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Mental architectures with selectively influenced but stochastically interdependent components

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Abstract

The way external factors influence distribution functions for the overall time required to perform a mental task (such as responding to a stimulus, or solving a problem) may be informative as to the underlying mental architecture, the hypothetical network of interconnected processes some of which are selectively influenced by some of the external factors. Under the assumption that all processes contributing to the overall performance time are stochastically independent, several basic results have been previously established. These results relate patterns of response time distribution functions produced by manipulating external factors to such questions as whether the hypothetical constituent processes in the mental architecture enter AND gates or OR gates, and whether pairs of processes are sequential or concurrent. The present study shows that all these results are also valid for stochastically interdependent component times, provided the selective dependence of these components upon external factors is understood within the framework of a recently proposed theory of selective influence. According to this theory each component is representable as a function of three arguments: the factor set selectively influencing it, a component-specific source of randomness, and a source of randomness shared by all the components.

Keywords: Mental processing architecture; Information processing time; Response time; Parallel (concurrent) processes; Serial (sequential) processes; Critical path network; Parallel-serial network; Wheatstone bridge; Selective influence; Stochastic interdependence

1. Introduction

Mental processing filling in the interval between presentation of a stimulus (or problem) and production of a response (solution) is often thought of as being effected by a network of variously interconnected processing units. These hypothetical processing units can be characterized in a variety of ways. Thus, each of them can be ascribed a specific function, or resulting output, such as "detection of the target", "retrieval of the next item from memory", or "comparison of the retrieved item with the target". For the purposes of the present analysis, however, the two relevant characteristics of a processing unit are its *processing time* and the *external factors* that influence this processing time. By external factors we understand any set Φ of observable variables (each having at least two distinct values) that can influence the distribution of the overall processing time $\mathbf{T}$, a random variable with observable values. External factors may be stimulus attributes, presentation conditions, experimental instructions, and even observable outputs of mental processing (i.e., responses or solutions). The latter example shows that the term "influence" should be understood in a covariational rather than causal sense: different values of an external factor correspond to different distributions of $\mathbf{T}$.

If the overall processing time $\mathbf{T}$ is known to depend on a set of external factors Φ, then the *processing time architecture* for the mental process in question (with respect to factors Φ) can be characterized by two relations:

$$\mathbf{T} = H(\mathbf{T}_1, \ldots, \mathbf{T}_n),$$
$$(\mathbf{T}_1, \ldots, \mathbf{T}_n) \hookleftarrow (\Gamma_1, \ldots, \Gamma_n), \tag{1}$$

where $\mathbf{T}_1, \ldots, \mathbf{T}_n$ ($n \geqslant 1$) are the processing times of the hypothetical component units, H is some *composition rule*, and $\Gamma_1, \ldots, \Gamma_n$ are subsets of the factor set Φ such

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that $\Gamma_1 \cup \cdots \cup \Gamma_n = \Phi$. The symbol $\hookleftarrow$ introduced in Dzhafarov (2003a) designates the *selective influence* relation between the factor subsets and the corresponding component times: Γ_i selectively influences $\mathbf{T}_i$ ($i = 1, \ldots, n$). In this notation some of the subsets $\Gamma_1, \ldots, \Gamma_n$ may be empty, and some pairs of these subsets may overlap. The most interesting cases of selective influence, however, involve disjoint $\Gamma_1, \ldots, \Gamma_n$ (each of which, except for empty subsets, may therefore be considered a single factor with multiple values).[1]

The meaning of selective influence is simple and clear when the constituting processing times $\mathbf{T}_1, \ldots, \mathbf{T}_n$ (taken at any fixed values of the external factors in Φ) are assumed to be *stochastically independent*.[2] In this case saying that Γ_i selectively influences $\mathbf{T}_i$ means that the marginal distribution of $\mathbf{T}_i$ is not influenced by any factor in Φ that falls outside the subset Γ_i.[3] The meaning of selective influence for *stochastically interdependent* $\mathbf{T}_1, \ldots, \mathbf{T}_n$ is less obvious. Its understanding in this paper is based on a general theory presented in Dzhafarov (2003a), whose relevant aspects will be recapitulated later, in Section 2.

The simplest example of a processing time architecture, not surprisingly, is provided by simple response time to stimulus of variable intensity I. The factor set here consists of I alone, $\Phi = \{I\}$, and it is reasonable to assume (Dzhafarov, 1992; Dzhafarov & Rouder, 1996) that the overall response time $\mathbf{T}$ is the sum of two components, an intensity-dependent one and intensity-independent one,

$$\mathbf{T} = \mathbf{A} + \mathbf{B},$$
$$(\mathbf{A}, \mathbf{B}) \hookleftarrow (\{I\}, \varnothing).$$

The second relationship says that $\mathbf{A}$ is selectively influenced by $\Gamma_1 = \{I\}$, whereas $\mathbf{B}$ is (formally speaking) selectively influenced by $\Gamma_2 = \varnothing$, the empty set.

Sternberg's (1969) classical theory (see also Ashby & Townsend, 1980; Roberts & Sternberg, 1993) can be presented in the form

$$\mathbf{T} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D},$$
$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \hookleftarrow (\{S_1, \ldots, S_k\}, \{M\}, \{R\}, \varnothing),$$

where $\mathbf{T}$ is the overall time required to (correctly) decide whether a given target belongs to a previously memorized list of items, $\mathbf{A}$ is a component time selectively influenced by physical characteristics $\{S_1, \ldots, S_k\}$ of the target (such as contrast, duration, etc.), $\mathbf{B}$ is a component time selectively influenced by the size M of the memorized list, the component time $\mathbf{C}$ is selectively influenced by the choice of response R (Yes or No), whereas $\mathbf{D}$ denotes the processing time for the rest of the hypothetical processing units, presumably not influenced by any of these factors. Here, in relation to (1), $\Phi = \{S_1, \ldots, S_k, M, R\}$ (i.e., $\mathbf{T}$ is considered a function of these $k+2$ factors only), $\Gamma_1 = \{S_1, \ldots, S_k\}$, $\Gamma_2 = \{M\}$, $\Gamma_3 = \{R\}$, $\Gamma_4 = \varnothing$.

Considering for a moment just two component times, $\mathbf{T}_1 = \mathbf{A}$ and $\mathbf{T}_2 = \mathbf{B}$, it is argued in Dzhafarov and Schweickert (1995), Dzhafarov and Cortese (1996), and Dzhafarov (1997) that one can contemplate an infinity of possible composition rules $\mathbf{T} = H(\mathbf{A}, \mathbf{B})$, such as

$$\mathbf{T} = a\mathbf{AB}, \quad a > 0,$$
$$\mathbf{T} = (\mathbf{A}^p + \mathbf{B}^p)^{1/p}, \quad p > 0,$$
$$\mathbf{T} = \exp\left(\log\frac{\mathbf{A}}{a} + \log\frac{\mathbf{B}}{a}\right), \quad a > 0,$$
$$\vdots$$

Of greatest traditional interest, however, are the composition rules

$$\mathbf{T} = \mathbf{A} + \mathbf{B},$$
$$\mathbf{T} = \mathbf{A} \max \mathbf{B},$$
$$\mathbf{T} = \mathbf{A} \min \mathbf{B}, \tag{2}$$

where we write $\mathbf{A} \max \mathbf{B}$ and $\mathbf{A} \min \mathbf{B}$ instead of the more conventional $\max(\mathbf{A}, \mathbf{B})$ and $\min(\mathbf{A}, \mathbf{B})$, respectively.[4] The reason the composition rules $+$, max, min are of special interest is that they are the combination rules for process durations when processes are, respectively, connected *in series*, connected *in parallel followed by an AND gate*, and connected *in parallel followed by an OR gate* (see Fig. 1). The literature related to these composition rules is very large (for surveys, see Luce, 1986; Massaro & Cowan, 1993; Schweickert, 1993; Townsend, 1990; Townsend & Ashby, 1983).

The networks shown in Fig. 1 are the simplest nontrivial examples of a *directed acyclic network*. The nodes of a directed acyclic network represent component processes, except for the initial node o and the terminal node e that mark the beginning and end of the entire processing (they can be viewed as processes with zero durations). Some of the nodes are connected by arrows indicating precedence in time. An arrow directed from one node to another indicates that the process represented by the first node precedes the process represented by the second. A single process can be viewed as a trivial directed acyclic network (Fig. 2A). Two examples of a multicomponent directed acyclic network are shown in Figs. 2B and C. Any two component processes (nodes) in such a network are

[1] Note that the factor set Φ does not include constants: each factor has at least two distinct values.

[2] Here and throughout the paper the term stochastically independent always means *mutually* stochastically independent. The same applies to the term *conditionally* (stochastically) independent used later on.

