Replacing Nothing with Something Special: Contextuality-by-Default and Dummy Measurements

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The object of contextuality analysis is a set of random variables each of which is uniquely labeled by a content and a context. In the measurement terminology, the content is that which the random variable measures, whereas the context describes the conditions under which this content is measured (in particular, the set of other contents being measured "together" with this one). Such a set of random variables is deemed noncontextual or contextual depending on whether the distributions of the context-sharing random variables are or are not compatible with certain distributions imposed on the content-sharing random variables. In the traditional approaches, contextuality is either restricted to only consistently-connected systems (those in which any two content-sharing random variables have the same distribution) or else all inconsistently-connected systems (those not having this property) are considered contextual. In the Contextuality-by-Default theory, an inconsistently connected system may or may not be contextual. There are several arguments for this understanding of contextuality, and this note adds one more. It is related to the fact that generally not each content is measured in each context, so there are "empty" content-context pairs. It is convenient to treat each of these empty pairs as containing a dummy random variable, one that does not change the degree of contextuality in a system. These dummy random variables are deterministic ones, attaining a single value with probability 1. The replacement of absent random variables with deterministic ones, however, can only be made if one allows for inconsistently-connected systems.

KEYWORDS: contextuality, dummy measurements, inconsistent connectedness, random variables.

Replacing "nothing" with "something" chosen for its special properties is one of the main ways a mathematical theory develops. One speaks of "nothing" when one chooses no elements from a set, adds no number to a total, or leaves a function unchanged; but a more sophisticated way of speaking of these "nothings" would be to take an empty subset of the set, to add a zero to the total, and to apply an identity operator to the function. As a rule, these "somethings" provide not only greater convenience, but also a greater insight. Mature set theory cannot be constructed without empty sets, nor can algebra be developed without neutral elements of operations. One faces an analogous situation in the theory of contextuality: "nothing" here means that certain things are not measured in certain contexts, and the "special somethings" to replace these "nothings" are deterministic random variables.

Contextuality analysis applies to systems of random variables R_q^c representing the outcomes of measuring a *content* q (property, object, thing, question, sensory stimulus) in a *context* c (circumstances, conditions, setup). An example is the matrix below, with three contents and four contexts:

$$
\begin{array}{|c|c|c|c}
\hline
R_1^1 & R_2^1 & \cdot & c=1 \\
\hline
R_1^2 & R_2^2 & \cdot & c=2 \\
\hline
R_1^3 & \cdot & R_3^3 & c=3 \\
\hline
\cdot & R_2^4 & R_3^4 & c=4 \\
\hline
\multicolumn{1}{c}{q=1} & \multicolumn{1}{c}{q=2} & \multicolumn{1}{c|}{q=3} & \boxed{\mathcal{R}}
\end{array}
.
$$

The rules such a matrix obeys are: (i) all random variables in the same column have the same set of values (and sigma-algebras); (ii) all random variables within a row are *jointly distributed*; (iii) random variables in different rows are not jointly distributed (are *stochastically unrelated* to each other) [5, 8, 11]. The system is considered *noncontextual* if the joint distributions of the random variables within the rows are compatible with the joint distributions imposed on the random variable within each column (the compatibility meaning that both the observed row-wise distributions and the imposed column-wise ones can be viewed as marginals of a single probability distribution imposed on the entire system). Otherwise the system is *contextual.*

We will use the system $\mathcal{R}$ throughout to illustrate our points, but the three points we make below hold for all systems of random variables indexed by contents and contexts.

As we see in the matrix, not every content is measured in every context, there are cells with "nothing" in them. It is natural to posit, however, that for a random variable being undefined is logically equivalent to being defined as always attaining a value labeled "undefined." If so, we can fill in the empty cells with deterministic random variables,

$$
\begin{array}{|c|c|c|c}
\hline
R_1^1 & R_2^1 & U_3^1 \equiv u & c=1 \\
\hline
R_1^2 & R_2^2 & U_3^2 \equiv u & c=2 \\
\hline
R_1^3 & U_2^3 \equiv u & R_3^3 & c=3 \\
\hline
U_1^4 \equiv u & R_2^4 & R_3^4 & c=4 \\
\hline
\multicolumn{1}{c}{q=1} & \multicolumn{1}{c}{q=2} & \multicolumn{1}{c|}{q=3} & \boxed{\mathcal{R}'}
\end{array}
,
$$

where u is interpreted as "undefined," and $U \equiv u$ means that random variable U equals u with probability 1. In order to comply with the rule (i) above, this value u then should be added to the set of possible values of all other random variables, as attained by each of them with probability zero.

The first point of this note is that a well-designed contextuality theory should allow the addition of these deterministic Us to any system without changing whether the

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system is contextual or noncontextual. One can even implement the addition of the deterministic Us empirically, e.g., by setting the procedure/device measuring $q = 3$ in contexts $c = 3$ and $c = 4$ to produce a fixed outcome interpreted as "undefined" in contexts $c = 1$ and $c = 2$.

