < \frac{\tilde{C}_{a_x}(x)\, \tilde{C}_{b_x}(1)}{\widetilde{C_{a_xb_x}}(x, 1)} < \infty.$$

By Lemma A3(iii), such position pairs a_x, b_x can indeed be found for all $x \in U$.

Necessity is proved by using Definition 5 and expressing all choice functions in (A1) and (A2) through m-generating functions $\{q_1(x), ..., q_k(x)\}$ and $q_{1\ldots k}(x)$. Denoting $\sum o_i$ by O, observe that for any $x \in U$,

$$\begin{aligned} &\frac{\tilde{C}_{a_x}(x)\, \tilde{C}_{b_x}(1)}{\widetilde{C_{a_xb_x}}(x, 1)} \\ &= \left[\frac{q_{a_x}(o_{a_x} + x)}{q_{1\ldots k}(O + x)} \frac{q_{b_x}(o_{b_x} + 1)}{q_{1\ldots k}(O + 1)} \Big/ \frac{q_{a_x}(o_{a_x} + x)\, q_{b_x}(o_{b_x} + 1)}{q_{1\ldots k}(O + x + 1)} \right] \\ &\quad \times \left[\frac{\prod_{i=1}^k q_i(o_i)}{q_{a_x}(o_{a_x})} \frac{\prod_{i=1}^k q_i(o_i)}{q_{b_x}(o_{b_x})} \Big/ \frac{\prod_{i=1}^k q_i(o_i)}{q_{a_x}(o_{a_x})\, q_{b_x}(o_{b_x})} \right] \\ &= \frac{q_{1\ldots k}(O + x + 1)}{q_{1\ldots k}(O + x)} \left(\frac{\prod_{i=1}^k q_i(o_i)}{q_{1\ldots k}(O + 1)} \right) \\ &= \frac{q_{1\ldots k}(O + x + 1)}{q_{1\ldots k}(O + x)} A, \end{aligned}$$

where A replaces the ratio in parentheses. Using this expression in (A2), one derives that for $x > 0$ ($x \in U$),

$$\begin{aligned} P(x) &= \prod_{i=0}^{x-1} \frac{q_{1\ldots k}(O + i + 1)}{q_{1\ldots k}(O + i)} A \\ &= \frac{A^x}{q_{1\ldots k}(O)} q_{1\ldots k}(O + x). \end{aligned}$$

For $x < 0$ ($x \in U$) the resulting expression is the same:

$$\begin{aligned} P(x) &= \prod_{i=1}^{|x|} \frac{q_{1\ldots k}(O - i)}{q_{1\ldots k}(O - i + 1)} A^{-1} \\ &= \frac{A^{-|x|}}{q_{1\ldots k}(O)} q_{1\ldots k}(O - |x|) \\ &= \frac{A^x}{q_{1\ldots k}(O)} q_{1\ldots k}(O + x). \end{aligned}$$

This expression satisfies the requirement that $P(0) = 1$. Substituting in the numerator of the right-hand ratio in (A1), one gets

$$\begin{aligned} &\prod_{i=1}^k \frac{q_i(o_i + x_i)[\prod_{j=1}^k q_j(o_j)/q_i(o_i)]}{q_{1\ldots k}(O + x_i)} q_{1\ldots k}(O + x_i) \frac{A^{x_i}}{q_{1\ldots k}(O)} \\ &= \left(\prod_{i=1}^k q_i(o_i) \right)^{k-1} \frac{A^{\sum x_i}}{q_{1\ldots k}(O)^k} \prod_{i=1}^k q_i(o_i + x_i). \end{aligned}$$

Equation (A1) then transforms into

$$\begin{aligned} \tilde{C}(x_1, ..., x_k) &= \frac{\prod_{i=1}^k q_i(o_i + x_i)}{\sum_{\sum u_i = \sum x_i} \prod_{i=1}^k q_i(o_i + u_i)} \\ &= \frac{\prod_{i=1}^k q_i(o_i + x_i)}{q_{1\ldots k}(O + \sum_{i=1}^k x_i)}, \end{aligned}$$

which, due to Definition 5, holds identically across all k-vectors $\{x_1, ..., x_k\} \in X_1^{o_1} \times \cdots \times X_k^{o_k}$. This completes the proof of the necessity.

