Empirical Recovery of Response Time Decomposition Rules
I. Sample-Level Decomposition Tests

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E. N. Dzhafarov and R. Schweickert (1995, *Journal of Mathematical Psychology*, **39**, 285–314) developed a mathematical theory for the decomposability of response time (RT) into two component times that are selectively influenced by different factors and are either stochastically independent or perfectly positively stochastically interdependent (in which case they are increasing functions of a common random variable). In this theory, RT is obtained from its component times by means of an associative and commutative operation. For any such operation, there is a decomposition test, a relationship between observable RT distributions that holds if and (under mild constraints) only if the RTs are decomposable by means of this operation. In this paper, we construct a sample-level version of these decomposition tests that serve to determine whether RTs that are represented by finite samples are decomposable by means of a given operation (under a given form of stochastic relationship between component times, independence or perfect positive interdependence). The decision is based on the asymptotic p-values associated with the maximal distance between empirical distribution functions computed by combining in a certain way the RT samples corresponding to different treatments. © 1996 Academic Press, Inc.

PROBLEM

In this paper, we construct and investigate a sample-level version of the decomposition tests proposed on a population level by Dzhafarov and Schweickert (1995). The decomposition tests serve to decide in what way a response time (RT), whose distribution depends on two experimental factors, can be decomposed into two component times that are selectively influenced by these factors and have a specified form of stochastic relationship between them. Denoting the RT by $\mathbf{T}(\alpha, \beta)$, where α and β are the two experimental factors mentioned, the decompositions investigated by Dzhafarov and Schweickert can be presented as

$$\mathbf{T}(\alpha, \beta) \stackrel{\mathrm{d}}{=} \mathbf{A}(\alpha) \blacklozenge \mathbf{B}(\beta), \quad \mathbf{A}(\alpha) \stackrel{\mathrm{s}}{-\!\!-} \mathbf{B}(\beta), \tag{1}$$

where $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$ are component times selectively influenced by α and β, $\stackrel{\mathrm{d}}{=}$ means "is distributed as," and $\blacklozenge$ is an associative and commutative algebraic operation, referred to as the decomposition rule. The symbol $\stackrel{\mathrm{s}}{-\!\!-}$

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denotes one of two simple forms of stochastic relationship between $\mathbf{A}(\alpha)$ and $\mathbf{B}(\beta)$: either these component times are stochastically independent, which is denoted by $\mathbf{A}(\alpha) \perp \mathbf{B}(\beta)$, or they are perfectly positively stochastically interdependent, $\mathbf{A}(\alpha) \parallel \mathbf{B}(\beta)$, which means that for any values of α and β, they are increasing functions of each other (that is, they can be viewed as increasing functions of a single "internal source of variability"). Following Dzhafarov and Schweickert (1995), we abbreviate stochastic independence as s.-independence, and perfect positive stochastic interdependence as p.p.s.-interdependence. Either of these forms of stochastic relationship makes $\mathbf{A}(\alpha) \blacklozenge \mathbf{B}(\beta)$ uniquely determined by the marginal distributions of the component times, for any $\blacklozenge$, with no restrictions imposed on these marginal distributions. (To avoid technical difficulties, however, we impose in this paper some smoothness constraints on both the distribution functions and their inverses, the quantile functions.)

In Dzhafarov and Schweickert's theory, the decomposition rule $\blacklozenge$ is the only thing to be determined. That the component times are influenced by α and β selectively, and that their stochastic relationship $\stackrel{\mathrm{s}}{-\!\!-}$ has a particular form ($\perp$ or $\parallel$) are not falsifiable assumptions; rather they should be considered parts of the definition of what kind of component times the decomposition rule sought is applied to. (See, however, Dzhafarov, 1992, and Dzhafarov & Rouder, 1996, for an empirical test of s.-independence versus p.p.s.-interdependence in a different context.)

In a 2×2 (subset of a) crossed factorial design, $(\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$, one can denote

$$\mathbf{T}_{ij} = \mathbf{T}(\alpha_i, \beta_j), \quad \mathbf{A}_i = \mathbf{A}(\alpha_i), \quad \mathbf{B}_j = \mathbf{B}(\beta_j) \quad (i = 1, 2, j = 1, 2)$$

and represent (1) as a system of four distributional equations,

$$\begin{cases} \mathbf{T}_{11} \stackrel{\mathrm{d}}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_1 \ (\mathbf{A}_1 \stackrel{\mathrm{s}}{-\!\!-} \mathbf{B}_1) \\ \mathbf{T}_{12} \stackrel{\mathrm{d}}{=} \mathbf{A}_1 \blacklozenge \mathbf{B}_2 \ (\mathbf{A}_1 \stackrel{\mathrm{s}}{-\!\!-} \mathbf{B}_2) \\ \mathbf{T}_{21} \stackrel{\mathrm{d}}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_1 \ (\mathbf{A}_2 \stackrel{\mathrm{s}}{-\!\!-} \mathbf{B}_1) \\ \mathbf{T}_{22} \stackrel{\mathrm{d}}{=} \mathbf{A}_2 \blacklozenge \mathbf{B}_2 \ (\mathbf{A}_2 \stackrel{\mathrm{s}}{-\!\!-} \mathbf{B}_2), \end{cases} \tag{2}$$

where $-\!\!^{\text{s}}\!\!-$ has the same meaning in all four equations, either $\perp$ or $\|$. A necessary condition for such a decomposability is

$$\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22} \stackrel{\text{d}}{=} \mathbf{T}_{12} \blacklozenge \mathbf{T}_{21} \ (\mathbf{T}_{11} -\!\!^{\text{s}}\!\!- \mathbf{T}_{22}, \mathbf{T}_{12} -\!\!^{\text{s}}\!\!- \mathbf{T}_{21}). \quad (3)$$

This proposition is referred to as the (population-level) decomposition test for $\blacklozenge$, or ($\blacklozenge$)-test, because if (3) does not hold, then the RT is not ($\blacklozenge$)-decomposable (i.e., it is not decomposable by means of the operation $\blacklozenge$). For a given $\blacklozenge$ and a given form of $-\!\!^{\text{s}}\!\!-$, the distributions of $\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22}$ and $\mathbf{T}_{12} \blacklozenge \mathbf{T}_{21}$ are observable, as they are uniquely determined by those of the observable random variables $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$. Namely,

$$\text{Prob}\{\mathbf{T}_{ij} \blacklozenge \mathbf{T}_{3-i,\,3-j} \leqslant t \mid \mathbf{T}_{ij} \perp \mathbf{T}_{3-i,\,3-j}\}$$

$$= \iint_{u \,\blacklozenge\, v \leqslant t} dF_{ij}(u) \, dF_{3-i,\,3-j}(v)$$

$$\text{Prob}\{\mathbf{T}_{ij} \blacklozenge \mathbf{T}_{3-i,\,3-j} \leqslant t \mid \mathbf{T}_{ij} \| \mathbf{T}_{3-i,\,3-j}\} \qquad (4)$$

$$= \int_{T_{ij}(p) \,\blacklozenge\, T_{3-i,\,3-j}(p) \leqslant t} dp,$$

where $i = 1, 2$, $j = 1, 2$, $F_{ij}(t)$ is the distribution function for $\mathbf{T}_{ij}$, and $T_{ij}(p) = F_{ij}^{-1}(p)$ is its quantile function ($0 \leqslant p \leqslant 1$).

In the special case of additive decompositions into s.-independent components (i.e., when $\blacklozenge$ is $+$ and $-\!\!^{\text{s}}\!\!-$ is $\perp$), the decomposition test (3) becomes the familiar "summation test," proposed by Ashby and Townsend (1980) and elaborated by Roberts and Sternberg (1992).

In the context of the present work, the central problem investigated in Dzhafarov and Schweickert (1995) is that of the uniqueness of the decomposition rule: if $\blacklozenge$ and $*$ are two different operations, can a RT be both ($\blacklozenge$)-decomposable and ($*$)-decomposable, assuming the same form of stochastic relationship between component times? Under certain constraints, listed below, the answer to this questions turns out to be negative: a successful ($\blacklozenge$)-test excludes the possibility of a success for another, ($*$)-test (under the same form of $-\!\!^{\text{s}}\!\!-$), excluding thereby the possibility that the RT in question is ($*$)-decomposable. The constraints are as follows.

First, the class of the decomposition rules to which $\blacklozenge$ and $*$ belong must be confined to a proper subclass of associative and commutative operations, termed the simple operations. This subclass consists of the operations $\min\{a, b\}$, $\max\{a, b\}$, and all operations $a \oplus b$ with "addition-like" properties: continuous in both arguments, strictly increasing in both arguments, and mapping onto their domains. One can think of the simple operations as obtained by the following algorithm: choose a strictly monotonic continuous function g, and define an operation $a \blacklozenge b$ as

$$a \blacklozenge b \equiv g^{-1}[\,g(a) \wedge g(b)\,], \qquad (5)$$

where $\wedge$ is one of the three "prototypical" operations, $\min\{a, b\}$, $\max\{a, b\}$, or $a + b$. Examples of addition-like operations $g^{-1}[\,g(a) + g(b)\,]$ (on the domain of positive reals) are $a + b$, $a \times b$, $(a^k + b^k)^{1/k}$, etc.; however, all "maximum-like" and "minimum-like" operations, $g^{-1}[\max\{g(a), g(b)\}]$ and $g^{-1}[\min\{g(a), g(b)\}]$, simply coincide with $\max\{a, b\}$ and $\min\{a, b\}$ themselves (not necessarily respectively). The operations $\max\{a, b\}$ and $\min\{a, b\}$ can also be construed as limiting cases for addition-like operations: for positive a and b, $(a^k + b^k)^{1/k}$ tends to $\max\{a, b\}$ and $\min\{a, b\}$ as k tends to ∞ and $-\infty$, respectively.

Second, the competing decomposition rules $\blacklozenge$ and $*$ must be "algebraically distinct." This means that for any u and v, there is at most one unordered pair (a, b) that satisfies the system of equations

$$\begin{cases} a \blacklozenge b = u \\ a * b = v. \end{cases}$$

The examples of simple operations given above, $\min\{a, b\}$, $\max\{a, b\}$, $a + b$, $a \times b$, $(a^k + b^k)^{1/k}$, etc., are all pairwise algebraically distinct.

The third and final constraint is imposed on the quadruple of observable RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$. Let $F_{11}(t)$, $F_{12}(t)$, $F_{21}(t)$, $F_{22}(t)$ be their respective distribution functions. If

$$\begin{cases} \max\{F_{12}(t), F_{21}(t)\} \equiv \max\{F_{11}(t), F_{22}(t)\} \\ \min\{F_{12}(t), F_{21}(t)\} \equiv \min\{F_{11}(t), F_{22}(t)\}, \end{cases}$$

then $F_{12}(t)$ and $F_{21}(t)$ are called "cross-over rearrangements" of $F_{11}(t)$ and $F_{22}(t)$ (see Dzhafarov & Schweickert, 1995, for a detailed explanation). The requirement is that this should not be the case. In particular, it should not be the case that

$$\begin{cases} F_{11}(t) \equiv F_{21}(t) \\ F_{22}(t) \equiv F_{12}(t) \end{cases} \quad \text{or} \quad \begin{cases} F_{11}(t) \equiv F_{12}(t) \\ F_{22}(t) \equiv F_{21}(t). \end{cases} \quad (6)$$

In either of the latter cases, at least one of the index factors is ineffective: changes in its level do not affect the RT distribution at any level of the other factor. When this happens, the RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ trivially satisfy (3) for any operation $\blacklozenge$, and nothing can be deduced about their decomposability. Note that cross-over rearrangements, as well as their special case, ineffective index factors, are defined solely in terms of observable RTs, and are therefore empirically identifiable in principle.