[3] One could add: "and is influenced by any factor within Γ_i" (*effectiveness requirement*, see Dzhafarov, 2001). In the present context, however, this requirement is not needed.

[4] This notation is more convenient in the present context as it (a) shows that max and min are treated as binary operations, and (b) significantly reduces the number of brackets in long expressions.

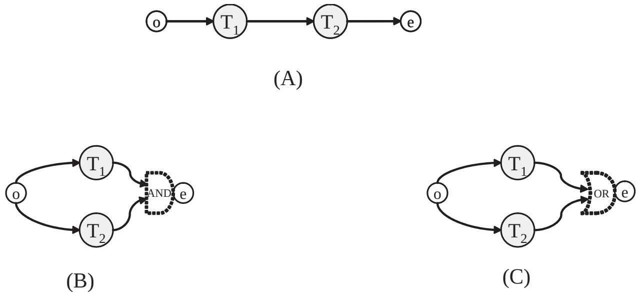

Fig. 1. Three simplest nontrivial directed acyclic networks. (A) Sequential connection, with the processing time architecture $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$. (B) Parallel-AND connection: the terminal node is reached when both processes 1 and 2 are terminated, i.e., the processing time architecture is $\mathbf{T} = \mathbf{T}_1 \max \mathbf{T}_2$. (C) Parallel-OR connection: the terminal node is reached when either of the processes 1 and 2 is terminated, $\mathbf{T} = \mathbf{T}_1 \min \mathbf{T}_2$. Note that the processes are denoted by nodes of the graphs, rather than by arrows (the latter only serve to show how the component processes are interconnected). The initial and terminal nodes, o and e, are "dummy" nodes; they can be viewed as processes with zero durations.

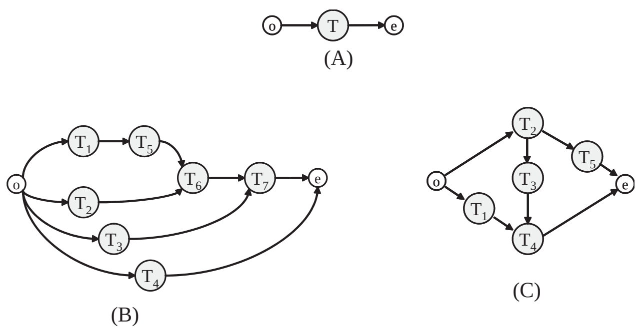

Fig. 2. The directed acyclic networks. (A) A trivial, one-component network. (B) A network whose processing time architecture is $\mathbf{T}_4 \circ (\mathbf{T}_3 \circ ((\mathbf{T}_1 + \mathbf{T}_5) \circ \mathbf{T}_2 + \mathbf{T}_6) + \mathbf{T}_7)$, where $\circ$ can be replaced by max (AND gate) or min (OR gate). This is a serial–parallel network considered in Schweickert and Giorgini (1999). Note that each component time enters in this expression only once. (C) A network whose processing time architecture is $(\mathbf{T}_2 + \mathbf{T}_5) \circ (\mathbf{T}_1 \circ (\mathbf{T}_2 + \mathbf{T}_3) + \mathbf{T}_4)$, with the same meaning of $\circ$. This is a Wheatstone bridge network (not serial–parallel) considered in Schweickert and Giorgini (1999). Note that $\mathbf{T}_2$ enters in this expression twice, and the expression cannot be rewritten to eliminate a repeated component time.

uniquely characterized as being *sequential* (if one can get from one of them to another following the arrows) or *concurrent* (if this cannot be done). A directed acyclic network is called a *serial–parallel network* if it consists of a single process (node) or if it can be obtained from two disjoint serial–parallel networks (with no common nodes) by connecting them serially or in parallel. A formal introduction to directed acyclic networks can be found in Fisher and Glasser (1996) and Schweickert, Fisher, and Goldstein (1992).

If every gate in a directed acyclic network is either an AND gate or an OR gate, the corresponding composition rule H in (1) can be written as an algebraic expression with $\mathbf{T}_1, \dots, \mathbf{T}_n$ being variously interconnected by operations $+, \max, \min$ (see the legend to Fig. 2 for examples). For a serial–parallel network this expression can be written so that each of the component times $\mathbf{T}_1, \dots, \mathbf{T}_n$ enters in it only once.[5] Thus, the directed acyclic networks shown in Figs. 2A and B are serial–parallel networks, whereas the one shown in Fig. 2C is not. A serial–parallel network whose composition rule involves only the operations $+$ and max is referred to as a serial–parallel network with *all-AND gates*; a serial–parallel network with *all-OR gates* is defined analogously (with min replacing max). The respective composition rules are denoted by SP_{AND} and SP_{OR}. The composition rule for a serial–parallel network that has all-AND gates or all-OR gates (without specifying which) will be denoted by $SP_{\mathrm{AND/OR}}$.

In the present study, we focus on processing time architectures of the type

$$\mathbf{T} = SP_{\mathrm{AND/OR}}(\mathbf{A}, \mathbf{B}, \mathbf{T}_1, \dots, \mathbf{T}_k),$$
$$(\mathbf{A}, \mathbf{B}, \underbrace{\mathbf{T}_1, \dots, \mathbf{T}_k}) \hookleftarrow (\alpha, \beta, \underbrace{\varnothing, \dots, \varnothing}), \tag{3}$$

where $k \geqslant 0$, $\alpha \cap \beta = \varnothing$, $\alpha \cup \beta \neq \varnothing$, and all remaining factor subsets are empty. In other words, we deal with serial–parallel networks with all-AND gates or all-OR gates in which two component times are selectively influenced by two specific, distinct factor subsets (that can be viewed as two distinct single factors, unless one of the subsets α, β is empty). The distribution of the overall duration $\mathbf{T}$ is viewed here as depending on factors belonging to $\Phi = \alpha \cup \beta$ only, all other factors that could potentially influence $\mathbf{T}$ being held constant.

The problem we address is this: *given the distribution functions for* $\mathbf{T}$ *at different values of factors* $\Phi = \alpha \cup \beta$, *and assuming the processing time architecture* (3), *how can one decide* (a) *whether the composition rule is* SP_{AND} *or* SP_{OR}; *and* (b) *whether* $\mathbf{A}, \mathbf{B}$ *are sequential or concurrent*?

Townsend and Nozawa (1995) considered a special case of this problem, confined to (using our notation) the composition rules $\mathbf{A} + \mathbf{B}$, $\mathbf{A} \max \mathbf{B} + \mathbf{C}$, and $\mathbf{A} \min \mathbf{B} + \mathbf{C}$, with $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \hookleftarrow (\alpha, \beta, \varnothing)$. Schweickert and Giorgini (1999) and Schweickert, Giorgini and Dzhafarov (2000) analyzed this problem for arbitrary serial–parallel networks (and for the Wheatstone bridge network shown in Fig. 2C) under the assumption that the component times $\mathbf{A}, \mathbf{B}, \mathbf{T}_1, \dots, \mathbf{T}_k$ are stochastically independent.

[5] A serial–parallel composition rule can, in fact, be given an inductive definition analogous to that for serial–parallel networks (given in the previous paragraph). A composition rule is serial–parallel if it is $H(\mathbf{T}) \equiv \mathbf{T}$, or if it is the sum, minimum, or maximum of two serial–parallel composition rules $H_1(\mathbf{A}_1, \dots, \mathbf{A}_n)$ and $H_2(\mathbf{B}_1, \dots, \mathbf{B}_m)$ with disjoint sets of arguments. Clearly, an expression consisting of arguments variously interconnected by operations $+, \min, \max$ can be constructed in this way if and only if no argument enters in it more than once.

In the present study this analysis is extended to serial–parallel networks with stochastically interdependent components. The earlier results established for stochastically independent component times are shown to hold for interdependent component times, provided they are selectively influenced in the sense clarified below. In particular, the patterns predicted for composition rules SP_{AND} and SP_{OR}, and for sequential and concurrent processes are qualitatively the same for the independent and interdependent cases.

2. Selective influence under interdependence

The first systematic attempt to explain how the selectiveness of influence can coexist with stochastic interdependence of the influenced random variables (here, component times) was made by Townsend, (1984) (see also Townsend & Thomas, 1994). A mathematical theory for a generalized version of Townsend's solution was presented in Dzhafarov (1999). As pointed out in that work and in Dzhafarov (2001, 2003a), certain properties of Townsend's solution prevent it from being regarded a viable definition of selective influence. Thus, if $\mathbf{T}_1, \ldots, \mathbf{T}_n$ are selectively influenced by $\Gamma_1, \ldots, \Gamma_n$ (respectively) in Townsend's sense, it does not follow that, say, $\mathbf{T}_1, \mathbf{T}_2$ are selectively influenced by Γ_1, Γ_2 in the same sense, or even that the distribution of $\mathbf{T}_1$ is influenced by Γ_1 alone.