Sufficiency is obtained immediately, by putting

$$q_i(o_i + x) = \begin{Bmatrix} \tilde{C}_i(x)\, P(x) & \text{for} \quad x \in U \\ 0 & \text{for} \quad x \notin U \end{Bmatrix}, \qquad i = 1, ..., k.$$

Definition 5 is obviously satisfied:

$$\frac{C(o_1 + x_1, ..., o_k + x_k)}{\sum_{\sum u_i = \sum x_i} C(o_1 + u_1, ..., o_k + u_k)} = \frac{\prod_{i=1}^k q_i(o_i + x_i)}{q_{1\ldots k}(O + \sum_{i=1}^k x_i)}$$

holds as a numerical equality if at least one partition $\{u_1, ..., u_k\}$ of $\sum x_i$ belongs to $X_1^{o_1} \times \cdots \times X_k^{o_k}$; if otherwise, then both sides of the equation are indeterminate forms 0/0. This completes the proof of the sufficiency and of the theorem. ∎

Comments on Theorem A1. The reason for presenting the denominator in (A1) as $\sum \prod \tilde{C}_i(u_i)\, P(u_i)$ rather than $P(\sum x_i)$, as in (6) of Theorem 3, is that $P(x)$ in (A2) is only defined for $x \in U$, and if U is finite, it does not include all possible values of $\sum x_i$. Without elaborating, it is easy to show that

$$\sum_{\sum u_i = \sum x_i} \prod_{i=1}^{k} \tilde{C}_i(u_i)\, P(u_i) = P\left(\sum_{i=1}^{k} x_i\right)$$

whenever $\sum x_i \in U$. In particular, this is always the case when U is infinite (i.e., not bounded from above).

Continuing the analogy with Theorem 3, one can prove now, as a corollary, that Theorem 1 generalizes to all connected m-separable choice functions. In addition, the corollary generalizes to connected choice functions the constructive algorithm for computing m-generating functions $\{q_1(x), ..., q_k(x)\}$. The corollary is presented without proof, because, given the modifications in its formulation, the proof is a virtually verbatim replication of that for everywhere-positive choice functions.

Corollary to Theorem A1. For an m-separable choice function $C(x_1, ..., x_k)$ connected to $\{o_1, ..., o_k\} \in X_1 \times \cdots \times X_k$, the m-generating functions $\{q_1(x), ..., q_k(x)\}$ are determined as follows:

(i) compute the auxiliary function $P(x)$ according to (A2), for all $x \in U$; the values of $P(x)$ are determined uniquely; that is, they are the same for any choice of the sequence a_x, b_x in (A2), provided that this sequence satisfies the inequality of Lemma A3(iii);

(ii) set $q_i(o_i)$, $i = 1, ..., k$, equal to arbitrary positive values, $\{\alpha_1, ..., \alpha_k\}$;

(iii) choose an arbitrary real λ, and compute $q_{1\cdots k}(O+1)$ (where O stands for $\sum o_i$) from

$$\lambda = \log \frac{\prod_{i=1}^{k} \alpha_i}{q_{1\cdots k}(O+1)}.$$

(iv) compute $q_i(o_i + x)$ and $q_{1\cdots k}(O+x)$ as

$$q_i(o_i + x) = \begin{cases} \alpha_i C_i(o_i + x)\, P(x)\, e^{-\lambda x}, & \text{if } x \in U \\ 0, & \text{if } x \notin U \end{cases},$$

$$i = 1, ..., k,$$

$$q_{1\cdots k}(O+x) = \sum_{\sum u_i = x} \prod_{i=1}^{k} q_i(o_i + u_i).$$

$C(x_1, ..., x_k)$ is not m-generated by any k-vector of functions that does not satisfy (8). Hence different k-vectors of m-generating functions $\{q_1(x), ..., q_k(x)\}$ are interrelated by transformations (10).