In the subsequent discussion these constraints are assumed to be satisfied implicitly. There is hardly a cause for concern here: theoretically interesting competing decomposition rules are likely to be algebraically distinct simple

operations, and when both factors are effective, cross-over rearrangements are highly artificial and unlikely to occur. The degenerate case when one of the factors is ineffective can, however, take place, and it should be excluded from consideration. In the context of the present research, the concern associated with this case is that even when both factors are effective on a population level, the effect of one of them may be too weak to enable one to recover the true decomposition rule in a competition with other operations.

To sum up the relevant aspects of Dzhafarov and Schweickert's theory: if RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ are de facto ($\blacklozenge$)-decomposable, as in (2), then the decomposition tests uniquely determine whether a given operation is the true decomposition rule $\blacklozenge$. Indeed, it is guaranteed that

$$
\begin{cases}
\mathbf{T}_{11} \blacklozenge \mathbf{T}_{22} \overset{d}{=} \mathbf{T}_{12} \blacklozenge \mathbf{T}_{21} \ (\mathbf{T}_{11} \overset{s}{-\cdot-} \mathbf{T}_{22}, \ \mathbf{T}_{12} \overset{s}{-\cdot-} \mathbf{T}_{21}) \\
\mathbf{T}_{11} * \mathbf{T}_{22} \overset{d}{\neq} \mathbf{T}_{12} * \mathbf{T}_{21} \ (\mathbf{T}_{11} \overset{s}{-\cdot-} \mathbf{T}_{22}, \ \mathbf{T}_{12} \overset{s}{-\cdot-} \mathbf{T}_{21})
\end{cases}
\tag{7}
$$

for any operation $*$ other than $\blacklozenge$.

Note that the unique recovery of the decomposition rule does not imply a unique recovery of the component times: generally, ($\blacklozenge$)-decomposable RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ allow for more than one quadruple of component times $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$. Recall also that the decomposition tests are not designed to test the form of stochastic relationship. The latter should be taken as part of the definition of the component times to be connected by the operation sought.

Now we can specify the sample-level problem we focus on in this work. Let RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ be ($\blacklozenge$)-decomposable, under a given form of stochastic relationship $\overset{s}{-\cdot-}$. For any operation $\diamond$ (identical to or different from $\blacklozenge$), denote by $C_\diamond(t)$ the distribution function of the "cross" combination $\mathbf{T}_{12} \diamond \mathbf{T}_{21}$ ($\mathbf{T}_{12} \overset{s}{-\cdot-} \mathbf{T}_{21}$) and by $U_\diamond(t)$ the distribution function of the "uncross" combination $\mathbf{T}_{11} \diamond \mathbf{T}_{22}$ ($\mathbf{T}_{11} \overset{s}{-\cdot-} \mathbf{T}_{22}$). Let $d[U_\diamond(t), C_\diamond(t)]$ be some dissimilarity function on the space of distribution functions (vanishing if and only if the two distribution functions are identical). We can restate (7) in the following form:

$$
d[U_\blacklozenge(t), C_\blacklozenge(t)] = 0, \quad d[U_*(t), C_*(t)] > 0. \tag{8}
$$

Here $*$ is any operation other than the true decomposition rule $\blacklozenge$. Obviously, this proposition cannot be verified directly if the RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ are only represented by finite-size samples. All one can do in this case is to construct consistent estimators $\mathbf{D}_\diamond$ for the inter-distributional dissimilarities $d[U_\diamond(t), C_\diamond(t)]$, and to base one's decisions on these estimators' values.

The decision considered in this paper is whether a given operation $\diamond$ is the true decomposition rule $\blacklozenge$ (that is

assumed to exist).[1] In statistical terms, the null hypothesis that $\diamond$ is $\blacklozenge$ is tested against the generic alternative that $\diamond$ is different from $\blacklozenge$. Intuitively, if the null hypothesis is correct, then the observed value of $\mathbf{D}_\diamond = \mathbf{D}_\blacklozenge$ should be sufficiently small. One justifies this intuition by showing that, as the sample sizes for the RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ increase beyond bound,

$$
\begin{cases}
\text{Prob}\{\mathbf{D}_\blacklozenge < \varepsilon\} \to 1, & \text{for all} \quad \varepsilon > 0 \\
\text{Prob}\{\mathbf{D}_* < \varepsilon\} \to 0, & \text{for some} \quad \varepsilon > 0
\end{cases}
\tag{9}
$$

for any operation $*$ other than $\blacklozenge$.

In the next section we construct inter-distributional dissimilarities (in fact, distances) and their estimators in such a way that p-values associated with the asymptotic sampling distribution of $\mathbf{D}_\blacklozenge$ can be analytically evaluated (or enclosed between bounds), for any operation $\blacklozenge$ and for either of the two forms of stochastic relationship.

To avoid technical difficulties we assume that the distribution functions $F_{ij}(t)$ of RTs $\mathbf{T}_{ij}$ ($i = 1, 2$, $j = 1, 2$) are piecewise differentiable and have piecewise differentiable inverses $T_{ij}(p)$, $0 \leq p \leq 1$, the quantile functions. Then the same is true for the distribution and quantile functions

$$
\begin{array}{lll}
U_\diamond(t)_\perp, \ U_\diamond^{-1}(p)_\perp, & \text{for} & \mathbf{T}_{11} \diamond \mathbf{T}_{22} \ (\mathbf{T}_{11} \perp \mathbf{T}_{22}), \\
C_\diamond(t)_\perp, \ C_\diamond^{-1}(p)_\perp, & \text{for} & \mathbf{T}_{12} \diamond \mathbf{T}_{21} \ (\mathbf{T}_{12} \perp \mathbf{T}_{21}), \\
U_\diamond(t)_{||}, \ U_\diamond^{-1}(p)_{||}, & \text{for} & \mathbf{T}_{11} \diamond \mathbf{T}_{22} \ (\mathbf{T}_{11} \parallel \mathbf{T}_{22}), \\
C_\diamond(t)_{||}, \ C_\diamond^{-1}(p)_{||}, & \text{for} & \mathbf{T}_{12} \diamond \mathbf{T}_{21} \ (\mathbf{T}_{12} \parallel \mathbf{T}_{21}).
\end{array}
$$

The notation refers to "uncross" and "cross" combinations of s.-independent ($\perp$) and p.p.s.-interdependent ($\parallel$) RTs by means of the operation $\diamond$. We will also assume piecewise differentiability of all simple operations in both arguments: this is trivially satisfied when $\diamond$ is minimum or maximum, and for the addition-like operations this is equivalent to the piecewise differentiability of the function g in $a \diamond b = g^{-1}[g(a) + g(b)]$.

All proofs and comments of a mathematical nature are relegated to the Appendices.

SAMPLE-LEVEL DECOMPOSITION TESTS: ASYMPTOTIC p-VALUES

Let $\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$ be a random sample from $\mathbf{T}_{ij}$ (i.e., independent random variables distributed as $\mathbf{T}_{ij}$), and let

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[1] Dzhafarov and Schweickert determined that a successful ($\blacklozenge$)-test guarantees that the RTs in questions are ($\blacklozenge$)-decomposable when the form of stochastic relationship is p.p.s.-interdependence, or if $\blacklozenge$ is minimum or maximum under s.-independence; however, if $\blacklozenge$ is an addition-like operation, then under s.-independence, it is possible that RTs are not decomposable by means of any simple operation even when the ($\blacklozenge$)-test is successful.

$\{\mathbf{T}_{ij}^{(1)}, ..., \mathbf{T}_{ij}^{(n)}\}$ be the same random sample arranged in an increasing order ($i = 1, 2, j = 1, 2$). Then the sequences

$$\{\mathbf{T}_{11}^1 \diamondsuit \mathbf{T}_{22}^1, ..., \mathbf{T}_{11}^n \diamondsuit \mathbf{T}_{22}^n\}$$

and

$$\{\mathbf{T}_{12}^1 \diamondsuit \mathbf{T}_{21}^1, ..., \mathbf{T}_{12}^n \diamondsuit \mathbf{T}_{21}^n\}$$

are random samples from $\mathbf{T}_{11} \diamondsuit \mathbf{T}_{22}$ ($\mathbf{T}_{11} \perp \mathbf{T}_{22}$) and $\mathbf{T}_{12} \diamondsuit \mathbf{T}_{21}$ ($\mathbf{T}_{12} \perp \mathbf{T}_{21}$), respectively. Denoting the (random) empirical distribution functions corresponding to these two sequences by $\mathbf{U}_{\diamondsuit}^n(t)_{\perp}$ and $\mathbf{C}_{\diamondsuit}^n(t)_{\perp}$, respectively, we have from classical statistical theory (see, e.g., Csörgo, 1983)

$$\begin{cases} \mathbf{U}_{\diamondsuit}^n(t)_{\perp} \xrightarrow{\text{a.s.}} U_{\diamondsuit}(t)_{\perp} \\ \mathbf{C}_{\diamondsuit}^n(t)_{\perp} \xrightarrow{\text{a.s.}} C_{\diamondsuit}(t)_{\perp} \end{cases} \quad (10)$$

(a.s. stands for almost sure convergence).

In the case of p.p.s.-interdependence, we form the sequences of paired empirical quantiles

$$\{\mathbf{T}_{11}^{(1)} \diamondsuit \mathbf{T}_{22}^{(1)}, ..., \mathbf{T}_{11}^{(n)} \diamondsuit \mathbf{T}_{22}^{(n)}\}$$

and

$$\{\mathbf{T}_{12}^{(1)} \diamondsuit \mathbf{T}_{21}^{(1)}, ..., \mathbf{T}_{12}^{(n)} \diamondsuit \mathbf{T}_{21}^{(n)}\}.$$

These sequences cannot be viewed as ordered random samples from $\mathbf{T}_{11} \diamondsuit \mathbf{T}_{22}$ ($\mathbf{T}_{11} \parallel \mathbf{T}_{22}$) and $\mathbf{T}_{12} \diamondsuit \mathbf{T}_{21}$ ($\mathbf{T}_{12} \parallel \mathbf{T}_{21}$), because the paired values, say, $\mathbf{T}_{11}^{(i)}$ and $\mathbf{T}_{22}^{(i)}$, do not have the same population quantile rank. Nevertheless, as shown in Appendix 1, here too we have the almost sure convergence

$$\begin{cases} \mathbf{U}_{\diamondsuit}^n(t)_{\parallel} \xrightarrow{\text{a.s.}} U_{\diamondsuit}(t)_{\parallel} \\ \mathbf{C}_{\diamondsuit}^n(t)_{\parallel} \xrightarrow{\text{a.s.}} C_{\diamondsuit}(t)_{\parallel}, \end{cases} \quad (11)$$

where $\mathbf{U}_{\diamondsuit}^n(t)_{\parallel}$ and $\mathbf{C}_{\diamondsuit}^n(t)_{\parallel}$ denote the (random) empirical distribution functions corresponding to the two sequences above.

A convenient choice for an inter-distributional dissimilarity function $d[U_{\diamondsuit}(t), C_{\diamondsuit}(t)]$, under either form of stochastic relationship, is the supremal distance

$$\sup_t |U_{\diamondsuit}(t) - C_{\diamondsuit}(t)|.$$

Its familiar estimator $\mathbf{D}_{\diamondsuit}$ is the Smirnov distance

$$\mathbf{D}_{\diamondsuit} = \sup_t |\mathbf{U}_{\diamondsuit}^n(t) - \mathbf{C}_{\diamondsuit}^n(t)|.$$

It is easy to show (see Appendix 2) that (10) and (11) respectively imply

$$\sup_t |\mathbf{U}_{\diamondsuit}^n(t)_{\perp} - \mathbf{C}_{\diamondsuit}^n(t)_{\perp}| \xrightarrow{\text{a.s.}} \sup_t |U_{\diamondsuit}(t)_{\perp} - C_{\diamondsuit}(t)_{\perp}|$$

$$\sup_t |\mathbf{U}_{\diamondsuit}^n(t)_{\parallel} - \mathbf{C}_{\diamondsuit}^n(t)_{\parallel}| \xrightarrow{\text{a.s.}} \sup_t |U_{\diamondsuit}(t)_{\parallel} - C_{\diamondsuit}(t)_{\parallel}|. \quad (12)$$

From this, (9) follows immediately, and it becomes clear that the asymptotic p-values associated with values $d_{\diamondsuit}$ of $\mathbf{D}_{\diamondsuit}$ should be computed as $\text{Prob}\{\mathbf{D}_{\blacklozenge} > d_{\diamondsuit}\}$; one may decide to reject the hypothesis that $\diamondsuit$ is $\blacklozenge$ (the true decomposition rule) if and only if the computed p-value falls below one's idea of a small probability (significance level). We come now to the problem of deriving the asymptotic distribution of $\mathbf{D}_{\blacklozenge}$.