The second point of this note is that this desideratum cannot be satisfied if one confines contextuality analysis to consistently-connected systems only, the systems in which all measurements of the same content (e.g., R_1^1, R_1^2, and R_1^3 in $\mathcal{R}$) have the same distribution [11]. With the exception of the Contextuality-By-Default theory, discussed below, and of Khrennikov's conditionalization approach [3, 13], this constraint is common in studies of quantum contextuality [1, 2, 4, 14, 15] (see Refs. [7, 9, 10, 12] for detailed discussions). Thus, if R_1^1, R_1^2, and R_1^3 in $\mathcal{R}$ do not have one and the same distribution (i.e. the system is inconsistently-connected), then, from the traditional point of view, either the notion of contextuality is not applicable to $\mathcal{R}$, or the system is considered contextual "automatically." In Refs. [5, 8, 9, 11] we provide several arguments against the necessity and desirability of the consistent connectedness constraint, and the present note adds one more. Namely, if one agrees that the transition from $\mathcal{R}$ to $\mathcal{R}'$ is a mere relabeling, one should consider it a flaw that in the traditional understanding of contextuality this transition has dramatic consequences: by adding the deterministic U's to a consistently-connected and noncontextual $\mathcal{R}$, one would "automatically" render it contextual, or else unanalyzable in contextuality terms.

The third point of this note is that the desideratum in question is satisfied in the *Contextuality-By-Default* (CbD) theory [5, 9–11]: adding the deterministic Us to $\mathcal{R}$ does not change the *degree of contextuality* computed in accordance with CbD. Moreover, the fixed value u in $\mathcal{R}'$ can be replaced with any other fixed values, and different fixed values can be chosen in different cells:

R_1^1	R_2^1	$Z_3^1 \equiv z_3^1$	$c = 1$
R_1^2	R_2^2	$Z_3^2 \equiv z_3^2$	$c = 2$
R_1^3	$Z_2^3 \equiv z_2^3$	R_3^3	$c = 3$
$Z_1^4 \equiv z_1^4$	R_2^4	R_3^4	$c = 4$
$q = 1$	$q = 2$	$q = 3$	$\boxed{\mathcal{R}^*}$

Since the choice is arbitrary, one can always avoid the necessity of adding, with zero probabilities, the values z_q^c to the set of possible values of all $R_q^{c'}$, in the same column. One can instead choose z_q^c to be one of these possible values (no matter which). Let, e.g., R_3^3 (hence also R_3^4) in $\mathcal{R}$ be a binary random variable with values +1/-1; then, Z_3^1 can be chosen either as $Z_3^1 \equiv 1$ or $Z_3^1 \equiv -1$.

The rest of the note demonstrates our third point. (Non)contextuality of the system $\mathcal{R}$ in the CbD theory is understood as follows.

(A) First we introduce a certain *statement* C that can be formulated for any pair of jointly distributed random variables. This statement should be chosen so that, for any column in $\mathcal{R}$, say, $\{R_1^1, R_1^2, R_1^3\}$ for $q = 1$, there is one and only one set of corresponding and jointly distributed random variables, $\left(T_1^1, T_1^2, T_1^3\right)$, such that (1) each of the Ts is distributed as the corresponding R; and (2) any two of the Ts in $\left(T_1^1, T_1^2, T_1^3\right)$ satisfy the statement C. This unique triple $\left(T_1^1, T_1^2, T_1^3\right)$ is called the C-*coupling* of $\{R_1^1, R_1^2, R_1^3\}$, and the C-couplings for other columns of $\mathcal{R}$ are defined analogously. Note that any part of the C-coupling of a set of random variables is the unique C-coupling of the corresponding subset of these random variables. In CbD, assuming all random variables in $\mathcal{R}$ are binary, the role of C is played by the statement "the two random variables are equal to each other with maximal possible probability." If the measurements are not dichotomous, then the system has to be dichotomized, as detailed in Ref. [6]. We need not go into these details, however, because we can make our point on a higher level of abstraction, for any C with the just stipulated properties.

(B) The system $\mathcal{R}$ is considered C-*noncontextual* if there is a random variable (vector) S with jointly distributed components corresponding to the components of $\mathcal{R}$,

S_1^1	S_2^1	$\cdot$	$c = 1$
S_1^2	S_2^2	$\cdot$	$c = 2$
S_1^3	$\cdot$	S_3^3	$c = 3$
$\cdot$	S_2^4	S_3^4	$c = 4$
$q = 1$	$q = 2$	$q = 3$	S

,

such that its rows are distributed as the corresponding rows of $\mathcal{R}$ and its columns are distributed as the C-couplings of the corresponding columns of $\mathcal{R}$. Otherwise, if such an S does not exist, the system is C-*contextual*. The intuition behind this definition is that the system is C-contextual if the distributions of the random variables within contexts prevent the random variables measuring one and the same content in different contexts from being coupled in compliance with C.