A different approach to selective influence was proposed in Dzhafarov (1997, 2001). The present study is based on the improved and generalized version of this approach, presented in Dzhafarov (2003a). The definition of the selective influence relation

$$(\mathbf{T}_1, \ldots, \mathbf{T}_n) \hookleftarrow (\Gamma_1, \ldots, \Gamma_n) \tag{4}$$

given in Dzhafarov (2003a) is as follows.

Definition 2.1. Selective influence relation (4) means that $\mathbf{T}_1, \ldots, \mathbf{T}_n$ can be presented as

$$\mathbf{T}_1 = f_1(\mathbf{R}, \mathbf{S}_1, \Gamma_1), \ldots, \mathbf{T}_n = f_n(\mathbf{R}, \mathbf{S}_n, \Gamma_n), \tag{5}$$

where $f_1, \ldots, f_n$ are some measurable functions, while $\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n$ are mutually stochastically independent random entities[6] whose distributions do not depend on any factors belonging to $\Phi = \Gamma_1 \cup \cdots \cup \Gamma_n$.

Eq. (5) says that the process by which any single realization $(t_1, \ldots, t_n)$ of the random vector $(\mathbf{T}_1, \ldots, \mathbf{T}_n)$ is generated for any given value $(G_1, \ldots, G_n)$ of factor subsets $(\Gamma_1, \ldots, \Gamma_n)$ consists in independently sampling realizations $r, s_1, \ldots, s_n$ of, respectively, $\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n$, and computing

$$t_1 = f_1(r, s_1, G_1), \ldots, t_n = f_n(r, s_n, G_n).$$

The stochastic interdependence of $(\mathbf{T}_1, \ldots, \mathbf{T}_n)$ is due to the fact that r is one and the same in all these expressions. To generate N realizations of $(\mathbf{T}_1, \ldots, \mathbf{T}_n)$, for a sequence of N values of $(\Gamma_1, \ldots, \Gamma_n)$ (not necessarily distinct), one should repeat this process N times.[7]

Note that if $\mathbf{R}$ can only attain a single value, then $\mathbf{T}_1, \ldots, \mathbf{T}_n$ (for any value of the factor set Φ) are stochastically independent. This shows that Definition 2.1 includes stochastic independence as a special case.

In the general case, for any fixed value r of $\mathbf{R}$, the conditional component times $\mathbf{T}_{1|\mathbf{R}=r}, \ldots, \mathbf{T}_{n|\mathbf{R}=r}$ can be presented as

$$\mathbf{T}_{1|\mathbf{R}=r} = f_1(r, \mathbf{S}_1, \Gamma_1), \ldots, \mathbf{T}_{n|\mathbf{R}=r} = f_n(r, \mathbf{S}_n, \Gamma_n).$$

These conditional component times are stochastically independent, and their selective dependence on $\Gamma_1, \ldots, \Gamma_n$ is obvious. As shown in Dzhafarov (2003a), this property is, in fact, equivalent to Definition 2.1. Namely, the following lemma holds.

Lemma 2.1. *Selective influence relation* (4) *holds if and only if one can find a random entity* $\mathbf{R}$ (*whose distribution does not depend on* $\Gamma_1 \cup \cdots \cup \Gamma_n$) *such that* $\mathbf{T}_1, \ldots, \mathbf{T}_n$ *are conditionally independent given any value of* $\mathbf{R}$, *with their conditional distributions depending on* $\Gamma_1, \ldots, \Gamma_n$, *respectively*.

It is this property (alternatively, definition) of selective influence that is most useful for our purposes. Fig. 3 provides a schematic illustration.

It follows from Lemma 2.1 that for any time values $t_1, \ldots, t_n$ (nonnegative reals),

$$Pr[\mathbf{T}_1 \leqslant t_1, \ldots, \mathbf{T}_n \leqslant t_n; \Phi] = \int_{Dom(\mathbf{R})} \left[\prod_{i=1}^{n} Pr(\mathbf{T}_i \leqslant t_i | \mathbf{R} = r; \Gamma_i) \right] d\omega(r), \tag{6}$$

where ω is the probability measure imposed on the domain of $\mathbf{R}$, $Dom(\mathbf{R})$. The probability $Pr(\mathbf{T}_i \leqslant t_i | \mathbf{R} = r; \Gamma_i)$ considered as a function of t_i is the conditional distribution function for $\mathbf{T}_i$, given $\mathbf{R} = r$. The selectiveness of influence manifests itself in the fact that, for any t_i and r, this function does not depend on factors outside Γ_i. If, for any $i = 1, \ldots, n$, this function is differentiable in t_i (at any r and any value of Γ_i), then, using ψ to

[6] The term "random entity" (rather than more familiar "random variable" or "random vector") is used to indicate that each of the $\mathbf{C}, \mathbf{S}_1, \ldots, \mathbf{S}_n$ may take on their values in arbitrary spaces (sets endowed with probability measures), which are not necessarily mappable on sets of reals or real-valued vectors.

[7] To obtain *random samples* of $(\mathbf{T}_1, \ldots, \mathbf{T}_n)$ (as in the simulations described in Section 7), the successive realizations of $(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$ should themselves be mutually independent. In general, however, while the components of $(\mathbf{R}, \mathbf{S}_1, \ldots, \mathbf{S}_n)$ must be mutually independent and have fixed distributions, interdependencies among their successive realizations are consistent with (5), and may have to be allowed to model sequential effects in the overall processing time $\mathbf{T}$.

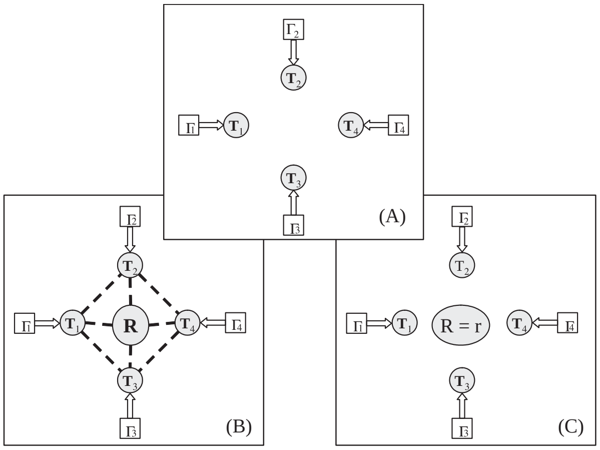

Fig. 3. (A) A network with stochastically independent components selectively influenced by different factor subsets. (B, C) A network with stochastically interdependent (dashed lines) and selectively influenced components. $\mathbf{R}$ is the hypothetical (unobservable) conditioning entity. Fixing its value makes the components stochastically independent and selectively influenced, as in (A).

denote probability densities,

$$\psi[\mathbf{T}_1 = t_1, \ldots, \mathbf{T}_n = t_n; \Phi] = \int_{Dom(\mathbf{R})} \left[\prod_{i=1}^{n} \psi(\mathbf{T}_i = t_i | \mathbf{R} = r; \Gamma_i) \right] d\omega(r). \quad (7)$$

It is obvious from (5) that if $(\mathbf{T}_1, \ldots, \mathbf{T}_n) \hookleftarrow (\Gamma_1, \ldots, \Gamma_n)$, then for any subvector of $(\mathbf{T}_1, \ldots, \mathbf{T}_n)$, say, the one comprised by its first k components, $(\mathbf{T}_1, \ldots, \mathbf{T}_k) \hookleftarrow (\Gamma_1, \ldots, \Gamma_k)$. Eq. (6) therefore holds for all subvectors of $(\mathbf{T}_1, \ldots, \mathbf{T}_n)$, and the same is true for (7), if the densities exist. In particular,

$$Pr[\mathbf{T}_i \leqslant t_i; \Phi] = \int_{Dom(\mathbf{R})} Pr(\mathbf{T}_i \leqslant t_i | \mathbf{R} = r; \Gamma_i)\, d\omega(r),$$

$$\psi[\mathbf{T}_i = t_i; \Phi] = \int_{Dom(\mathbf{R})} \psi(\mathbf{T}_i = t_i | \mathbf{R} = r; \Gamma_i)\, d\omega(r).$$

Also, for any function $\mathbf{T} = H(\mathbf{T}_1, \ldots, \mathbf{T}_n)$, we have

$$Pr[\mathbf{T} \leqslant t; \Phi] = \int_{Dom(\mathbf{R})} Pr(\mathbf{T} \leqslant t | \mathbf{R} = r; \Phi)\, d\omega(r),$$

$$\psi[\mathbf{T} = t; \Phi] = \int_{Dom(\mathbf{R})} \psi(\mathbf{T} = t | \mathbf{R} = r; \Phi)\, d\omega(r). \quad (8)$$

The importance of this observation is in the fact that even though $\mathbf{T}_1, \ldots, \mathbf{T}_n$ in $\mathbf{T} = H(\mathbf{T}_1, \ldots, \mathbf{T}_n)$ are stochastically interdependent, the subintegral expressions $Pr(\mathbf{T} \leqslant t | \mathbf{R} = r; \Phi)$ and $\psi(\mathbf{T} = t | \mathbf{R} = r; \Phi)$ (if the latter exists) are computed for stochastically independent conditional component times $\mathbf{T}_{1|\mathbf{R}=r}, \ldots, \mathbf{T}_{n|\mathbf{R}=r}$. As a result, certain properties of the subintegral expressions can be transmitted to $Pr[\mathbf{T} \leqslant t; \Phi]$ and $\psi[\mathbf{T} = t; \Phi]$ "automatically", which is what is made use of in the present study.