In the case of s.-independence, $\mathbf{D}_{\blacklozenge}$ is the classical Smirnov statistic, and it is known that

$$\mathbf{D}_{\blacklozenge} = \sup_t |\mathbf{U}_{\blacklozenge}^n(t)_{\perp} - \mathbf{C}_{\blacklozenge}^n(t)_{\perp}| \xrightarrow{\text{d}} \sqrt{\frac{2}{n}} \sup_p |\mathbf{B}(p)|, \quad (13)$$

where $\xrightarrow{\text{d}}$ indicates convergence in distribution (as n increases), and $\mathbf{B}(p)$, $0 \leq p \leq 1$, is a Gaussian process known as a Brownian bridge (see Appendix 3). The supremum of the absolute value of a Brownian bridge has a known distribution function (e.g., Kendall & Stuart, 1967),

$$B(z) = \text{Prob}\{\sup_p |\mathbf{B}(p)| \leq z\} = 1 - 2 \sum_{r=1}^{\infty} (-1)^{r-1} e^{-2(rz)^2},$$

which allows one to compute the asymptotic p-values as

$$\lim_{n \to \infty} \text{Prob}\{\mathbf{D}_{\blacklozenge} > d_{\diamondsuit}\} = 1 - B\left(\sqrt{\frac{n}{2}} d_{\diamondsuit}\right). \quad (14)$$

In the case of p.p.s.-interdependence the sampling theory is more complicated. As shown in Appendix 4, we have in this case

$$\mathbf{D}_{\blacklozenge} = \sup_t |\mathbf{U}_{\blacklozenge}^n(t)_{\parallel} - \mathbf{C}_{\blacklozenge}^n(t)_{\parallel}| \xrightarrow{\text{d}} \frac{1}{\sqrt{n}} \sup_p |\lambda_p \mathbf{B}_{11}(p) + (1 - \lambda_p) \mathbf{B}_{22}(p) - \eta_p \mathbf{B}_{12}(p) - (1 - \eta_p) \mathbf{B}_{21}(p)|, \quad (15)$$

where $\mathbf{B}_{11}(p)$, $\mathbf{B}_{22}(p)$, $\mathbf{B}_{21}(p)$, $\mathbf{B}_{12}(p)$, $0 \leq p \leq 1$, are four stochastically independent Brownian bridges, and λ_p and η_p are deterministic functions of p whose values are confined to the interval $[0, 1]$. Although for terminological simplicity we continue to refer to this statistic as the Smirnov distance, its asymptotic distribution is different from (13). Asymptotic p-values associated with this statistic are not
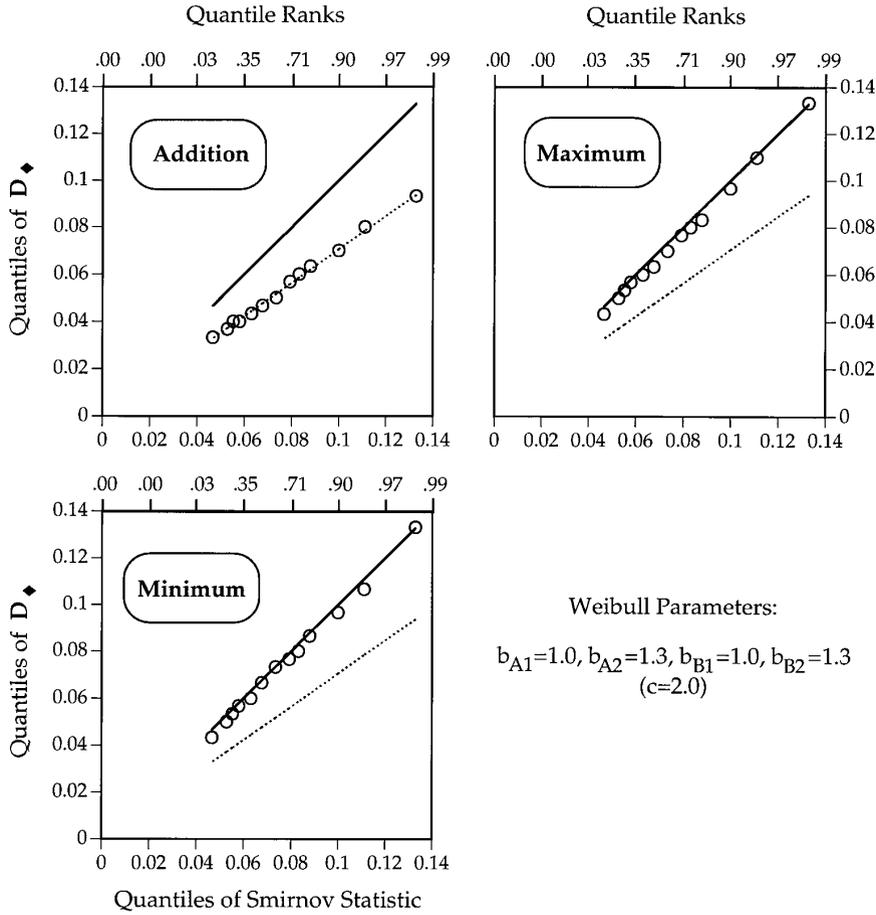
FIG. 1. Quantiles of $\mathbf{D}_\blacklozenge$ against the corresponding quantiles of the Smirnov statistic under p.p.s.-interdependence, for a quadruple of component time distributions having the same shape. The solid, unit-slope lines represent the values of $\sqrt{(2/n)}\,B^{-1}(1-\alpha)$, the dotted lines represent the values of $\sqrt{(1/n)}\,B^{-1}(1-\alpha)$.

available, but one can derive for them the following lower and upper bounds (Appendix 5):

$$
\begin{cases}
\lim_{n \to \infty} \text{Prob}\{\mathbf{D}_\blacklozenge > d_\diamond\} \\
\qquad \geq \text{Prob}\{\sup_p |\mathbf{B}(p)| > \sqrt{n}\,d_\diamond\} = 1 - B(\sqrt{n}\,d_\diamond) \\
\lim_{n \to \infty} \text{Prob}\{\mathbf{D}_\blacklozenge > d_\diamond\} \\
\qquad \leq \text{Prob}\{\sup_p |\mathbf{B}(p)| > \sqrt{\tfrac{n}{2}}\,d_\diamond\} = 1 - B\left(\sqrt{\tfrac{n}{2}}\,d_\diamond\right).
\end{cases}
\tag{16}
$$

where the notation is the same as in (13) and (14). This suggests that if one has to formally reject or retain the hypothesis that $\diamond$ is $\blacklozenge$ (against the generic alternative), then the decision rule should be tripartite:

$$
\alpha < 1 - B(\sqrt{n}\,d_\diamond) \Rightarrow \text{retain}
$$

$$
\alpha > 1 - B\left(\sqrt{\tfrac{n}{2}}\,d_\diamond\right) \Rightarrow \text{reject}
$$

$$
1 - B(\sqrt{n}\,d_\diamond) \leq \alpha \leq 1 - B\left(\sqrt{\tfrac{n}{2}}\,d_\diamond\right)
$$

$$
\Rightarrow \text{indefinite (more data needed)}.
$$

The evaluations given in (16) can sometimes be improved. The derivations in Appendix 4 show, in particular, that if the true decomposition rule $\blacklozenge$ is maximum, or minimum, then

$$
\lambda_p \equiv 1, \quad \eta_p \equiv 1,
$$

and the asymptotic p-values in (16) merge with their upper bounds, becoming thereby identical to the asymptotic p-values in the case of s.-independence, (14). If, however, $\blacklozenge$ is an addition-like operation, then in order to improve

the evaluations given in (16), one should know something about the population quantiles $T_{ij}(p)$ of RTs $\mathbf{T}_{ij}$ ($i = 1, 2$, $j = 1, 2$). For example, when the true decomposition rule is addition,

$$
\begin{cases}
\lambda_p = \dfrac{dT_{11}(p)/dp}{dT_{11}(p)/dp + dT_{22}(p)/dp} \\[3mm]
\eta_p = \dfrac{dT_{12}(p)/dp}{dT_{12}(p)/dp + dT_{21}(p)/dp}.
\end{cases}
$$

The situation simplifies if one has reason to believe that the RTs being combined, $\mathbf{T}_{ij}$ and $\mathbf{T}_{3-i,\,3-j}$ ($i = 1, 2, j = 1, 2$), are linearly related to each other:

$$
\begin{cases}
\mathbf{T}_{22} \stackrel{d}{=} b_u \mathbf{T}_{11} + a_u, & b_u > 0 \\[2mm]
\mathbf{T}_{21} \stackrel{d}{=} b_c \mathbf{T}_{12} + a_c, & b_c > 0.
\end{cases}
$$

This means that when a treatment changes in both factors, the RT distribution may only change its mean and variance,

but not its shape. This will follow, for example, from the assumption that the hypothetical component times, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$, all have the same shape. Then, as shown in Appendix 5,

$$
\lim_{n \to \infty} \operatorname{Prob}\{\mathbf{D}_\blacklozenge > d_\diamond\} = 1 - B\left(\sqrt{\frac{n}{\psi}}\, d_\diamond\right), \qquad (17)
$$

where the constant is

$$
\psi = \left(\frac{1}{1+b_u}\right)^2 + \left(\frac{b_u}{1+b_u}\right)^2 + \left(\frac{1}{1+b_c}\right)^2 + \left(\frac{b_c}{1+b_c}\right)^2.
$$

This constant can be estimated by comparing the sample variances for $\mathbf{T}_{ij}$ and $\mathbf{T}_{3-i,\,3-j}$ ($i = 1, 2, j = 1, 2$). It is always true that $1 \leqslant \psi \leqslant 2$, but if the differences in the variance values are not too large (say, less than by a factor of 10), then ψ is sufficiently close to 1. In such a case the asymptotic p-values in (16) are close to their lower bounds.