(C) If the system $\mathcal{R}$ is C-contextual, the degree of its contextuality is computed in the following way. The random variable S above is characterized by the probability masses

$$p\left(s_1^1, s_2^1, s_1^2, s_2^2, s_1^3, s_3^3, s_2^4, s_3^4\right)$$

assigned to every value $\left(S_1^1 = s_1^1, S_2^1 = s_2^1, \ldots, S_3^4 = s_3^4\right)$ of S. We redefine S into a *quasi-random variable* if we replace these probability masses with arbitrary real numbers

$$q\left(s_1^1, s_2^1, s_1^2, s_2^2, s_1^3, s_3^3, s_2^4, s_3^4\right)$$

summing to 1. We require that this *quasi-probability distribution* satisfy the same properties as the distribution of S in (B), namely, that it agrees with the distributions of the rows of $\mathcal{R}$ and with the distributions of the C-couplings of its columns. Thus, the agreement with the first row distribution means that, for any $R_1^1 = r_1^1, R_2^1 = r_2^1$, we should have

$$\begin{aligned}\textstyle\sum_{s_1^2, s_2^2, s_1^3, s_3^3, s_2^4, s_3^4} q\left(r_1^1, r_2^1, s_1^2, s_2^2, s_1^3, s_3^3, s_2^4, s_3^4\right) \\ = \Pr\left[R_1^1 = r_1^1, R_2^1 = r_2^1\right].\end{aligned} \tag{1}$$

The agreement with the distribution of the C-coupling $\left(T_1^1, T_1^2, T_1^3\right)$ for the first column means that, for any $R_1^1 = r_1^1, R_1^2 = r_1^2, R_1^3 = r_1^3$, we should have

$$\sum_{s_2^1,s_2^2,s_3^3,s_2^4,s_3^4} q\left(r_1^1, s_2^1, r_1^2, s_2^2, r_1^3, s_3^3, s_2^4, s_3^4\right) = \Pr\left[T_1^1 = r_1^1, T_1^2 = r_1^2, T_1^3 = r_1^3\right]. \tag{2}$$

Such quasi-random variables S always exist, and among them one can always find (generally non-uniquely) ones whose total variation is minimal [8]. The total variation is defined as

$$V[S] = \sum_{s_1^1,s_2^1,s_1^2,s_2^2,s_1^3,s_3^3,s_2^4,s_3^4} \left|q\left(s_1^1, s_2^1, s_1^2, s_2^2, s_1^3, s_3^3, s_2^4, s_3^4\right)\right|. \tag{3}$$

The quantity $\min V[S] - 1$ can be taken as a principled and universal measure of the *degree of contextuality.* If this quantity equals 0, which is the smallest possible value for $V[S] - 1$, then all quasi-probability masses q are nonnegative, and S^* is a proper random variable. The system then is C-noncontextual.

It is easy now to see the truth of our claim, that $\mathcal{R}^*$ has the same degree of contextuality as $\mathcal{R}$. On the right-hand side of (1),

$$\Pr\left[R_1^1 = r_1^1, R_2^1 = r_2^1\right]$$

$$= \Pr\left[R_1^1 = r_1^1, R_2^1 = r_2^1, Z_3^1 = z_3^1\right],$$

because $Z_3^1 \equiv z_3^1$. The same reasoning applies to other rows of $\mathcal{R}^*$. On the the right-hand side of (2), for any $\dot{Z}_1^4 \equiv z_1^4$,

$$\Pr\left[T_1^1 = r_1^1, T_1^2 = r_1^2, T_1^3 = r_1^3\right]$$

$$= \Pr\left[T_1^1 = r_1^1, T_1^2 = r_1^2, T_1^3 = r_1^3, \dot{Z}_1^4 = z_1^4\right].$$

Now, $\left(T_1^1, T_1^2, T_1^3, \dot{Z}_1^4\right)$ is the C-coupling of $\left\{R_1^1, R_1^2, R_1^3, Z_1^4\right\}$. Indeed, the C-coupling $\left(\dot{T}_1^1, \dot{T}_1^2, \dot{T}_1^3, \dot{Z}_1^4\right)$ of $\left\{R_1^1, R_1^2, R_1^3, Z_1^4\right\}$ exists and is unique. The part $\left(\dot{T}_1^1, \dot{T}_1^2, \dot{T}_1^3\right)$ is then the unique C-coupling of $\left\{R_1^1, R_1^2, R_1^3\right\}$, whence $\left(\dot{T}_1^1, \dot{T}_1^2, \dot{T}_1^3\right) = \left(T_1^1, T_1^2, T_1^3\right)$. The same reasoning applies to other columns of $\mathcal{R}^*$. So the right-hand sides in the equations exemplified by (1) and (2) do not change when $\mathcal{R}$ is replaced with $\mathcal{R}^*$. Since, under this replacement, the left-hand sides of these equations do not change either, except that each quasi-probability value

$$q\left(s_1^1, s_2^1, s_1^2, s_2^2, s_1^3, s_3^3, s_2^4, s_3^4\right)$$

in them is bijectively renamed into

$$q\left(s_1^1, s_2^1, z_3^1, s_1^2, s_2^2, z_3^2, s_1^3, z_2^3, s_3^3, z_1^4, s_2^4, s_3^4\right),$$

the set of the quasi-probability distributions solving (1) and (2) (and similar equations) in $\mathcal{R}^*$ remains the same as in $\mathcal{R}$, and the minimum value of $V[S]$ in (3) therefore remains unchanged.

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