As shown in Dzhafarov (2003a), Definition 2.1 (equivalently, Lemma 2.1) is restrictive: one can find, for example, random variables $\mathbf{T}_1, \mathbf{T}_2$ whose joint distribution depends on $\Gamma_1 \cup \Gamma_2$, whose marginal distributions (for $\mathbf{T}_1$ and for $\mathbf{T}_2$) depend on Γ_1 and Γ_2, respectively, but such that there exists no conditioning random entity $\mathbf{R}$ such that

$$\mathbf{T}_{1|\mathbf{R}=r} = f_1(r, \mathbf{S}_1, \Gamma_1), \quad \mathbf{T}_{2|\mathbf{R}=r} = f_2(r, \mathbf{S}_2, \Gamma_2).$$

Thus

$$(\mathbf{T}_1, \mathbf{T}_2) \not\hookleftarrow (\Gamma_1, \Gamma_2).$$

With Definition 2.1 and Lemma 2.1 in place, the general meaning of (1) and (3) is completely specified, and we can begin our study of the problem posed at the end of Section 1.

3. Ancillary assumptions

The theory presented in Schweickert, Giorgini, and Dzhafarov (2000) makes use of certain ancillary assumptions, of which the main one, the assumption of stochastic independence among $\mathbf{T}_1, \ldots, \mathbf{T}_n$, is now being dropped. The remaining two assumptions are that component times possess densities and means, and that a component time at one factor level stochastically dominates the same component time at another. The use of the stochastic dominance assumption in the context of processing time architectures has a long history. It was pioneered by Sternberg (1973). Some early results on stochastic dominance can be found in Townsend and Ashby (1978, 1983, Chapter 8). Schweickert (1977, 1982) introduced an assumption later shown to be equivalent to stochastic dominance (Townsend & Schweickert, 1989) and used to analyze mean response times in directed acyclic networks (Schweickert & Townsend, 1989).

These ancillary assumptions have to be revised to reflect the new meaning in which we understand the selective influence relation

$$(\mathbf{A}, \mathbf{B}, \underbrace{\mathbf{T}_1, \ldots, \mathbf{T}_k}) \hookleftarrow (\alpha, \beta, \underbrace{\varnothing, \ldots, \varnothing}).$$

The essence of this revision is that the properties of $\mathbf{A}, \mathbf{B}, \mathbf{T}_1, \ldots, \mathbf{T}_k$ needed to derive the results in Schweickert et al. (2000) are now ascribed to the independent conditional component times $\mathbf{A}_{|\mathbf{R}=r}$, $\mathbf{B}_{|\mathbf{R}=r}$, $\mathbf{T}_{1|\mathbf{R}=r}, \ldots, \mathbf{T}_{k|\mathbf{R}=r}$, for all values r of $\mathbf{R}$.[8] This will ensure that the results derived in Schweickert et al. (2000) will

[8] Here and throughout this paper the quantification "for all r" can always be replaced with "for ω-almost all r", where ω is the probability measure that defines $\mathbf{R}$. The same applies to quantifications "for all t" (with respect to Lebesgue measure) and "for all t, r" (with respect to the product measure).

be valid for the conditional overall processing times

$$\mathbf{T}_{|\mathbf{R}=r} = SP_{\mathrm{AND/OR}}(\mathbf{A}_{|\mathbf{R}=r}, B_{|\mathbf{R}=r}, T_{1|\mathbf{R}=r}, \ldots, \mathbf{T}_{k|\mathbf{R}=r}),$$

whence the validity of these results for unconditional $\mathbf{T}$ will follow by a straightforward argument.

3.1. Conditional densities and means

We assume that all processing times have conditional densities

$$\psi(\mathbf{A} = \mathbf{t}|\mathbf{R} = r; \alpha), \psi(\mathbf{B} = t|\mathbf{R} = r; \beta),$$
$$\psi(\mathbf{T}_1 = t|\mathbf{R} = r; \varnothing), \ldots, \psi(\mathbf{T}_k = t|\mathbf{R} = r; \varnothing)$$

and finite conditional means

$$E(\mathbf{A}|\mathbf{R} = r; \alpha), E(\mathbf{B}|\mathbf{R} = r; \beta),$$
$$E(\mathbf{T}_1|\mathbf{R} = r; \varnothing), \ldots, E(\mathbf{T}_k|\mathbf{R} = r; \varnothing).$$

The existence of densities and finite means for sums, maxima, minima, or combinations thereof of the conditional component times then follows by standard argument.

3.2. Stochastic dominance

Let $\alpha \neq \varnothing$. We assume then that α has at least two distinct values α_1 and α_2 such that, for every value r of $\mathbf{R}$ and for every nonnegative real t,

$$Pr(\mathbf{A} \leqslant t|\mathbf{R} = r; \alpha = \alpha_1) \geqslant Pr(\mathbf{A} \leqslant t|\mathbf{R} = r; \alpha = \alpha_2). \quad (9)$$

One can say that $\mathbf{A}_{|\mathbf{R}=r}$ at α_1 *stochastically dominates* $\mathbf{A}_{|\mathbf{R}=r}$ at α_2, for every r. The analogous assumption applies to $\mathbf{B}_{|\mathbf{R}=r}$ and β, provided $\beta \neq \varnothing$:

$$Pr(\mathbf{B} \leqslant t|\mathbf{R} = r; \beta = \beta_1) \geqslant Pr(\mathbf{B} \leqslant t|\mathbf{R} = r; \beta = \beta_2). \quad (10)$$

Occasionally we assume that (9) and (10) hold in conjunction with both or one of the following properties: on some interval $0 \leqslant t \leqslant \tau$,

$$\psi(\mathbf{A} = t|\mathbf{R} = r; \alpha_1) \geqslant \psi(\mathbf{A} = t|\mathbf{R} = r; \alpha_2), \quad (11)$$

$$\psi(\mathbf{B} = t|\mathbf{R} = r; \beta_1) \geqslant \psi(\mathbf{B} = t|\mathbf{R} = r; \beta_2) \quad (12)$$

for all r.

3.3. Convention

In the following all the ancillary assumptions above will be assumed tacitly, with the exception of (11) and (12) which should not be assumed unless stated explicitly.

4. Single-component results

To make the logic of our analysis transparent, consider first the special case of (3) when one of the two factor subsets (say, β) is empty. (This is equivalent to considering (3) at some fixed value of β.) The process $\mathbf{B}$ in this case has no special status among the rest of the component times, and (3) can be replaced with

$$\mathbf{T} = SP_{\mathrm{AND/OR}}(\mathbf{A}, \mathbf{T}_1, \ldots, \mathbf{T}_k),$$
$$(\mathbf{A}, \underbrace{\mathbf{T}_1, \ldots, \mathbf{T}_k}) \hookleftarrow (\alpha, \underbrace{\varnothing, \ldots, \varnothing}), \quad (13)$$

with $k \geqslant 0$. The question is: how is the dependence of $\mathbf{A}$ on α reflected in the dependence of the overall processing time $\mathbf{T}$ on α?

Let the value r of the conditioning random entity $\mathbf{R}$ be fixed. Define

$$G_1(t|r) = Pr(\mathbf{T} \leqslant t|\mathbf{R} = r; \alpha = \alpha_1),$$
$$G_2(t|r) = Pr(\mathbf{T} \leqslant t|\mathbf{R} = r; \alpha = \alpha_2).$$

These are the distribution functions for the conditional overall processing time $\mathbf{T}_{|\mathbf{R}=r}$, taken at the two values of α satisfying the stochastic dominance condition (9) with respect to the target component $\mathbf{A}$.