Analogous considerations can be applied to other addition-like operations, such as multiplication, as they can all
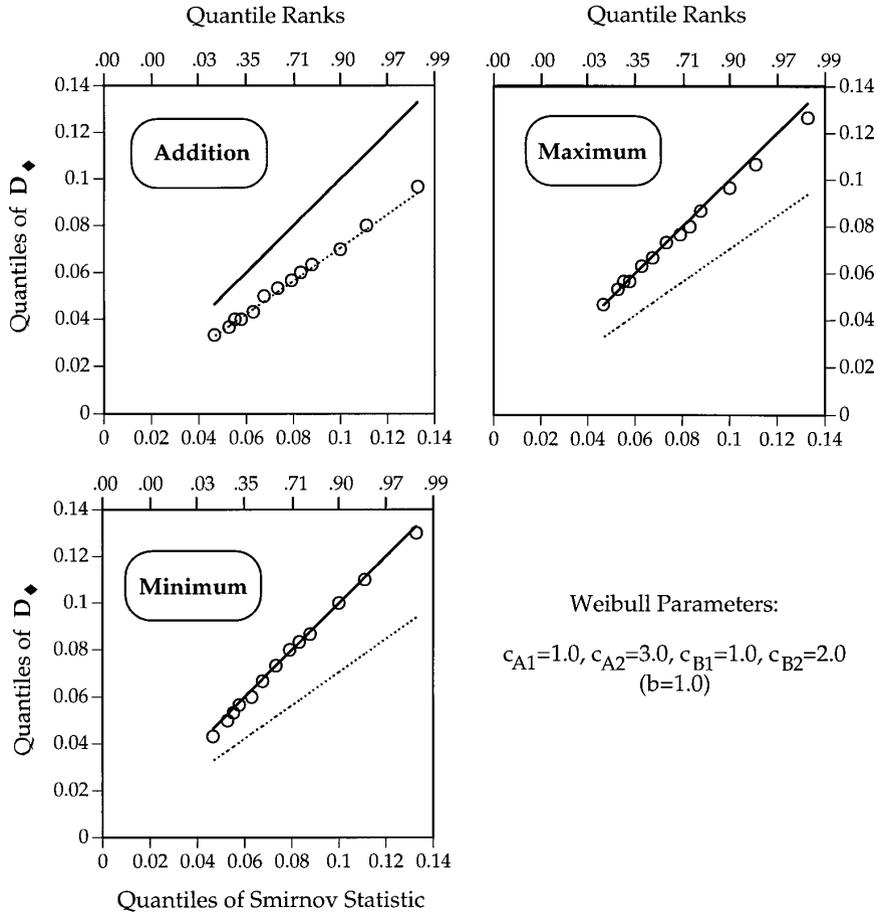
FIG. 2. Quantiles of $\mathbf{D}_\blacklozenge$ against the corresponding quantiles of the Smirnov statistic under p.p.s.-interdependence, for a quadruple of component time distributions having different shapes. The solid and dotted lines are as in Fig. 1.

be reduced to addition by g-transforming the values of $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ (see (5)). The assumption of shape invariance, however, then becomes

$$\begin{cases} g(\mathbf{T}_{22}) \stackrel{\mathrm{d}}{=} b_u\, g(\mathbf{T}_{11}) + a_u, \quad b_u > 0 \\ g(\mathbf{T}_{21}) \stackrel{\mathrm{d}}{=} b_c\, g(\mathbf{T}_{12}) + a_c, \quad b_c > 0, \end{cases}$$

which may or may not be considered justifiable.

Figures 1 and 2 illustrate some of the points just made about the distribution of $\mathbf{D}_{\blacklozenge}$ under the assumption of p.p.s.-interdependence between component times. These figures show the relationship between quantiles of $\mathbf{D}_{\blacklozenge}$, obtained by a Monte-Carlo simulation, and the corresponding quantiles of the classical Smirnov statistic. The component times were chosen to be Weibull-distributed,

$$\begin{cases} \mathbf{A}_1 \stackrel{\mathrm{d}}{=} b_{A1}[-\log(1-\mathbf{P})]^{c_{A1}^{-1}} \\ \mathbf{A}_2 \stackrel{\mathrm{d}}{=} b_{A2}[-\log(1-\mathbf{P})]^{c_{A2}^{-1}} \\ \mathbf{B}_1 \stackrel{\mathrm{d}}{=} b_{B1}[-\log(1-\mathbf{P})]^{c_{B1}^{-1}} \\ \mathbf{B}_2 \stackrel{\mathrm{d}}{=} b_{B2}[-\log(1-\mathbf{P})]^{c_{B2}^{-1}}, \end{cases} \tag{18}$$

where $\mathbf{P}$ is the unit-uniformly distributed quantile rank. The RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ are then computed as

$$\mathbf{T}_{ij} \stackrel{\mathrm{d}}{=} b_{Ai}[-\log(1-\mathbf{P})]^{c_{Ai}^{-1}} \blacklozenge\, b_{Bj}[-\log(1-\mathbf{P})]^{c_{Bj}^{-1}}$$
$$(i = 1, 2, j = 1, 2). \tag{19}$$

Samples of size $n = 300$ were selected from these populations, $\{t_{ij}^1, ..., t_{ij}^n\}$, ordered, $\{t_{ij}^{(1)}, ..., t_{ij}^{(n)}\}$, and combined as

$$\{t_{11}^{(1)} \blacklozenge t_{22}^{(1)}, ..., t_{11}^{(n)} \blacklozenge t_{22}^{(n)}\}$$

and

$$\{t_{12}^{(1)} \blacklozenge t_{21}^{(1)}, ..., t_{12}^{(n)} \blacklozenge t_{21}^{(n)}\}.$$

Then the supremal distance $d_{\blacklozenge}$ was computed between the empirical distribution functions corresponding to these two sequences. This procedure was repeated 2500 times, yielding the same number of $d_{\blacklozenge}$-values, from which we estimated the asymptotic quantiles $d_{\blacklozenge}^{1-\alpha}$, for quantiles ranks $(1-\alpha)$ ranging from 0.1 though 0.99:

$$\lim_{n \to \infty} \mathrm{Prob}\{\mathbf{D}_{\blacklozenge} > d_{\blacklozenge}^{1-\alpha}\} = \alpha.$$

We used three operations to serve, in a succession, as the true decomposition rule: plus, maximum, and minimum. In the latter two cases the quantiles $d_{\blacklozenge}^{1-\alpha}$, as predicted, simply coincide with the corresponding quantiles of the classical Smirnov statistic,

$$d_{\blacklozenge}^{1-\alpha} = \sqrt{\frac{2}{n}}\, B^{-1}(1-\alpha).$$

For additive decompositions, Fig. 1 represents a case when the distributions of $\mathbf{T}_{ij}$ and $\mathbf{T}_{3-i,\,3-j}$ ($i = 1, 2, j = 1, 2$) have identical shapes, so that (17) applies, with the true value of ψ only negligibly different from 1:

$$d_{\blacklozenge}^{1-\alpha} \approx \sqrt{\frac{1}{n}}\, B^{-1}(1-\alpha).$$

Figure 2 represents a case when the shapes of $\mathbf{T}_{ij}$ and $\mathbf{T}_{3-i,\,3-j}$ ($i = 1, 2, j = 1, 2$) are different, and when the operation is addition we cannot improve the evaluations given in (16) theoretically. However, the same approximation holds here too, which probably indicates that it is sufficiently robust with respect to violations of the shape invariance assumption, provided that the variances of the RTs being combined are not greatly different.

SAMPLE-LEVEL DECOMPOSITION TESTS: POWER ANALYSIS

For an incorrect operation $*$ (i.e., any operation different from the true decomposition rule $\blacklozenge$) the asymptotic statistical power at a significance level α is

$$\lim_{n \to \infty} \mathrm{Prob}\{\mathbf{D}_* > d_{\blacklozenge}^{(1-\alpha)}\}.$$

Unfortunately, under both forms of stochastic relationship, the asymptotic sampling distribution of $\mathbf{D}_*$ depends on the entire course of the distribution functions $U_*(t)$ and $C_*(t)$, and cannot be evaluated. Indeed, as shown in Appendix 3, in the case of s.-independence,

$$\mathbf{D}_* = \sup_t |\mathbf{U}_*^n(t)_\perp - \mathbf{C}_*^n(t)_\perp| \xrightarrow{\mathrm{d}} \sup_t \Big| U_*(t)_\perp - C_*(t)_\perp $$
$$+ \sqrt{\frac{1}{n}}\, \mathbf{B}[\,U_*(t)_\perp] - \sqrt{\frac{1}{n}}\, \mathbf{B}[\,C_*(t)_\perp]\Big|, \tag{20}$$

whereas in the case of p.p.s.-interdependence,

$$\mathbf{D}_* = \sup_t |\mathbf{U}_*^n(t)_{||} - \mathbf{C}_*^n(t)_{||}| \xrightarrow{\mathrm{d}} \sup_t \Big| U_*(t)_{||} - C_*(t)_{||}$$
$$+ \frac{1}{\sqrt{n}} \{\lambda_p \mathbf{B}_{11}[\,U_*(t)_{||}] + (1-\lambda_p)\, \mathbf{B}_{22}[\,U_*(t)_{||}]$$
$$- \eta_p \mathbf{B}_{12}[\,C_*(t)_{||}] - (1-\eta_p)\, \mathbf{B}_{21}[\,C_*(t)_{||}]\}\Big|, \tag{21}$$

with the same notation as in (13) and (15).

One can, however, derive lower bounds for the asymptotic power in terms of the distance

$$\Delta_* = \sup_t |U_*(t) - C_*(t)|.$$

If, however, the evaluation of $d_{\bullet}^{(1-\alpha)}$ is improved, as discussed in the previous section, then the lower bound (23) for the asymptotic power can be improved accordingly. In particular, if (17) holds, and the coefficient ψ $(1 \leqslant \psi \leqslant 2)$ can be estimated (e.g., as being close to 1), then the right-hand expression in (23) increases to become

$$1 - \Phi[\sqrt{2n}\,\Delta_* + \sqrt{2\psi}\,B^{-1}(1-\alpha)]$$

$$+ \Phi[\sqrt{2n}\,\Delta_* - \sqrt{2\psi}\,B^{-1}(1-\alpha)]. \qquad (24)$$

In the usual way, the lower bounds (22) and (23) can be used to compute the minimum sample sizes for the RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ that would guarantee a given power at a given significance level for a given value of Δ_* (or conversely, to compute the minimum value of Δ_* that is guaranteed to correspond to a given power at a given significance level). For instance, it is easy to estimate that one would need at most

$$n = 2\left(\frac{B^{-1}(1-\alpha)}{\Delta_*}\right)^2 \qquad (25)$$

observations per treatments to guarantee that the power exceeds 0.5 at a conventional significance level α (say, below 0.2). If Δ_* is on the order of 0.1, this means hundreds of observations per treatment, which is a relatively small experiment. If, however, Δ_* is on the order of 0.01, this figure rises to tens of thousand—clearly unrealistic for an
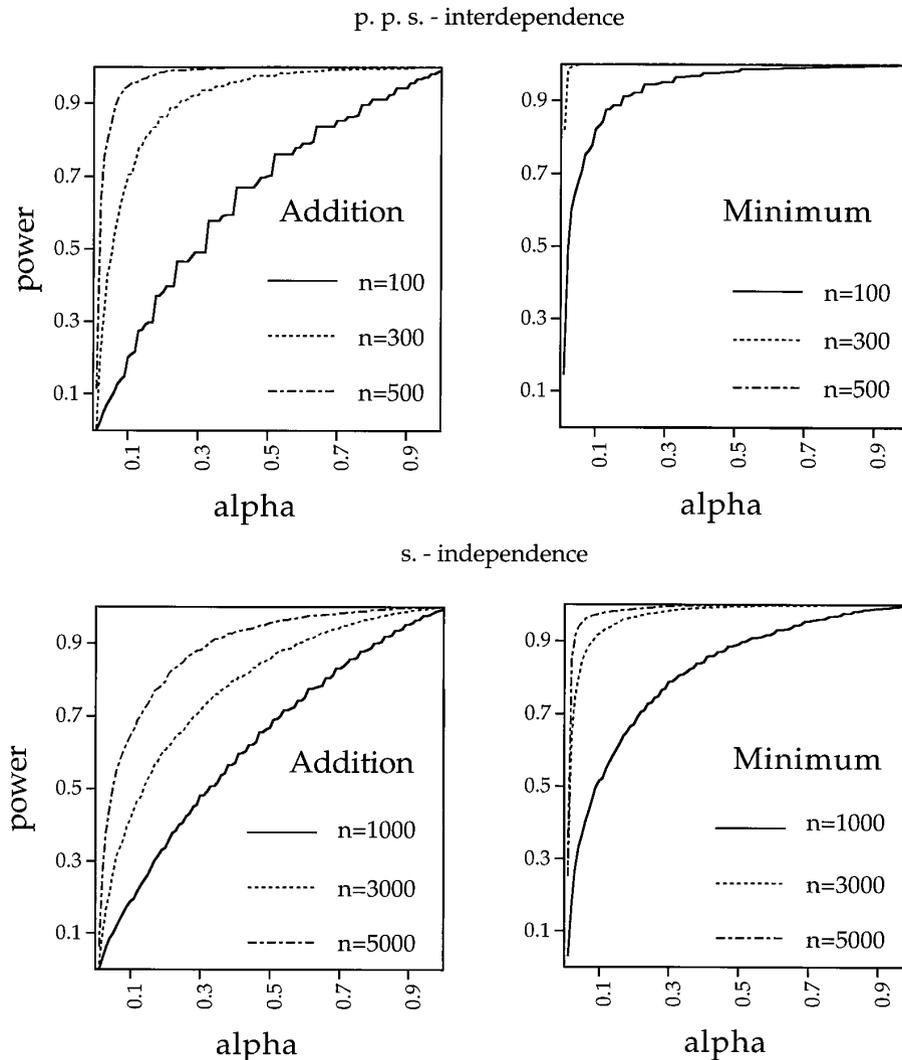
FIG. 4. Same as in Fig. 3, except that the true decomposition rule is maximum. The missing or incomplete curves in one of the panels are due to their closeness to the plot frame.

experiment. In the case of p.p.s.-interdependence, provided one can estimate the parameter ψ in (24), these figures can be lowered, but not by more than a factor of 2 (which is achieved at $\psi = 1$).