To complete the theory of independent decomposability of connected choice functions, it remains to observe the following. Because the assumption of Theorem 4(iii) is satisfied for all connected m-separable choice functions, independent decompositions of such a function exist if, and only if, $\max\{\text{ord } q_1(x), ..., \text{ord } q_k(x)\}$ is finite, and then all such decompositions are computed according to (11) and (12). Put differently, *Theorem 2 applies in its entirety to all connected m-separable choice function.*

EXAMPLES. (E5, continued) The connected m-separable choice function $C(x_1, x_2)$ is independently decomposable. It is easy to show that $\text{ord } \mathscr{D}^{a_1, a_2}_{p_1, p_2}(x) = |\log p_1| + |\log p_2|$, and that for any λ exceeding this value,

$$\widetilde{\mathscr{D}^{a_1, a_2}_{p_1, p_2}}(\lambda) = \sum_{x=0}^{\infty} (p_1^{x-a_1} + p_1^{a_1-x} - 2)(p_2^{x-a_2} + p_2^{a_2-x} - 2)\, e^{-\lambda x}$$

$$= \sum_{\iota_1, \iota_2 = -1, 0, 1} \frac{4 p_1^{\iota_1 a_1} p_2^{\iota_2 a_2}}{(1 + |\iota_1|)(1 + |\iota_2|)(1 - p_1^{-\iota_1} p_2^{-\iota_2} e^{-\lambda})}.$$

Hence $C(x_1, x_2)$ is independently decomposed by

$$R_1(x) = \frac{\mathscr{D}^{1,2}_{p_1, p_2}(x)}{\widetilde{\mathscr{D}^{1,2}_{p_1, p_2}}(\lambda)},$$

$$R_2(x) = \frac{\mathscr{D}^{1,4}_{q_1, q_2}(x)}{\widetilde{\mathscr{D}^{1,4}_{q_1, q_2}}(\lambda)},$$

$$R(x) = \frac{\sum_{u_1 + u_2 = x_1 + x_2} \mathscr{D}^{1,2}_{p_1, p_2}(u_1)\, \mathscr{D}^{1,4}_{q_1, q_2}(u_2)}{\widetilde{\mathscr{D}^{1,2}_{p_1, p_2}}(\lambda)\, \widetilde{\mathscr{D}^{1,4}_{q_1, q_2}}(\lambda)},$$

where $\lambda > \max\{|\log p_1| + |\log p_2|, |\log q_1| + |\log q_2|\}$. By the corollary to Theorem A1 and by Theorem 4(iii), these are the only possible triples of decomposing functions for $C(x_1, x_2)$.

(E6, continued) For $g \leqslant h$, the m-separable choice function $C(x_1, x_2)$ is connected; due to the corollary to Theorem A1 and by Theorem 4(iii), it is independently decomposable if, and only if, $\text{ord } q_2(x)$ is finite. Indeed, since $X_1 = \{0, 1, ..., h\}$ is finite, $\text{ord } q_1(x)$ is necessarily finite. For $g > h$, $C(x_1, x_2)$ is not connected. One can still apply Theorem 4(ii) to conclude that if $\text{ord } q_2(x)$ is finite, then $C(x_1, x_2)$ is independently decomposable. The "only if" counterpart of this statement also happens to be true, but it does not follow from the corollary to Theorem A1, and the class of decomposing count p.m.f.'s here is not restricted to those given by (11) and (12).