In accordance with Lemma 2.1, the conditional component times $\mathbf{A}_{|\mathbf{R}=r}, \mathbf{T}_{1|\mathbf{R}=r}, \ldots, \mathbf{T}_{k|\mathbf{R}=r}$ are mutually stochastically independent. *As a consequence, all the results established in Schweickert et al. (2000) for the distribution of the overall processing time in a serial–parallel network with stochastically independent component times must be valid for the distribution of* $\mathbf{T}_{|\mathbf{R}=r}$. We have then the following fact.

Lemma 4.1. *If* (13) *holds, then* $G_1(t|r) - G_2(t|r) \geqslant 0$, *for all* $t \geqslant 0$ *and for all* r.

Proof. See Theorem 1 in Schweickert et al. (2000).[9] □

Denote now the unconditional distribution functions for $\mathbf{T}$ as

$$G_1(t) = Pr(\mathbf{T} \leqslant t|\alpha = \alpha_1),$$
$$G_2(t) = Pr(\mathbf{T} \leqslant t|\alpha = \alpha_2).$$

In accordance with (8),

$$G_1(t) = \int_{Dom(\mathbf{R})} G_1(t|r)\, d\omega(r),$$
$$G_2(t) = \int_{Dom(\mathbf{R})} G_2(t|r)\, d\omega(r),$$

whence it immediately follows that if $G_1(t|r) - G_2(t|r) \geqslant 0$, for all $t \geqslant 0$ and for all r, then

$$G_1(t) - G_2(t) = \int_{Dom(\mathbf{R})} [G_1(t|r) - G_2(t|r)]\, d\omega(r) \geqslant 0$$

for all $t \geqslant 0$. Thus we have established the following result.

[9] The theorem in Schweickert et al. (2000) translates, in fact, into a stronger proposition: if (13) holds, and stochastic dominance (9) holds on an interval $0 \leqslant t < \tau$, then $G_1(t|r) \geqslant G_2(t|r)$ on the same interval (where τ is allowed to be infinite).

Theorem 4.1. *If* (13) *holds, then* $G_1(t) - G_2(t) \geqslant 0$, *for all* $t \geqslant 0$.

The significance of this theorem is easy to understand. It shows that insofar as the stochastic dominance (9) is concerned, any sub-expression of $SP_{\text{AND/OR}}(\mathbf{A}, \mathbf{T}_1, \ldots, \mathbf{T}_k)$ that contains $\mathbf{A}$ can be treated as a single component time selectively influenced by α.

Note the logic by which this result is obtained. In Schweickert et al. (2000) the statement of Theorem 4.1 was shown to be valid if $\mathbf{A}, \mathbf{T}_1, \ldots, \mathbf{T}_k$ in (13) are stochastically independent. Due to Lemma 2.1, this immediately translates into the statement of Lemma 4.1, conditional upon a single (but arbitrary) value of $\mathbf{R}$. Then we use the simple integration relation in (8) and achieve, essentially "automatically", the generalization of the result by Schweickert et al. (2000) to (13) with stochastically interdependent $\mathbf{A}, \mathbf{T}_1, \ldots, \mathbf{T}_k$.

To demonstrate that the same logic applies also to density functions, consider the following statement, in which $g_1(t|r)$ and $g_2(t|r)$ denote probability densities for $\mathbf{T}_{|\mathbf{R}=r}$ at, respectively, $\alpha = \alpha_1$ and $\alpha = \alpha_2$. (The existence of these densities is guaranteed by the ancillary assumptions, Section 3.)

Lemma 4.2. *If* (13) *holds with the composition rule* SP_{AND} (*all-AND gates*), *and if* (11) *holds on some interval* $0 \leqslant t \leqslant \tau$, *for all* r, *then* $g_1(t|r) - g_2(t|r) \geqslant 0$ *on the same interval*.

Proof. See Theorem 2 in Schweickert et al. (2000).[10] □

Denote the unconditional densities for $\mathbf{T}$ by

$$g_1(t) = \psi(\mathbf{T} = t | \alpha = \alpha_1),$$
$$g_2(t) = \psi(\mathbf{T} = t | \alpha = \alpha_2).$$

In accordance with (8),

$$g_1(t) = \int_{Dom(\mathbf{R})} g_1(t|r)\, d\omega(r),$$
$$g_2(t) = \int_{Dom(\mathbf{R})} g_2(t|r)\, d\omega(r),$$

whence it immediately follows that if $g_1(t|r) - g_2(t|r) \geqslant 0$, for some t and for all r, then

$$g_1(t) - g_2(t) = \int_{Dom(\mathbf{R})} [g_1(t|r) - g_2(t|r)]\, d\omega(r) \geqslant 0,$$

for the same t. This proves the following proposition.

[10] This theorem does not require, in fact, the validity of the stochastic dominance relation (9) for all possible t (which is tacitly assumed in all other results; see the Convention in Section 3). Note also that this theorem is valid for SP_{AND} but not for SP_{OR}.

Theorem 4.2. *If* (13) *holds with the composition rule* SP_{AND} (*all-AND gates*), *and if* (11) *holds on some interval* $0 \leqslant t \leqslant \tau$, *for all* r, *then* $g_1(t) - g_2(t) \geqslant 0$ *on the same interval*.

5. Two general lemmas

The logic by which the results obtained in Schweickert et al. (2000) are generalized from networks with stochastically independent components to networks with stochastically interdependent but selectively influenced components can itself be generalized and presented in the form of two exceedingly simple lemmas. In these lemmas r, $\omega(r)$, and $Dom(\mathbf{R})$ are as before, while $f(t,r)$ is a measurable function[11] whose integral

$$F(t) = \int_{Dom(\mathbf{R})} f(t,r)\, d\omega(r)$$

exists for all t.

Lemma 5.1. *Let* $f(t,r) \geqslant 0$ *for some value of* t *and for all values of* r. *Then at the same value of* t,

$$F(t) = \int_{Dom(\mathbf{R})} f(t,r)\, d\omega(r) \geqslant 0.$$

The statement remains valid if both occurrences of $\geqslant 0$ *are replaced with* >0 (*or* $\leqslant 0$, *or* <0, *or* $=0$).

This is an elementary property of Lebesgue integration. We did, in fact, use a variant of this property in both theorems of the previous section. For some of the results to follow we will need another property.

Lemma 5.2. *Let, for all* r *and for some measurable subset* A *of nonnegative reals*,

$$\int_A f(t,r)\, dt \geqslant 0.$$

Then

$$\int_A F(t)\, dt = \int_A \int_{Dom(\mathbf{R})} f(t,r)\, d\omega(r)\, dt \geqslant 0,$$

provided the integral exists. The statement remains valid if both occurrences of $\geqslant 0$ *are replaced with* >0 (*or* $\leqslant 0$, *or* <0, *or* $=0$).

The proof obtains by exchanging the order of integration, whose validity is guaranteed by the Fubini theorem (see, e.g., Hewitt & Stromberg, 1965, pp. 386).

[11] with respect to the product of the conventional Lebesgue measure and ω.

6. Interaction contrasts for two selectively influenced components

Consider now the processing architecture described by (3), assuming both $\alpha \neq \varnothing$, $\beta \neq \varnothing$. As a result, both stochastic dominance relations (9) and (10) hold (see the Convention subsection of Section 3).

Denote the (conditional) distribution and density functions for $\mathbf{T}_{|\mathbf{R}=r}$ by

$$\begin{matrix} G_{ij}(t|r) = Pr(\mathbf{T} \leqslant t|\mathbf{R} = r; \alpha = \alpha_i, \beta = \beta_j), \\ g_{ij}(t|r) = \psi(\mathbf{T} = t|\mathbf{R} = r; \alpha = \alpha_i, \beta = \beta_j), \end{matrix} \quad i = 1,2, \ j = 1,2$$

and the (unconditional) distribution and density functions for $\mathbf{T}$ by

$$\begin{matrix} G_{ij}(t) = Pr(\mathbf{T} \leqslant t|\alpha = \alpha_i, \beta = \beta_j), \\ g_{ij}(t) = \psi(\mathbf{T} = t|\alpha = \alpha_i, \beta = \beta_j), \end{matrix} \quad i = 1,2, \ j = 1,2.$$

The conditional and unconditional *interaction contrasts* are defined as, respectively,

$$c(t|r) = G_{11}(t|r) - G_{12}(t|r) - G_{21}(t|r) + G_{22}(t|r)$$

and

$$c(t) = G_{11}(t) - G_{12}(t) - G_{21}(t) + G_{22}(t).$$

Clearly,

$$c(t) = \int_{Dom(\mathbf{R})} c(t|r)\, d\omega(r). \tag{14}$$

In accordance with Lemma 2.1, the validity of the following lemma immediately follows from Schweickert et al. (2000).