One should keep in mind that the estimates just given are only those of the upper bounds for the required sample sizes, based on the lower bounds for the asymptotic power. One cannot infer from these estimates any conclusions concerning the factual asymptotic power of a ($\blacklozenge$)-test against an alternative operation $*$, or the relative asymptotic powers of such a test under the two forms of stochastic relationship. For any two operations, $\blacklozenge$ (the true one) and $*$ (the incorrect one), the magnitude of Δ_* depends, in a complicated way, on the distributions of the RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ and on the form of stochastic relationship. Moreover, the magnitude of Δ_* is by far not the sole determinant of the power: the latter, as shown in (20) and (21),

is determined by the entire course of the functions $U_*(t)$ and $C_*(t)$.

Figures 3–5 show examples of conventional significance versus power curves obtained by means of a Monte-Carlo simulation. The simulation is essentially the same as described in relation to Figs. 1 and 2, except that now we deal with both forms of stochastic relationship, and the distributions for the Smirnov distances were computed not only for the true decomposition rule, but also for the competing operations. The RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ were computed from the Weibull-distributed component times (18): they were computed according to (19) in the case of p.p.s.-interdependence, and as

$$\mathbf{T}_{ij} \stackrel{\mathrm{d}}{=} b_{Ai}[-\log(1-\mathbf{P})]^{c_{Ai}^{-1}} \blacklozenge b_{Bj}[-\log(1-\mathbf{Q})]^{c_{Bj}^{-1}}, \quad \mathbf{P} \perp \mathbf{Q} \tag{26}$$
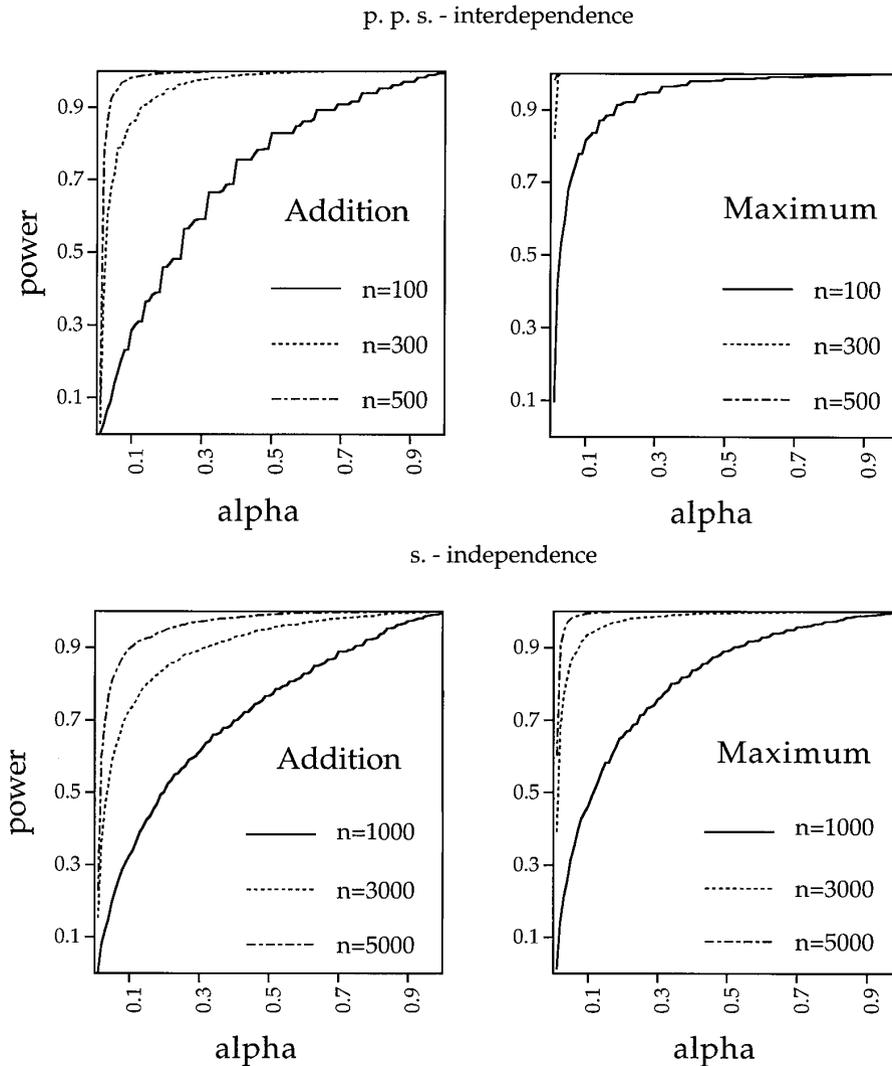
FIG. 5. Same as in Fig. 3, except that the true decomposition rule is minimum.

in the case of s.-independence ($i = 1, 2, j = 1, 2$; $\mathbf{P}$ and $\mathbf{Q}$ are unit-uniformly distributed). The true decomposition rule ◆ was, in a succession, plus, maximum, or minimum, the remaining two serving as the competing alternatives.

Samples of a given size n were selected from these populations and combined by means of all three operations and according to both forms of stochastic relationship,

$$\{t_{11}^{(1)} + t_{22}^{(1)}, ..., t_{11}^{(n)} + t_{22}^{(n)}\}$$
$$\text{and} \quad \{t_{12}^{(1)} + t_{21}^{(1)}, ..., t_{12}^{(n)} + t_{21}^{(n)}\}$$
$$\{\min[t_{11}^{(1)}, t_{22}^{(1)}], ..., \min[t_{11}^{(n)}, t_{22}^{(n)}]\}$$
$$\text{and} \quad \{\min[t_{12}^{(1)}, t_{21}^{(1)}], ..., \min[t_{12}^{(n)}, t_{21}^{(n)}]\}$$
$$\{\max[t_{11}^{(1)}, t_{22}^{(1)}], ..., \max[t_{11}^{(n)}, t_{22}^{(n)}]\}$$
$$\text{and} \quad \{\max[t_{12}^{(1)}, t_{21}^{(1)}], ..., \max[t_{12}^{(n)}, t_{21}^{(n)}]\}$$

in the case of p.p.s.-interdependence, and

$$\{t_{11}^{1} + t_{22}^{1}, ..., t_{11}^{n} + t_{22}^{n}\}$$
$$\text{and} \quad \{t_{12}^{1} + t_{21}^{1}, ..., t_{12}^{n} + t_{21}^{n}\}$$
$$\{\min[t_{11}^{1}, t_{22}^{1}], ..., \min[t_{11}^{n}, t_{22}^{n}]\}$$
$$\text{and} \quad \{\min[t_{12}^{1}, t_{21}^{1}], ..., \min[t_{12}^{n}, t_{21}^{n}]\}$$
$$\{\max[t_{11}^{1}, t_{22}^{1}], ..., \max[t_{11}^{n}, t_{22}^{n}]\}$$
$$\text{and} \quad \{\min[t_{12}^{1}, t_{21}^{1}], ..., \max[t_{12}^{n}, t_{21}^{n}]\}$$

in the case of s.-independence. Then the supremal distances $d_{\blacklozenge}$ (for the operation chosen to serve as the true decomposition rule), d_{*_1}, and d_{*_2} (for the two remaining operations) were computed between the empirical distribution functions corresponding to these "cross" and "uncross" sequences in each pair. This procedure was repeated 2500 times, allowing us to estimate the distributions of $\mathbf{D}_{\blacklozenge}$, $\mathbf{D}_{*_1}$, and $\mathbf{D}_{*_2}$ for every choice of the true decomposition rule, form of stochastic relationship, and sample size per treatment.

The results are only shown for the same parameters of the Weibull-distributed component times as in Fig. 1, but they are essentially the same for a broad spectrum of other parameter values (both scale and shape). In the case of s.-independence, in order to achieve comparable significance versus power curves, one needs by an order of magnitude larger sample sizes per treatment than in the case of p.p.s.-interdependence. At the same time, even in the case of s.-independence the sample sizes (say, 5000 per treatment) are within the reach of an experiment, though a large one.

Direct computation of the functions

$$|U_{*_1}(t) - C_{*_1}(t)| \quad \text{and} \quad |U_{*_2}(t) - C_{*_2}(t)|$$

for the simulated RTs in Figs. 3–5 shows that they are by an order of magnitude larger in the case of p.p.s.-interdependence (reaching values on the order of 0.1 at the supremum) than they are in the case of s.-independence (on the order of 0.01 at the supremum). This provides an obvious explanation for the observed power superiority of the p.p.s.-interdependence. One should be careful, however, not to overgeneralize this observation: with different distributions the power superiority effect might very well be reversed. To see this clearly, consider the following example.

Let $F_{A1}(t) < F_{B1}(t) < F_{A2}(t) < F_{B2}(t)$ be distribution functions for component times $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$, stochastically ordered as shown. If the true decomposition rule is maximum, while the competing incorrect operation is minimum, it is easy to derive that

$$|U_{\min}(t)_{||} - C_{\min}(t)_{||}| = |F_{A2}(t) - F_{B1}(t)|$$
$$|U_{\min}(t)_{\perp} - C_{\min}(t)_{\perp}| = |F_{A1}(t) - F_{A2}(t)| \, |F_{B1}(t) - F_{B2}(t)|.$$

Obviously, it is possible that $|F_{A2}(t) - F_{B1}(t)| < |F_{A1}(t) - F_{A2}(t)| \, |F_{B1}(t) - F_{B2}(t)|$, at all moments t, in which case the power will be higher under s.-independence. If, however, the successive intervals between the distribution functions $F_{A1}(t) < F_{B1}(t) < F_{A2}(t) < F_{B2}(t)$ are of the same order of magnitude (at any moment t), then it is easy to see that $|F_{A1}(t) - F_{A2}(t)| \, |F_{B1}(t) - F_{B2}(t)|$ will be by an order of magnitude smaller than $|F_{A2}(t) - F_{B1}(t)|$, and the power will be higher under p.p.s.-interdependence, as it was in our simulations.