(E7, continued) For $h = 1$, $C(x_1, x_2)$ is everywhere-positive, and the corollary to Theorem A1 applies in the form of the corollary to Theorem 3. For $h > 1$, it follows from Theorem 4(ii) that $C(x_1, x_2)$ is independently decomposable if both $\text{ord } q_1(x)$ and $\text{ord } q_2(x)$ are finite. As in the previous example, the "only if" counterpart of this statement also happens to be true, but it does not follow from the corollary to Theorem A1, and the class of decomposing count p.m.f.'s here is not restricted to those given by (11) and (12).

(E8, continued) The m-separable choice function $C(x_1, x_2)$ is not connected, and the theory above does not apply. It is obvious that $C(x_1, x_2)$ is independently decomposed by any pair of p.m.f.'s whose positive domains coincide with those of $\{q_1(x), q_2(x)\}$. ∎

In conclusion, to prevent false conjectures, it should be pointed out that connectedness of m-separable choice functions is only sufficient, but not necessary, for admissible transformations of m-generating functions to be limited to transformations (10). As an example, $\{q_1(x), q_2(x)\}$ with the following properties,

$$\left\{\begin{array}{lll} q_i(x) > 0 & \text{if} & x \leqslant h \quad \text{or} \quad x \geqslant h+g \\ q_i(x) = 0 & \text{if} & h < x < h+g \end{array}\right\}, \qquad i = 1, 2,$$

can be shown to be determined up to transformations (10), for any non-negative h and g. In particular, this is true when $g > h$, even though in this case the choice function m-generated by $\{q_1(x), q_2(x)\}$ is not connected (compare this example with E6 above).

References

Aczél, J. (1966). *Lectures on functional equations and their applications.* New York: Academic Press.

Aczél, J. (1975). General solution of a functional equation connected with a characterization of statistical distributions. In G. P. Patil, S. Kotz, & J. K. Ord (Eds.). *Statistical distributions in scientific work* (Vol. 3, pp. 47–55). Dordrecht: Riedel.

Bol'shev, L. N. (1965). On characterization of the Poisson distribution and its statistical applications. *Teoria Veroyatnostey i ee Primeneniya*, **10**, 488–499.

Böckenholt, U. (1993). Estimating latent distributions in recurrent choice data. *Psychometrika*, **58**, 489–509.

Gerber, H. U. (1980). A characterization of certain families of distributions via Esscher transforms and independence. *Journal of the American Statistical Association*, **75**, 1015–1018.

Hameister, E. (1946). *Laplace-transformation.* Ann Arbor, MI: Edwards.

Janardan, K. G. (1974). A characterization of multinomial and negative multinomial distributions. *Scandinavian Actuarial Journal*, 58–62.

Janardan, K. G. (1975). Characterizations of certain discrete distributions. In G. P. Patil, S. Kotz, & J. K. Ord (Eds.). *Statistical Distributions in Scientific Work* (Vol. 3, pp. 359–364). Dordrecht: Riedel.

Johnson, N. L., and Kotz, S. (1969). *Discrete distributions.* New York: Houghton Mifflin.

Kabe, D. G. (1969). On characterizing the multivariate linear exponential distribution. *Canadian Mathematical Bulletin*, **12**, 567–572.

Mathai, A. M. (1967). On the structural properties of the conditional distributions. *Canadian Mathematical Bulletin*, **10**, 239–245.

Menon, M. V. (1966). Characterization theorems for some univariate probability distributions. *Journal of the Royal Statistical Society B*, **28**, 143–145.

Moran, P. A. P. (1952). A characteristic property of the Poisson distribution. *Proceedings of the Cambridge Philosophical Society*, **48**, 206–207.

Panaretos, J. (1982). On a structural property of finite distributions. *Journal of the Royal Statistical Society B*, **44**, 209–211.

Patil, G. P., and Seshadri, V. (1964). Characterization theorems for some univariate probability distributions. *Journal of the Royal Statistical Society B*, **26**, 286–292.

Received: April 14, 1993