Lemma 6.1. *Assume the processing time architecture* (3). *Then the following statements hold true, for all* $t \geqslant 0$ *and for all* r.

1. *If the composition rule is* SP_{AND} *and if* **A** *and* **B** *are concurrent, then*

   $$c(t|r) \geqslant 0.$$

2. *If the composition rule is* SP_{OR} *and if* **A** *and* **B** *are concurrent, then*

   $$c(t|r) \leqslant 0.$$

3. *If the composition rule is* SP_{AND} *and if* **A** *and* **B** *are sequential, then*

   $$\int_t^\infty c(x|r)\, dx \leqslant 0.$$

   In particular,

   $$\int_0^\infty c(x|r)\, dx \leqslant 0.$$

4. *If the composition rule is* SP_{OR} *and if* **A** *and* **B** *are sequential, then*

   $$\int_0^t c(x|r)\, dx \geqslant 0.$$

   At the limit,

   $$\int_0^\infty c(x|r)\, dx \geqslant 0.$$

Proof. See Theorems 4 and 6 in Schweickert et al. (2000). $\square$

The unconditional statements about networks with stochastically interdependent components are obtained from this lemma by using (14) and applying Lemma 5.1 to propositions 1, 2 and Lemma 5.2 to propositions 3, 4.

Theorem 6.1. *Assume the processing time architecture* (3). *Then the following statements hold true for any* $t \geqslant 0$.

1. *If the composition rule is* SP_{AND} *and if* **A** *and* **B** *are concurrent, then*

   $$c(t) \geqslant 0.$$

2. *If the composition rule is* SP_{OR} *and if* **A** *and* **B** *are concurrent, then*

   $$c(t) \leqslant 0.$$

3. *If the composition rule is* SP_{AND} *and if* **A** *and* **B** *are sequential, then*

   $$\int_t^\infty c(x)\, dx \leqslant 0,$$

   including

   $$\int_0^\infty c(x)\, dx \leqslant 0.$$

4. *If the composition rule is* SP_{OR} *and if* **A** *and* **B** *are sequential, then*

   $$\int_0^t c(x)\, dx \geqslant 0,$$

   with the limit case

   $$\int_0^\infty c(x)\, dx \geqslant 0.$$

One can see that, under all the assumptions made, one can use $c(t)$ to distinguish the concurrent and sequential cases if one knows the gates (all-AND or all-OR), and to distinguish the all-AND case from the all-OR case if one knows whether the two influenced components are sequential or concurrent. The new achievement is that this can be done without assuming that the component times are stochastically independent.

One can also see, however, that based on this theorem one may fail to tell apart some of the four possible cases. In addition to the trivial observation that if the interaction contrast is identically zero it is compatible with all four cases, one also faces two nontrivial entanglements: the SP_{AND} case with sequential $\mathbf{A}, \mathbf{B}$ is compatible with the SP_{OR} case and concurrent $\mathbf{A}, \mathbf{B}$; whereas the SP_{OR} case with sequential $\mathbf{A}, \mathbf{B}$ is compatible with the SP_{AND} case and concurrent $\mathbf{A}, \mathbf{B}$. The following results employ an additional assumption to enable one to tell apart the first two cases.

Lemma 6.2. *Assume the processing time architecture* (3) *with the composition rule* SP_{AND}. *Let also, on some interval* $0 \leqslant t \leqslant \tau$, *either* (11) *or* (12) *hold. Then* $c(t|r) \geqslant 0$ *on the same interval, for all* r.

Proof. See Theorem 5 in Schweickert et al. (2000). □

By (14) and Lemma 5.1, we have then

Theorem 6.2. *Assume the processing time architecture* (3) *with the composition rule* SP_{AND}. *Let also, on some interval* $0 \leqslant t \leqslant \tau$, *either* (11) *or* (12) *hold. Then* $c(t) \geqslant 0$ *on the same interval.*

With the additional assumption of either (11) or (12), therefore, concurrent $\mathbf{A}, \mathbf{B}$ in an SP_{OR}-network (where the interaction contrast never exceeds 0) cannot be confused with sequential $\mathbf{A}, \mathbf{B}$ in an SP_{AND}-network: in the latter case $c(t) \geqslant 0$ prior to some moment τ (but, if the inequality is strict, it has to attain negative values on some subsequent intervals of time, in order to ensure the inequality $\int_t^\infty c(x)\,dx \leqslant 0$).

Unfortunately, no results analogous to Lemma 6.2 and Theorem 6.2 exist for SP_{OR}-networks. As a result, the problem of telling the SP_{OR} case with sequential $\mathbf{A}, \mathbf{B}$ apart from the SP_{AND} case with concurrent $\mathbf{A}, \mathbf{B}$ remains unsolved, for both independent and interdependent component times.

7. Illustrations

To illustrate results of Theorem 6.1, consider the network in Fig. 4. The gate in the network is either an AND gate or an OR gate, so the composition rule is either $(\mathbf{T}_1 + \mathbf{T}_2)\max \mathbf{T}_3$ or $(\mathbf{T}_1 + \mathbf{T}_2)\min \mathbf{T}_3$. The durations $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ are constructed to be stochastically interdependent but selectively influenced in the sense of Definition 2.1 and Lemma 2.1. Specifically, let the

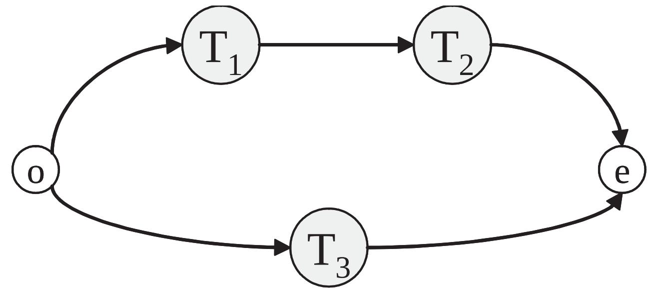

Fig. 4. The network used for illustrations of Theorem 6.1. The corresponding composition rule is $(\mathbf{T}_1 + \mathbf{T}_2) \circ \mathbf{T}_3$, where $\circ$ is max or min.

component times be generated as

$$\begin{aligned} \mathbf{T}_1 &= \gamma_1 \mathbf{S}_1 \mathbf{R}, \\ \mathbf{T}_2 &= \gamma_2 \mathbf{S}_2 \mathbf{R}, \\ \mathbf{T}_3 &= \gamma_3 \mathbf{S}_3 \mathbf{R}, \end{aligned} \tag{15}$$

where γ_i $(i = 1, 2, 3)$ are factors with numerical values, $\mathbf{R}$ is uniformly distributed, and $\mathbf{S}_i$ $(i = 1, 2, 3)$ are identically standard-gamma-distributed, with the density $t^{\mu-1}e^{-t}/\Gamma(\mu)$ (where μ is the mean and Γ is gamma function). Information on cumulative distribution functions for directed acyclic networks can be found in Fisher and Goldstein (1983).

The random variables ($\mathbf{R}$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$) are mutually independent and their distributions do not depend on the factors $\gamma_1, \gamma_2, \gamma_3$. The stochastic interdependence of the random variables $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ is due to their dependence on the common source of randomness $\mathbf{R}$. For any given value of $\mathbf{R}$, however, $\mathbf{T}_{1|\mathbf{R}=r}$, $\mathbf{T}_{2|\mathbf{R}=r}$ and $\mathbf{T}_{3|\mathbf{R}=r}$ are stochastically independent, with their (conditional) distributions selectively depending on the respective factors, $\gamma_1, \gamma_2, \gamma_3$.

Note that all the ancillary assumptions made in Section 3 are satisfied. In particular, the distribution function for $\mathbf{T}_i$ at a larger value of factor γ_i $(i = 1, 2, 3)$ dominates the distribution function for $\mathbf{T}_i$ at a smaller value of the same factor, in the sense of (9) and (10). The dominance relations (11) and (12) too can be easily checked to hold.

To selectively influence the *sequential* processes with durations $\mathbf{T}_1$ and $\mathbf{T}_2$, we varied $\gamma_1 \times \gamma_2$ in a 2×2 design and kept γ_3 fixed. Since constants are never included among the factors influencing the overall processing time, the selective influence scheme in this case is

$$(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \looparrowleft (\{\gamma_1\}, \{\gamma_2\}, \varnothing).$$

To selectively influence the *concurrent* processes with durations $\mathbf{T}_1$ and $\mathbf{T}_3$, we varied $\gamma_1 \times \gamma_3$ in a 2×2 design and kept γ_2 fixed. The selective influence scheme in this case is

$$(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \looparrowleft (\{\gamma_1\}, \varnothing, \{\gamma_3\}).$$

The results shown in Fig. 5 are computed analytically for the case when μ (the mean value for $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$) is 1.