The sampling distributions of the Smirnov distances $\mathbf{D}_{\diamond}$ (whether or not $\diamond$ is the true decomposition rule) are critically determined by the pairing schemes employed in forming the "uncross" and "cross" combinations of the RTs $\mathbf{T}_{ij}$ and $\mathbf{T}_{3-i, 3-j}$ ($i = 1, 2, j = 1, 2$). In the case of p.p.s.-interdependence there seem to be no reasonable alternatives to the pairing of the sample values having identical ordinal positions,

$$\{\mathbf{T}_{ij}^{(1)} \diamond \mathbf{T}_{3-i, 3-j}^{(1)}, ..., \mathbf{T}_{ij}^{(n)} \diamond \mathbf{T}_{3-i, 3-j}^{(n)}\}.$$

In the case of s.-independence, however, one might wonder whether the power of a sample-level (◆)-test could be increased by forming a Cartesian product of the two samples,

$$\{\mathbf{T}_{ij}^{(k)} \diamond \mathbf{T}_{3-i, 3-j}^{(l)}\}, \quad k = 1, ..., n; l = 1, ..., n, i = 1, 2, j = 1, 2$$

instead of their random "linear" pairing,

$$\{\mathbf{T}_{ij}^{1} \diamond \mathbf{T}_{3-i, 3-j}^{1}, ..., \mathbf{T}_{ij}^{n} \diamond \mathbf{T}_{3-i, 3-j}^{n}\}.$$

The empirical distribution functions, "uncross" and "cross," between which one is to compute the Smirnov distance, then could be constructed over the two sets of n^2 values.

The Cartesian product scheme may indeed seem natural. Roberts and Sternberg (1992) used this scheme in their empirical analysis of additive decompositions into s.-independent components. In its general form, for all possible operations, this scheme was also presented in Dzhafarov and Schweickert (1995) as the "operational meaning" of s.-independence. However, the n^2 values $\mathbf{T}_{ij}^{(k)} \diamond \mathbf{T}_{3-i,\,3-j}^{(l)}$ in a Cartesian product scheme are stochastically interdependent, which makes the empirical distribution of statistics computed over these values (including the Smirnov distance between the "uncross" and "cross" empirical distribution functions) both complicated and ill-behaved.

To understand the latter, refer to Fig. 6 that shows the sampling distributions of Smirnov distance for the operations plus, maximum, and minimum. These distribution were obtained by the same Monte-Carlo simulation as described in relation to Figs. 3–5, except that now we only deal with the case of s.-independence, and the pairing used in forming the "uncross" and "cross" empirical distribution functions followed the Cartesian product scheme. The upper panels illustrate the expected fact that the Smirnov distance

for any of the three operations is stochastically smaller when this operation serves as the true decomposition rule than when it serves as a competing operation. The difference is very small, however, and quite comparable with what one would observe under a random "linear" pairing for the same sample size (200 observations per treatment). This shows that the n^2 values in the Cartesian product scheme do not yield any power advantage.

The focal information, illustrating the "ill-behavedness" alluded to earlier, is presented in the lower panels of Fig. 6. We see here that the Smirnov distance for the operation plus is always stochastically smaller than that for the operations maximum or minimum, even when the true decomposition rule is de facto maximum or minimum, respectively. This means that, at any given significance level, it will be more likely to reject the true operation maximum or minimum than it will be to reject the incorrect operation plus. It is not difficult to account for this anomalous situation: when applied to a Cartesian product of two samples, the operation minimum and maximum create long strings of tied values (the number of different clusters of tied values being between n and $2n$, among the total of n^2 pairs, assuming no ties in the two samples themselves). Put informally, the presence of such clusters counters the intermixing of the two
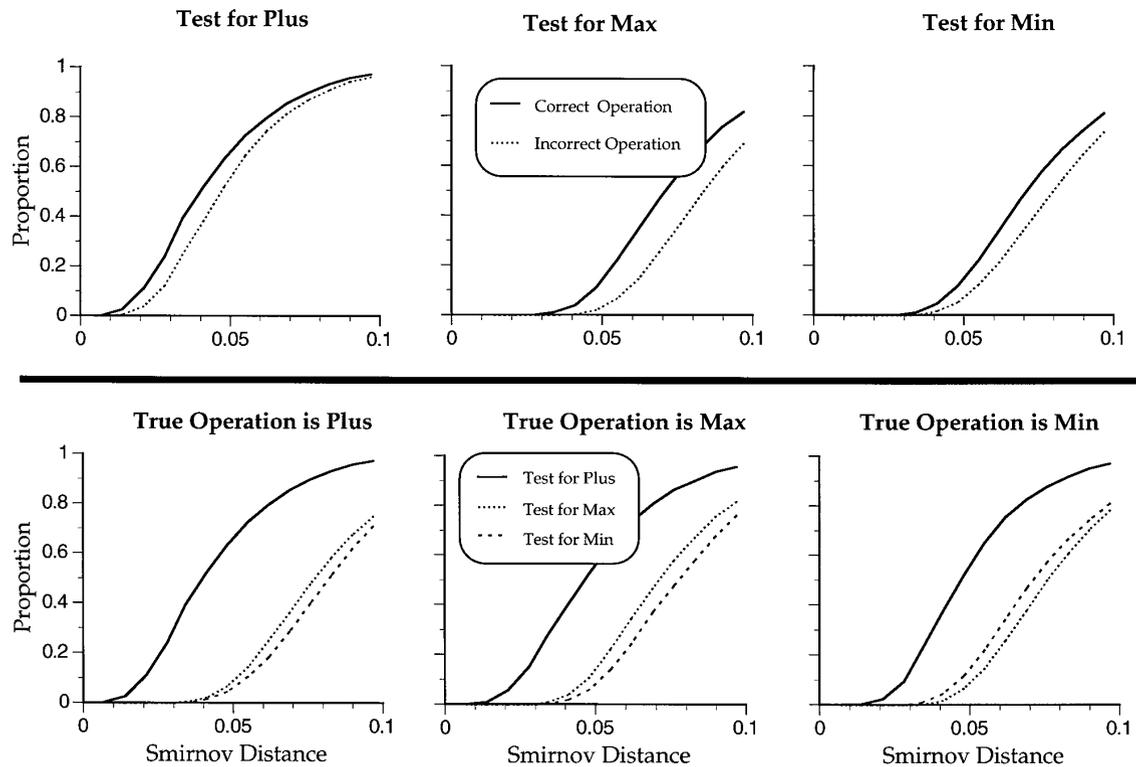
FIG. 6. Distribution functions for the Smirnov distance under s.-independence obtained by the Cartesian product scheme. In each of the upper panels, the solid curve represents the distribution function for an operation serving as the true decomposition rule; the dotted line represents the arithmetic mean of the two distribution functions for the same operation when it is incorrect. In the lower panels, the three curves (solid, dotted, and dashed) correspond to three fixed operations (as shown in inset) irrespective of their being true or incorrect.

samples in the ordered sequence of $2n^2$ values, increasing thereby the Smirnov distance magnitude (see, e.g., Hettmansperger, 1984). The situation would be the same with all operations that tend to create clusters of close values, such as "Minkowski-norm" operations, $(a^k + b^k)^{1/k}$ with large value of $|k|$.

CONCLUSION

The problem of deciding whether a given operation is a true decomposition rule has a relatively straightforward statistical solution. This solution is based on the asymptotic p-values associated with the Smirnov distance between empirical distribution functions, the functions being computed by combining in a certain way the RT samples corresponding to opposite treatments (i.e., the treatments with differing levels in both factors). Under s.-independence, the distribution of the Smirnov distance coincides with that of the classical Smirnov statistic; under p.p.s.-interdependence, this distance is stochastically "sandwiched" between the classical Smirnov statistic and the same statistic scaled by a factor of $1/\sqrt{2}$ (the bounds that may sometimes be improved).

The statistical power of a sample-level ($\blacklozenge$)-test depends on the alternative operation $*$, the form of stochastic relationship, and the distributions of the RTs corresponding to different treatments. Conservative bounds for the power can be derived in terms of the Smirnov distance $\varDelta_*$ between the empirical distribution functions resulting from combining the RT samples by means of the alternative operation $*$. One cannot, however, estimate $\varDelta_*$ based only on the operations $\blacklozenge$ and $*$ and the form of stochastic relationship; even if one could, $\varDelta_*$ is not the sole determinant of the factual power.

APPENDICES: PROOFS AND MATHEMATICAL COMMENTS

Appendix 1

To prove (11),

$$\mathbf{U}_\diamond^n(t)_{\|} \xrightarrow{\text{a.s.}} U_\diamond(t)_{\|}$$

(the proof for $\mathbf{C}_\diamond^n(t)_{\|}$ and $C_\diamond(t)_{\|}$ is analogous), choose a quantile rank p ($0 \leqslant p \leqslant 1$) and consider a sequence of integers k such that $k/n \to p$ as $n \to \infty$. We have then (omitting the subscript $\|$ for convenience)

$$\frac{k}{n} = \mathbf{U}_\diamond^n(\mathbf{T}_{11}^{(k)} \diamond \mathbf{T}_{22}^{(k)}) \to p.$$

Since $\mathbf{T}_{11}^{(k)} \xrightarrow{\text{a.s.}} T_{11}(p)$, $\mathbf{T}_{22}^{(k)} \xrightarrow{\text{a.s.}} T_{22}(p)$, and $\diamond$ is continuous in both arguments,

$$\mathbf{T}_{11}^{(k)} \diamond \mathbf{T}_{22}^{(k)} \xrightarrow{\text{a.s.}} T_{11}(p) \diamond T_{22}(p),$$

and we have

$$\mathbf{U}_\diamond^n(T_{11}(p) \diamond T_{22}(p)) \xrightarrow{\text{a.s.}} p.$$

This is equivalent to

$$\mathbf{U}_\diamond^n(U_\diamond^{-1}(p)) \xrightarrow{\text{a.s.}} p,$$

from which (11) follows immediately, by putting $U_\diamond^{-1}(p) = t$.

Appendix 2

To prove (12),

$$\sup_t |\mathbf{U}_\diamond^n(t) - \mathbf{C}_\diamond^n(t)| \xrightarrow{\text{a.s.}} \sup_t |U_\diamond(t) - C_\diamond(t)|$$

(the proof is the same for both $\perp$ and $\|$, so the subscripts are omitted), observe that by the classical Glivenko–Cantelli argument (pointwise a.s. convergence in the space of distribution functions implies uniform a.s. convergence),

$$\begin{cases} \mathbf{U}_\diamond^n(t) \xrightarrow{\text{a.s.}} U_\diamond(t) \\ \mathbf{C}_\diamond^n(t) \xrightarrow{\text{a.s.}} C_\diamond(t) \end{cases} \Rightarrow \begin{cases} \sup_t |\mathbf{U}_\diamond^n(t) - U_\diamond(t)| \xrightarrow{\text{a.s.}} 0 \\ \sup_t |\mathbf{C}_\diamond^n(t) - C_\diamond(t)| \xrightarrow{\text{a.s.}} 0 \end{cases}$$

Statement (12) then follows from the following triangle inequalities, with $d[\dots, \dots]$ denoting the supremal metric,

$$d[\mathbf{U}_\diamond^n(t), \mathbf{C}_\diamond^n(t)] \leqslant d[U_\diamond(t), C_\diamond(t)] + d[\mathbf{U}_\diamond^n(t), U_\diamond(t)]$$
$$+ d[\mathbf{C}_\diamond^n(t), C_\diamond(t)]$$
$$d[\mathbf{U}_\diamond^n(t), \mathbf{C}_\diamond^n(t)] \geqslant d[U_\diamond(t), C_\diamond(t)] - d[\mathbf{U}_\diamond^n(t), U_\diamond(t)]$$
$$- d[\mathbf{C}_\diamond^n(t), C_\diamond(t)].$$

Appendix 3

Here we comment on (13), describing the distribution of the classical Smirnov statistic, and we present a few mathematical facts we need subsequently. Asymptotically, the difference (multiplied with $\sqrt{n}$) between a continuous distribution function and a corresponding empirical distribution function is distributed as a Gaussian stochastic process called the Brownian Bridge, $\mathbf{B}(p)$, $0 \leqslant p \leqslant 1$: it has the mean of zero, autocovariance function $p(1-q)$, $p \leqslant q$, and its endpoints are fixed, $\mathbf{B}(0) = \mathbf{B}(1) = 0$ (which is the reason it is called a "bridge"). We need the following

elementary facts (given here without proof) about Gaussian stochastic processes (on a unit interval) in general and Brownian bridges in particular.

First, given several identically distributed independent Gaussian processes, $\mathbf{X}_i(p)$, and the same number of functions $f_i(p)$, the autocovariance function of $\sum f_i(p)\,\mathbf{X}_i(p)$ is $\sum f_i(p)\,f_i(q)\,C(p, q)$, where $C(p, q)$ is the autocovariance function of $\mathbf{X}_i(p)$, for all i.

Second, if $\mathbf{X}_1(p)$ and $\mathbf{X}_2(p)$ have the variance functions $\sigma_1^2(p) \leqslant \sigma_2^2(p)$ but the same autocorrelation function (the term used here to denote $C(p, q)/\sigma(p)\,\sigma(q)$, as opposed to the autocovariance function $C(p, q)$), then for any positive c,

$$\text{Prob}\{-c \leqslant \mathbf{X}_1(p) \leqslant c, \text{ for all } p\}$$
$$\geqslant \text{Prob}\{-c \leqslant \mathbf{X}_2(p) \leqslant c, \text{ for all } p\}.$$

The same inequality holds if $\mathbf{X}_1(p)$ and $\mathbf{X}_2(p)$ have the same variance function $\sigma^2(p)$ but

$$C_1(p, q)/\sigma_1(p)\,\sigma_1(q) \geqslant C_2(p, q)/\sigma_2(p)\,\sigma_2(q).$$

Third, a linear combination $\sum c_i \mathbf{B}_i(p)$ of several independent Brownian bridges is a scaled Brownian bridge

$$\sqrt{\sum c_i^2}\,\mathbf{B}(p).$$

In particular, under s.-independence, the difference between the empirical distribution functions $\mathbf{U}_\blacklozenge^n(t)_\perp$ and $\mathbf{C}_\blacklozenge^n(t)_\perp$, with $U_\blacklozenge(t)_\perp \equiv C_\blacklozenge(t)_\perp$, is asymptotically distributed as $\sqrt{(2/n)}\,\mathbf{B}(p)$, because

$$[\mathbf{U}_\blacklozenge^n(t)_\perp - \mathbf{C}_\blacklozenge^n(t)_\perp] = [\mathbf{U}_\blacklozenge^n(t)_\perp - U_\blacklozenge(t)_\perp]$$
$$- [\mathbf{C}_\blacklozenge^n(t)_\perp - C_\blacklozenge(t)_\perp]$$
$$\xrightarrow{\text{d}} \sqrt{\frac{1}{n}}\,\mathbf{B}_1(p) - \sqrt{\frac{1}{n}}\,\mathbf{B}_2(p)$$
$$\overset{\text{d}}{=} \sqrt{\frac{2}{n}}\,\mathbf{B}(p),$$

where p denotes $U_\blacklozenge(t)_\perp \equiv C_\blacklozenge(t)_\perp$. This explains (13). For an incorrect operation $*$, however, the analogous formula does not simplify in the same way: using the same logic as above we only arrive at (20),

$$\mathbf{U}_*^n(t)_\perp - \mathbf{C}_*^n(t)_\perp \xrightarrow{\text{d}} U_*(t)_\perp - C_*(t)_\perp$$
$$+ \sqrt{\frac{1}{n}}\,\mathbf{B}[U_*(t)_\perp] - \sqrt{\frac{1}{n}}\,\mathbf{B}[C_*(t)_\perp],$$

Appendix 4

Here, we prove statement (15) for $\sup_t |\mathbf{U}_\blacklozenge^n(t)_{||} - \mathbf{C}_\blacklozenge^n(t)_{||}|$, statement (21) for $\sup_t |\mathbf{U}_*^n(t)_{||} - \mathbf{C}_*^n(t)_{||}|$, and we derive explicit expressions for the functions λ_p and η_p. Recall that all distribution and quantiles functions are assumed to be piecewise differentiable.

We prove first that for any operation $\diamond$,

$$\mathbf{U}_\diamond^n(t)_{||} - U_\diamond(t)_{||} \xrightarrow{\text{d}} \frac{1}{\sqrt{n}}\{\lambda_p \mathbf{B}_{11}[U_\diamond(t)_{||}]$$
$$+ (1 - \lambda_p)\,\mathbf{B}_{22}[U_\diamond(t)_{||}]\}$$
$$\mathbf{C}_\diamond^n(t)_{||} - C_\diamond(t)_{||} \xrightarrow{\text{d}} \frac{1}{\sqrt{n}}\{\eta_p \mathbf{B}_{12}[C_\diamond(t)_{||}]$$
$$+ (1 - \eta_p)\,\mathbf{B}_{21}[C_\diamond(t)_{||}]\}, \quad\text{(A1)}$$

where λ_p and η_p are some functions to be specified. Statement (21),

$$\sup_t |\mathbf{U}_*^n(t)_{||} - \mathbf{C}_*^n(t)_{||}|$$
$$\xrightarrow{\text{d}} \sup_t \left| U_*(t)_{||} - C_*(t)_{||} + \frac{1}{\sqrt{n}}\{\lambda_p \mathbf{B}_{11}[U_*(t)_{||}] \right.$$
$$+ (1 - \lambda_p)\,\mathbf{B}_{22}[U_*(t)_{||}] - \eta_p \mathbf{B}_{12}[C_*(t)_{||}]$$
$$\left. - (1 - \eta_p)\,\mathbf{B}_{21}[C_*(t)_{||}]\} \right|,$$

then follows immediately, by renaming $\diamond$ to $*$, whereas statement (15),

$$\sup_t |\mathbf{U}_\blacklozenge^n(t)_{||} - \mathbf{C}_\blacklozenge^n(t)_{||}| \xrightarrow{\text{d}} \frac{1}{\sqrt{n}} \sup_p |\lambda_p \mathbf{B}_{11}(p)$$
$$+ (1 - \lambda_p)\,\mathbf{B}_{22}(p) - \eta_p \mathbf{B}_{12}(p)$$
$$- (1 - \eta_p)\,\mathbf{B}_{21}(p)|,$$

follows by renaming $\diamond$ to $\blacklozenge$ and denoting $U_\blacklozenge(t)_{||} \equiv C_\blacklozenge(t)_{||}$ by p.

To prove (A1) for $\mathbf{U}_\diamond^n(t)_{||} - U_\diamond(t)_{||}$ (the proof for $\mathbf{C}_\diamond^n(t)_{||} - C_\diamond(t)_{||}$ is analogous), choose a sequence of integers k such that $k/n \to p$ $(0 \leqslant p \leqslant 1)$ as $n \to \infty$, and observe that (omitting the subscript $||$ for convenience)

$$\mathbf{U}_\diamond^n[T_{11}(p) \diamond T_{22}(p)] \xrightarrow{\text{a.s.}} U_\diamond[T_{11}(\mathbf{P}_{11}^{(k)}) \diamond T_{22}(\mathbf{P}_{22}^{(k)})],$$

where $\mathbf{P}_{11}^{(k)}$ and $\mathbf{P}_{22}^{(k)}$ are kth order statistics from two independent samples of size n from a unit-uniform distribution. Indeed, both expressions a.s. converge to p, the left-hand one by Appendix 1, and the right-hand one because all functions involved are continuous, whereas $\mathbf{P}_{ii}^{(k)}$ $(i = 1, 2)$ a.s.

converge to p. As a result, putting $U_\diamond(t) = p$ and using the fact that $U_\diamond^{-1}(p) = T_{11}(p) \diamond T_{22}(p)$, we have

$$\mathbf{U}_\diamond^n(t) - U_\diamond(t) = \mathbf{U}_\diamond^n[T_{11}(p) \diamond T_{22}(p)] - p$$
$$\xrightarrow{\text{d}} U_\diamond[T_{11}(\mathbf{P}_{11}^{(k)}) \diamond T_{22}(\mathbf{P}_{22}^{(k)})] - p.$$

Taylor-expanding the latter expression in terms of $\mathbf{P}_{11}^{(k)} - p$ and $\mathbf{P}_{22}^{(k)} - p$, we conclude that for all t, except perhaps for a countable number of isolated points,

$$\mathbf{U}_\diamond^n(t) - U_\diamond(t)$$
$$\xrightarrow{\text{d}} \frac{\partial[T_{11}(x) \diamond T_{22}(p)]/\partial x|_{x=p}}{d[T_{11}(p) \diamond T_{22}(p)]/dp} (\mathbf{P}_{11}^{(k)} - p)$$
$$+ \frac{\partial[T_{11}(p) \diamond T_{22}(x)]/\partial x|_{x=p}}{d[T_{11}(p) \diamond T_{22}(p)]/dp} (\mathbf{P}_{22}^{(k)} - p).$$

From Appendix 3, we know that $\sqrt{n}\,(\mathbf{P}_{ii}^{(k)} - p)$ converges in distribution to a Brownian bridge, and denoting the latter by $\mathbf{B}_{ii}(p)$ ($i = 1, 2$), we come to

$$\mathbf{U}_\diamond^n(t) - U_\diamond(t)$$
$$\xrightarrow{\text{d}} \frac{1}{\sqrt{n}} \left\{ \frac{\partial[T_{11}(x) \diamond T_{22}(p)]/\partial x|_{x=p}}{d[T_{11}(p) \diamond T_{22}(p)]/dp} \mathbf{B}_{11}(p) \right.$$
$$\left. + \frac{\partial[T_{11}(p) \diamond T_{22}(x)]/\partial x|_{x=p}}{d[T_{11}(p) \diamond T_{22}(p)]/dp} \mathbf{B}_{22}(p) \right\}, \quad \text{(A2)}$$

where p at the right denotes $U_\diamond(t)$, and the two Brownian bridges are mutually independent.

To compute the coefficients at the bridges, assume first that $\diamond$ is $+$. Then

$$\mathbf{U}_\diamond^n(t) - U_\diamond(t)$$
$$\xrightarrow{\text{d}} \frac{1}{\sqrt{n}} \left\{ \frac{dT_{11}(p)/dp}{dT_{11}(p)/dp + dT_{22}(p)/dp} \mathbf{B}_{11}(p) \right.$$
$$\left. + \frac{dT_{22}(p)/dp}{dT_{11}(p)/dp + dT_{22}(p)/dp} \mathbf{B}_{22}(p) \right\},$$

and we obtain (A1) with

$$\lambda_p = \frac{dT_{11}(p)/dp}{dT_{11}(p)/dp + dT_{22}(p)/dp}. \quad \text{(A3)}$$

If $\diamond$ is some addition-like operation, then there is a monotonic transformation g (that here is assumed to be piecewise differentiable) such that $a \diamond b \equiv g^{-1}[g(a) + g(b)]$.