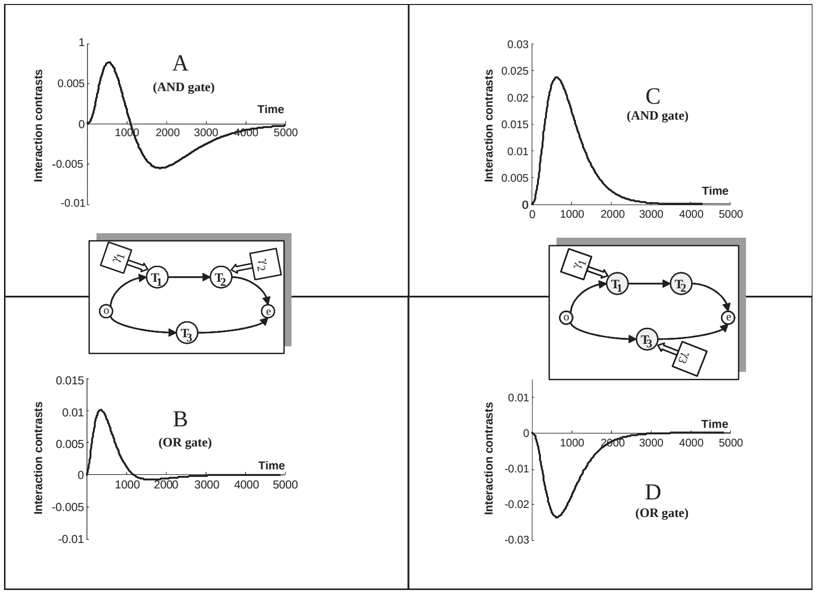

Fig. 5. Interaction contrasts for the network shown in Fig. 4. The component times are generated in accordance with (15), $\mathbf{S}_i$ $(i = 1, 2, 3)$ are identically standard-exponentially distributed (with the density e^{-t}), and $\mathbf{R}$ is uniformly distributed between 1 and 2. (A and B): Interaction contrast for selectively influenced sequential components, $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \hookleftarrow (\{\gamma_1\}, \{\gamma_2\}, \emptyset)$. Factorial design here is $\gamma_1 \times \gamma_2 = \{500, 700\} \times \{400, 600\}$ and $\gamma_3 = 1200$. (C and D): Interaction contrast for selectively influenced concurrent components, $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \hookleftarrow (\{\gamma_1\}, \emptyset, \{\gamma_3\})$. Factorial design here is $\gamma_1 \times \gamma_3 = \{500, 700\} \times \{600, 1200\}$ and $\gamma_2 = 400$. (Note that the scale of the vertical axes in the four panels is different.)

In this case the common distribution of $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ is standard exponential. When the selectively influenced components are sequential and the gate is AND (Fig. 5A), the interaction contrast begins positive, as predicted by Theorem 6.2, but later becomes negative. Since the total positive area under this curve is smaller than the total negative area, it is clear that the net area under the curve to the right of any time t is negative, in compliance with Theorem 6.1 (statement 3).

When the selectively influenced components are sequential and the gate is OR (Fig. 5B), the total positive area is clearly larger than the total negative area, whence it follows that the net area to the left of any time t is positive, in compliance with Theorem 6.1 (statement 4).

When the selectively influenced components are concurrent, then the interaction contrast is nonnegative at all times if the gate is AND (Fig. 5C), and it is nonpositive at all times if the gate is OR (Fig. 5D). This agrees with Theorem 6.1 (statements 1 and 2).

The results shown in Fig. 6 are computed by means of computer simulations (performed in EXCEL) for the case when μ is 4. For each combination of the values for $\gamma_1, \gamma_2, \gamma_3$ that was used in the simulations, the distribution function for the overall processing time was reconstructed from 2000 trials, which corresponds to a large but still manageable experiment. Each of these trials consisted of independently sampling a value of $\mathbf{R}$, a value of $\mathbf{S}_1$, a value of $\mathbf{S}_2$, and a value of $\mathbf{S}_3$ (with different trials being independent as well; see footnote 7). As in Fig. 5, the results are in good compliance with Theorem 6.1, in spite of the occasional "jitters" clearly attributable to sampling error.

8. Concluding remarks

8.1. Selective influence under stochastic interdependence

The notion of selective influence constructed in Dzhafarov (2003a) has desirable mathematical properties, some of which are mentioned in Section 2. This is not, however, the only advantage of this notion over the previous attempts to define selective influence under stochastic interdependence (Dzhafarov, 1997, 1999; Townsend, 1984; Townsend & Thomas, 1994). The additional claim made in Dzhafarov (2003a) is that the new theory has a considerably greater working power, with a greater variety of potential applications. The mathematical reason for this lies in the fact that probability distributions for interdependent (but selectively influenced) random variables and for functions thereof are obtained by integration over expressions obtained for stochastically independent conditional random variables. Integration over the domain of the conditioning random entity $\mathbf{R}$ will often preserve essential properties of the integrands, generalizing them thereby from stochastically independent variables to stochastically interdependent variables with the same pattern of selective influences. Even when this is not the case, the multiplicative decomposability of the integrands pertaining to (conditionally) stochastically

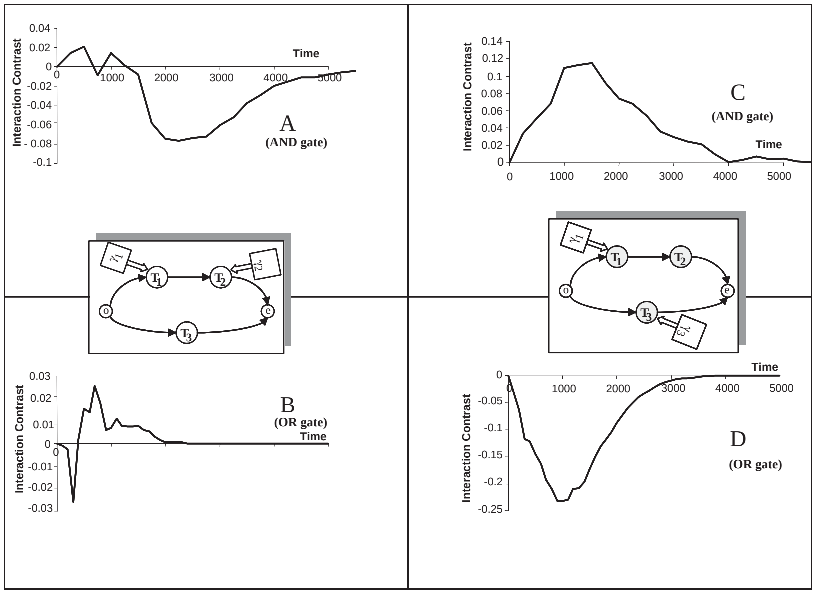

Fig. 6. Empirical interaction contrasts (based on simulations with 2000 observations per treatment) for the network shown in Fig. 4. The component times are generated in accordance with (15), but with distributions different from those used in Fig. 5: $\mathbf{S}_i$ ($i = 1, 2, 3$) are identically gamma-distributed (with the density $t^3e^{-t}/6$), and $\mathbf{R}$ is uniformly distributed between 0.00001 and 0.99999. (A and B): interaction contrast for selectively influenced sequential components, $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \hookleftarrow (\{\gamma_1\}, \{\gamma_2\}, \emptyset)$. Factorial design in (A) is $\gamma_1 \times \gamma_2 = \{200, 500\} \times \{100, 300\}$ and $\gamma_3 = 400$; in (B) it is the same except $\gamma_3 = 200$. (C and D): interaction contrast for selectively influenced concurrent components, $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \hookleftarrow (\{\gamma_1\}, \emptyset, \{\gamma_3\})$. Factorial design in (C) is $\gamma_1 \times \gamma_3 = \{200, 500\} \times \{400, 700\}$ and $\gamma_2 = 100$; in (D) it is the same except γ_3 has values $\{200, 700\}$. (Note that the scale of the vertical axes in the four panels is different.)

independent variables often greatly facilitates theoretical analysis of the resulting joint distributions.

The claim of greater applied power has been successfully tested on the traditional problem of selectively influenced components of a multivariate normal distribution (Dzhafarov, 2003a) and on the problem of stochastically interdependent random images selectively attributed to distinct stimuli. In the latter case, certain results were first obtained for independent images (Dzhafarov, 2003b) and then, using the theory of selective influence, were shown to almost immediately generalize to interdependent images (Dzhafarov, 2003c). The present paper provides a further demonstration of the same logic, and of the same effectiveness. All mathematical complexity in the proofs of our theorems is absorbed by the proofs of the results for independent component times (given in Schweickert et al., 2000), their subsequent generalization to interdependent component times becoming straightforward.

8.2. Interaction contrast for serial–parallel networks

The main results obtained in this paper need no summarizing as they are clearly presented as statements of Theorem 6.1.

One of the weaknesses of these results is that they are all formulated as nonstrict inequalities. For example, none of the considered cases prevents $c(t)$ from being identically zero. To investigate the conditions under which the obtained inequalities are strict, however, is more tedious than complex. It requires postulating certain intervals upon which the stochastic dominance inequalities (9) and (10) are strict, followed by a meticulous analysis relating these intervals to the intervals upon which distribution functions for various component times are increasing.

Another weakness of our results, already mentioned, is that they do not allow to separate the case of concurrent selectively influenced components in a network with all-AND gates from the case of sequential selectively influenced components in a network with all-OR gates.

Finally, nothing is known about serial–parallel networks that may contain both AND and OR gates.

8.3. On Wheatstone bridge networks

Any directed acyclic network that is not serial–parallel has a Wheatstone bridge subnetwork imbedded in it (Dodin, 1985; Kaerkes & Mohring, 1978). An example of a Wheatstone bridge is given in Fig. 2C, with the composition rule

$$(\mathbf{T}_2 + \mathbf{T}_5)\circ(\mathbf{T}_1\circ(\mathbf{T}_2 + \mathbf{T}_3) + \mathbf{T}_4). \qquad (16)$$

Without getting into details, the Wheatstone bridge configuration is of interest not only because it is a directed acyclic network that is not serial–parallel, but also because some of the sequential pairs of processes in it (in our example, $\mathbf{T}_2, \mathbf{T}_4$) may sometimes behave like concurrent pairs (Schweickert, 1978).[12]

Schweickert and Giorgini (1999) report several results related to (16), both analytic and simulation-based. These results are predicated on the assumption of stochastic independence, and all of them can be shown to generalize to stochastically interdependent but selectively influenced components, using the same logic as in Section 5. We relegate details to an appendix because the analysis in Schweickert and Giorgini (1999) is confined to one specific Wheatstone bridge configuration only, described by (16). It is clear, however, that as it is for serial–parallel networks, the main difficulty in dealing with networks containing Wheatstone bridges lies in the case of stochastically independent components, the extension to interdependent selectively influenced components being relatively straightforward.

8.4. New developments

Two papers available too recently for us to discuss in detail use interaction contrasts based on entire distribution functions to uncover process arrangements. Townsend and Fific (in press) found evidence that some individuals change from serial to parallel memory search when the delay between the memory set and probe is increased. Schweickert, Fortin and Sung (2003) found that visual search and reproduction of a time interval can go on concurrently. Our results here broaden the class of systems to which the conclusions of these recent papers apply.

Acknowledgments

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Appendix A. On Wheatstone bridge network in Fig. 2C

Let

$$W(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4, \mathbf{T}_5) = (\mathbf{T}_2 + \mathbf{T}_5) \circ (\mathbf{T}_1 \circ (\mathbf{T}_2 + \mathbf{T}_3) + \mathbf{T}_4).$$

We will say that the composition rule is W_{AND} (or W_{OR}) if $\circ$ in this expression is max (respectively, min). If a statement applies to both, we write $W_{\mathrm{AND/OR}}$. We consider this composition rule in combination with the selective influence assumption

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3) \hookleftarrow (\alpha, \beta, \varnothing, \varnothing, \varnothing),$$

where $(\mathbf{A}, \mathbf{B})$ is some pair of processing times chosen from $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4, \mathbf{T}_5)$ and $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ are the remaining three. We make the same ancillary assumptions as before (Section 3).

A.1. Concurrent pairs of processes

Let $(\mathbf{A}, \mathbf{B})$ be one of the pairs $(\mathbf{T}_1, \mathbf{T}_2)$, $(\mathbf{T}_1, \mathbf{T}_3)$, $(\mathbf{T}_3, \mathbf{T}_5)$, $(\mathbf{T}_1, \mathbf{T}_5)$, $(\mathbf{T}_4, \mathbf{T}_5)$. Then the following statements hold true, for all $t \geqslant 0$:

1. If the composition rule is W_{AND}, then

   $$c(t) \geqslant 0.$$

2. If the composition rule is W_{OR}, then

   $$c(t) \leqslant 0.$$

The proof is obtained by applying Lemma 5.1 to the observation that these statements hold true for any given value r of $\mathbf{R}$ (which in turn follows from the proofs given in Schweickert & Giorgini, 1999, for stochastically independent component times).

A.2. Sequential pairs of processes

Let $(\mathbf{A}, \mathbf{B})$ be one of the pairs $(\mathbf{T}_1, \mathbf{T}_4)$, $(\mathbf{T}_2, \mathbf{T}_3)$, $(\mathbf{T}_2, \mathbf{T}_5)$, $(\mathbf{T}_3, \mathbf{T}_4)$. Then the following statements hold true, for all $t \geqslant 0$.

1. If the composition rule is W_{AND}, then

   $$\int_t^\infty c(x)\, dx \leqslant 0.$$

2. If the composition rule is W_{OR}, then

   $$\int_0^t c(x)\, dx \geqslant 0.$$

The proof is obtained by applying Lemma 5.2 to the observation (following from Schweickert & Giorgini, 1999) that these statements hold true for any given value r of $\mathbf{R}$.

A.3. The "special pair" $(\mathbf{T}_2, \mathbf{T}_4)$

The pair $(\mathbf{T}_2, \mathbf{T}_4)$ is "special" because in the composition rule W the two component times are related to each other by both operation $+$ and operation $\circ$ (max or min). Indeed, for constant values of other component times, W can be written as $[f(\mathbf{T}_2) + \mathbf{T}_4] \circ g(\mathbf{T}_2)$.

For $(\mathbf{A}, \mathbf{B}) = (\mathbf{T}_2, \mathbf{T}_4)$ Schweickert and Giorgini (1999) present no analytic results but their simulation

[12] Observe that, in (16), $\mathbf{T}_2$ and $\mathbf{T}_4$ are related to each other by both operation $+$ and operation $\circ$ (max or min). Indeed, for constant values of other component times, (16) can be written as $[f(\mathbf{T}_2) + \mathbf{T}_4] \circ g(\mathbf{T}_2)$.

study suggests the following. If the composition rule is W_{AND}, and if all the component times are stochastically independent, then

$$\int_0^t c(x)\,dx \geqslant 0,$$

for all $t \geqslant 0$. If this statement is true, then it immediately generalizes to stochastically interdependent networks with selectively influenced components.

Although no simulation (or analytic) results are available for the composition rule W_{OR}, it is plausible to conjecture (see Schweickert & Giorgini, 1999, for details) that in this case

$$\int_t^\infty c(x)\,dx \leqslant 0,$$

for all $t \geqslant 0$. Again, if this statement is true, its generalization to stochastically interdependent networks must be true as well.

A.4. Additional results

Theorem 6.2 has several analogues for our Wheatstone bridge configuration.

1. For the composition rule $W_{\mathrm{AND/OR}}$, let $\mathbf{A} = \mathbf{T}_2$ and $\mathbf{B} = \mathbf{T}_3$ or $\mathbf{B} = \mathbf{T}_5$. If (11) holds on some interval $0 \leqslant t \leqslant \tau$, then

   $c(t) \geqslant 0,$

   on the same interval.
2. For the composition rule $W_{\mathrm{AND/OR}}$, let $\mathbf{B} = \mathbf{T}_4$ and $\mathbf{A} = \mathbf{T}_1$ or $\mathbf{A} = \mathbf{T}_3$. If (12) holds on some interval $0 \leqslant t \leqslant \tau$, then

   $c(t) \geqslant 0,$

   on the same interval.
3. For the composition rule W_{AND}, if $(\mathbf{A}, \mathbf{B}) = (\mathbf{T}_2, \mathbf{T}_4)$ and if both (11) and (12) hold on some interval $0 \leqslant t \leqslant \tau$, then

   $c(t) \geqslant 0,$

   on the same interval.
4. For the composition rule W_{OR}, if $(\mathbf{A}, \mathbf{B}) = (\mathbf{T}_2, \mathbf{T}_4)$ and if both (11) and (12) hold on some interval $0 \leqslant t \leqslant \tau$, then

   $c(t) \leqslant 0,$

   on the same interval.

The proof consists in applying Lemma 5.1 to the observation (following from the proofs given in Schweickert & Giorgini, 1999, for stochastically independent component times) that these statements hold true for any given value r of $\mathbf{R}$.

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