Using this expression in (A2), we again obtain (A1), this time with

$$\lambda_p = \frac{g'[T_{11}(p)]\, dT_{11}(p)/dp}{g'[T_{11}(p)]\, dT_{11}(p)/dp + g'[T_{22}(p)]\, dT_{22}(p)/dp}. \quad \text{(A4)}$$

To compute the coefficients in (A2) when $\diamond$ is maximum, observe that we can always put $T_{11}(p) \geqslant T_{22}(p)$, at all p. Indeed, if this is not the case, then we can rename $\max\{T_{11}(p), T_{22}(p)\}$ to $T_{11}(p)$ and $\min\{T_{11}(p), T_{22}(p)\}$ to $T_{22}(p)$. This would not change the piecewise differentiability of the functions, and

$$\max\{\max\{T_{11}(p), T_{22}(p)\}, \min\{T_{11}(p), T_{22}(p)\}\}$$
$$\equiv \max\{T_{11}(p), T_{22}(p)\}.$$

Assume first that $T_{11}(p)$ never coincides with $T_{22}(p)$ on an interval. Then $T_{11}(p) > T_{22}(p)$ almost everywhere, which implies that almost everywhere $T_{11}(p) > T_{22}(x)$ and $T_{11}(x) > T_{22}(p)$ in some neighborhood of p. Using this in (A2), we obtain

$$\mathbf{U}_\diamond^n(t) - U_\diamond(t) \xrightarrow{\text{d}} \frac{1}{\sqrt{n}} \mathbf{B}_{11}(p),$$

which is (A1) with

$$\lambda_p \equiv 1. \quad \text{(A5)}$$

If $T_{11}(p)$ does coincide with $T_{22}(p)$ on some interval, then consider a sequence of functions $T_{11}^s(p) \to T_{11}(p)$ as $s \to \infty$, such that $T_{11}^s(p) \geqslant T_{22}(p)$ without coinciding with $T_{22}(p)$ on an interval, for all s. Constructing the distribution functions $\mathbf{U}_\diamond^{n,s}(t)$ and $U_\diamond^s(t)$ for every s, we have

$$\mathbf{U}_\diamond^{n,s}(t) - U_\diamond^s(t) \xrightarrow{\text{d}} \frac{1}{\sqrt{n}} \mathbf{B}_{11}(p).$$

Since an asymptotic sampling distribution must be stable with respect to infinitesimal variations in the population distributions, it must also be true that

$$\mathbf{U}_\diamond^{n,s}(t) - U_\diamond^s(t) \xrightarrow{\text{d}} \mathbf{U}_\diamond^n(t) - U_\diamond(t),$$

and we conclude that when $\diamond$ is maximum, (A5) holds universally. The derivation for the case when $\diamond$ is minimum is analogous and leads to the same result, $\lambda_p \equiv 1$.

This concludes the proof of (A1) and thereby of (15) and (21).

Appendix 5

Here, we derive the upper and lower bounds (16) and evaluation (17) for

$$\lim_{n \to \infty} \text{Prob}\{\mathbf{D}_{\blacklozenge} > d_{\diamond}\}$$

$$= \text{Prob}\left\{\frac{1}{\sqrt{n}}\sup_p |\lambda_p \mathbf{B}_{11}(p) \right.$$
$$+ (1-\lambda_p)\,\mathbf{B}_{22}(p) - \eta_p \mathbf{B}_{12}(p)$$
$$\left. - (1-\eta_p)\,\mathbf{B}_{21}(p)| > d_{\diamond}\right\}.$$

To obtain the lower bound, observe that

$$\mathbf{X}(p) = \lambda_p \mathbf{B}_{11}(p) + (1-\lambda_p)\,\mathbf{B}_{22}(p)$$
$$- \eta_p \mathbf{B}_{12}(p) - (1-\eta_p)\,\mathbf{B}_{21}(p)$$

is a Gaussian stochastic process with zero mean and autocovariance function

$$\{\lambda_p \lambda_q + (1-\lambda_p)(1-\lambda_q) + \eta_p \eta_q$$
$$+ (1-\eta_p)(1-\eta_q)\}\, p(1-q), \quad p \leqslant q.$$

The variance function of $\mathbf{X}(p)$,

$$\{\lambda_p^2 + (1-\lambda_p)^2 + \eta_p^2 + (1-\eta_p)^2\}\, p(1-p),$$

attains its minimum (at all points), and simultaneously, its autocorrelation function

$$\frac{\lambda_p \lambda_q + (1-\lambda_p)(1-\lambda_q) + \eta_p \eta_q + (1-\eta_p)(1-\eta_q)}{\sqrt{\lambda_p^2 + (1-\lambda_p)^2 + \eta_p^2 + (1-\eta_p)^2}\,\sqrt{\lambda_q^2 + (1-\lambda_q)^2 + \eta_q^2 + (1-\eta_q)^2}}$$
$$\times \sqrt{\frac{p}{1-p}}\,\sqrt{\frac{1-q}{q}}, \quad p \leqslant q,$$

attains its maximum (at all pairs of points) when

$$\lambda_p^2 = \eta_p^2 = \tfrac{1}{4},$$

at all points. As we know from Appendix 3, then

$$\text{Prob}\{\sup_p |\mathbf{X}(p)| > c\} = 1 - \text{Prob}\{-c \leqslant \mathbf{X}(p) \leqslant c, \text{ for all } p\}$$

attains its minimum, for any positive c. We also know from Appendix 3 that

$$\tilde{\mathbf{X}}(p) = \pm\tfrac{1}{2}\mathbf{B}_{11}(p) \pm \tfrac{1}{2}\mathbf{B}_{22}(p) \pm \tfrac{1}{2}\mathbf{B}_{12}(p) \pm \tfrac{1}{2}\mathbf{B}_{21}(p) \overset{\text{d}}{=} \mathbf{B}(p),$$

where $\mathbf{B}(p)$ is some Brownian bridge. As a result, the lower bound is obtained as

$$\text{Prob}\{\sup_p |\tilde{\mathbf{X}}(p)| > \sqrt{n}\,d_{\diamond}\}$$
$$= \text{Prob}\{\sup_p |\mathbf{B}(p)| > \sqrt{n}\,d_{\diamond}\} = 1 - B(\sqrt{n}\,d_{\diamond}).$$

To derive the upper bound, consider a fixed trajectory (realization of the stochastic process)

$$X(p) = \lambda_p B_{11}(p) + (1-\lambda_p)\,B_{22}(p)$$
$$- \eta_p B_{12}(p) - (1-\eta_p)\,B_{21}(p).$$

and let P be the point at which

$$|X(p)| = \sup_p |X(p)|.$$

All trajectories can be classified into eight disjunctive classes $\mathfrak{I}_1, ..., \mathfrak{I}_8$ depending on the combination of the inequalities

$$X(P)\begin{cases}>0\\\leqslant 0\end{cases}, \quad B_{11}(P) - B_{22}(P)\begin{cases}>0\\\leqslant 0\end{cases},$$
$$B_{12}(P) - B_{12}(P)\begin{cases}>0\\\leqslant 0\end{cases}.$$

For each of these classes one can choose a pair

$$\alpha_i = \begin{cases}0\\1\end{cases}, \quad \beta_i = \begin{cases}0\\1\end{cases}, \quad i = 1, ..., 8,$$

such that for all trajectories $X(p) \in \mathfrak{I}_i$,

$$\sup_p |X(p)| \leqslant \sup_p |X_i(p)|,$$

where

$$X_i(p) = \alpha_i B_{11}(p) + (1-\alpha_i)\,B_{22}(p) - \beta_i B_{12}(p)$$
$$- (1-\beta_i)\,B_{21}(p).$$

Indeed, the values of α_i and β_i can always be chosen so that the difference

$$X_i(P) - X(p) = [\alpha_i - \lambda_p][B_{11}(P) - B_{22}(P)]$$
$$+ [\beta_i - \eta_p][B_{21}(P) - B_{12}(P)],$$

has the same sign as $X(p)$.

We can write now

$$\text{Prob}\{\sup_p |\mathbf{X}(p)| > \sqrt{n}\,d_\diamond\}$$

$$= \sum_{i=1}^{8} \text{Prob}\{\sup_p |\mathbf{X}(p)| > \sqrt{n}\,d_\diamond \mid \mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$\times \text{Prob}\{\mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$\leqslant \sum_{i=1}^{8} \text{Prob}\{\sup_p |\mathbf{X}_i(p)| > \sqrt{n}\,d_\diamond \mid \mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$\times \text{Prob}\{\mathbf{X}(p) \in \mathfrak{I}_i\}. \tag{A6}$$

For every $i = 1, ..., 8$, the stochastic process $\mathbf{X}_i(p)$ is a difference between two independent Brownian bridges, which is distributed as a scaled Brownian bridge (see Appendix 3),

$$\mathbf{X}_i(p) \overset{\text{d}}{=} \sqrt{2}\,\mathbf{B}(p), \quad i = 1, ..., 8.$$

Since events conditioned on mutually exclusive possibilities are mutually independent, we can continue the chain in (A6) as

$$\sum_{i=1}^{8} \text{Prob}\{\sup_p |\mathbf{X}_i(p)| > \sqrt{n}\,d_\diamond \mid \mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$\times \text{Prob}\{\mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$= \sum_{i=1}^{8} \text{Prob}\{\sup_p \sqrt{2}\,|\mathbf{B}(p)| > \sqrt{n}\,d_\diamond \mid \mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$\times \text{Prob}\{\mathbf{X}(p) \in \mathfrak{I}_i\}$$

$$= \text{Prob}\left\{\sup_p |\mathbf{B}(p)| > \sqrt{\frac{n}{2}}\,d_\diamond\right\} = 1 - B\left(\sqrt{\frac{n}{2}}\,d_\diamond\right).$$

This completes the derivation of the upper bound in (16).

To derive (17), replace $T_{22}(p)$ and $T_{21}(p)$ in (A3) of Appendix 4 (and in the analogous formula for η_p) respectively by

$$\begin{cases} b_u T_{11}(p) + a_u, & b_u > 0 \\ b_c T_{12}(p) + a_c, & b_c > 0 \end{cases}$$

to obtain

$$\lambda_p = \frac{1}{1 + b_u}, \quad \eta_p = \frac{1}{1 + b_c}.$$

We know from Appendix 3 that

$$\frac{1}{1 + b_u}\mathbf{B}_{11}(p) + \frac{b_u}{1 + b_u}\mathbf{B}_{22}(p)$$

$$- \frac{1}{1 + b_c}\mathbf{B}_{12}(p) - \frac{b_c}{1 + b_c}\mathbf{B}_{21}(p) \overset{\text{d}}{=} \sqrt{\psi}\,\mathbf{B}(p),$$

where

$$\psi = \left(\frac{1}{1 + b_u}\right)^2 + \left(\frac{b_u}{1 + b_u}\right)^2 + \left(\frac{1}{1 + b_c}\right)^2 + \left(\frac{b_c}{1 + b_c}\right)^2.$$

This yields (17) immediately, because

$$\text{Prob}\{\sup_p |\sqrt{\psi}\,\mathbf{B}(p)| > \sqrt{n}\,d_\diamond\} = 1 - B\left(\sqrt{\frac{n}{\psi}}\,d_\diamond\right).$$

Appendix 6

Here we explain the logic behind the lower bound (22) for the statistical power. As we know from Appendix 3,

$$\mathbf{U}_*^n(t)_\perp - \mathbf{C}_*^n(t)_\perp \overset{\text{d}}{\longrightarrow} U_*(t)_\perp - C_*(t)_\perp$$

$$+ \sqrt{\frac{1}{n}}\mathbf{B}[U_*(t)_\perp] - \sqrt{\frac{1}{n}}\mathbf{B}[C_*(t)_\perp].$$

It is obvious that

$$\text{Prob}\left\{c \leqslant U_*(t)_\perp - C_*(t)_\perp + \sqrt{\frac{1}{n}}\mathbf{B}[U_*(t)_\perp]\right.$$

$$\left. - \sqrt{\frac{1}{n}}\mathbf{B}[C_*(t)_\perp] \leqslant c, \text{ for all } t\right\}$$

does not exceed the same probability computed for just one $t = \tilde{t}$, the one at which

$$|U_*(\tilde{t}) - C_*(\tilde{t})| = \sup_t |U_*(t) - C_*(t)|.$$

The difference $\mathbf{U}_*^n(\tilde{t})_\perp - \mathbf{C}_*^n(\tilde{t})_\perp$ is distributed asymptotically normally, with the mean $U_*(t) - C_*(t)$ and variance that cannot exceed $1/(4n) + 1/(4n)$. From this, expression (22) is obtained by straightforward transformations.

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