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Regular Minimality and Thurstonian-type modeling

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ABSTRACT

A Thurstonian-type model for pairwise comparisons is any model in which the response (e.g., "they are the same" or "they are different") to two stimuli being compared depends, deterministically or probabilistically, on the realizations of two randomly varying representations (perceptual images) of these stimuli. The two perceptual images in such a model may be stochastically interdependent but each has to be selectively dependent on its stimulus. It has been previously shown that all possible discrimination probability functions for same–different comparisons can be generated by Thurstonian-type models of the simplest variety, with independent percepts and deterministic decision rules. It has also been shown, however, that a broad class of Thurstonian-type models, called "well-behaved" (and including, e.g., models with multivariate normal perceptual representations whose parameters are smooth functions of stimuli) cannot simultaneously account for two empirically plausible properties of same–different comparisons, Regular Minimality (which essentially says that "being least discriminable from" is a symmetric relation) and nonconstancy of the minima of discrimination probabilities (the fact that different pairs of least discriminable stimuli are discriminated with different probabilities). These results have been obtained for stimulus spaces represented by regions of Euclidean spaces. In this paper, the impossibility for well-behaved Thurstonian-type models to simultaneously account for Regular Minimality and nonconstancy of minima is established for a much broader notion of well-behavedness applied to a much broader class of stimulus spaces (any Hausdorff arc-connected ones). The universality of Thurstonian-type models with independent perceptual images and deterministic decision rules is shown (by a simpler proof than before) to hold for arbitrary stimulus spaces.

1. Introduction

If pairs of stimuli (x, y) are presented to an observer with the request to determine whether x and y are the same or different ("overall" or in a specified respect, such as brightness or shape), the observer's response to a given stimulus pair (x, y) will generally be different in different trials. If one assumes, as an approximable but unattainable idealization, that all external factors other than (x, y) are kept constant, precise physical identities of x and y are still insufficient to predict the response. At best one can posit that (x, y) determines the probabilities of the responses,

$$(x, y) \mapsto \begin{cases} \Pr[\text{`different'}] = \psi(x, y), \\ \Pr[\text{`same'}] = 1 - \psi(x, y). \end{cases}$$

The most common way of explaining the probabilistic nature of discrimination judgments is to employ the modeling scheme first proposed by Thurstone (1927a,b): stimuli x and y are mapped into random variables taking their values in some space of internal representations (perceptual images), and the response is determined by the values (realizations) of these two random variables in a given trial. Thurstone's original theory was designed for "greater–less" comparisons, but its adaptation to the "same–different" judgments is straightforward, in principle if not in technical details (Ennis, 1992; Ennis, Palen, & Mullen, 1988; Luce & Galanter, 1963; Suppes & Zinnes, 1963; Thomas, 1996, 1999; Zinnes & MacKey, 1983). Fig. 1 illustrates the model proposed by Luce and Galanter (1963), the simplest and probably the earliest of Thurstonian-type models for same–different comparisons.

Dzhafarov (2003a,b) argued that since we do not know what mathematical structures are appropriate for describing perceptual representations of stimuli (e.g., it is far from obvious that stimuli presented for same–different comparisons are represented by vectors of "features" with real-valued components), it is profitable to consider Thurstonian-type models on a very high level of abstraction, with perceptual images of stimuli being random

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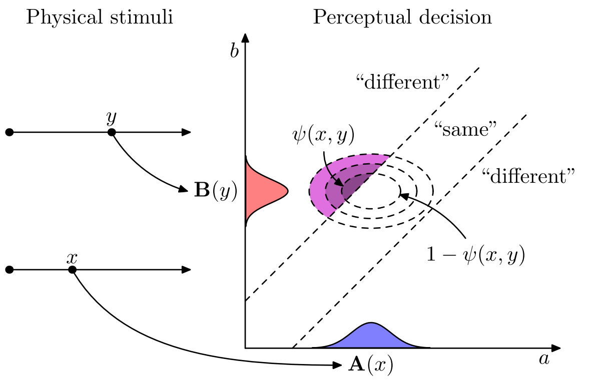

Fig. 1. A prototypical example of a Thurstonian-type model, proposed by Luce and Galanter (1963). When a stimulus pair (x, y) is presented for a same–different judgment, x is mapped into $\mathbf{A}(x)$ and y into $\mathbf{B}(y)$, two independent random variables normally distributed on the set of reals. The notation $\mathbf{A}(x)$ indicates that the parameters of the variable $\mathbf{A}$ (in this case, its mean and variance) depend on x; and analogously for $\mathbf{B}(y)$. The response in a given trial is "different" if and only if the realization (a, b) of $(\mathbf{A}(x), \mathbf{B}(y))$ in this trial falls within the area labeled "different" in the figure, i.e., if $|a - b|$ exceed a certain constant ε (one half of the vertical, or horizontal, separation between the two dashed lines). Thus, the probability $\psi(x, y)$ of judging x and y different (i.e., the discrimination probability function "generated" by the Luce–Galanter model) equals the probability of $|\mathbf{A}(x) - \mathbf{B}(y)| > \varepsilon$.

entities in probability spaces of arbitrary nature.[1] Thurstonian-type models then can be classified into four groups ("varieties"), depending on

(a) whether the response is evoked by realizations of random variables *deterministically* or *probabilistically* (*Det* vs *Prob* varieties), and
(b) whether the two random variables are stochastically *independent* or *interdependent* (*Ind* vs *Int* varieties).[2]

The four combinations of these characteristics can then be abbreviated as *DetInd*, *DetInt*, *ProbInd*, and *ProbInt* varieties of Thurstonian-type models. It was shown in Dzhafarov (2003a) that if one imposes no a priori restrictions on the hypothetical random variables, their dependence on stimuli, and possible decision rules, then

> *every discrimination probability function $\psi(x, y)$ can be generated by an appropriately constructed Thurstonian-type model of the simplest, DetInd variety.*

We will refer to this result as the "*DetInd universality statement*". In this paper it will be proved in a much simpler way than in Dzhafarov (2003a), as a corollary to a new, also surprisingly simple result, referred to as the "*Prob-Det equivalence statement*":

> *every ProbInt (as a special case, ProbInd) model is equivalent, in the sense of generating the same discrimination probability function, to a DetInt (respectively, DetInd) model.*

This result is of interest for its own sake, as the assumption that the mapping of random perceptual images of stimuli into possible responses is probabilistic rather than determined by a rule seems to be defeating the very purpose of Thurstonian-type models, to explain the probabilistic nature of responses by stochasticity in the mapping of stimuli into perceptual representations.

The main result established in Dzhafarov (2003a,b) is, however, of a very different nature. It was shown there that

> *if stimuli can change "continuously" and their hypothetical representations in a Thurstonian-type model are sufficiently "well-behaved" in response to these changes, then the generated discrimination probability functions cannot simultaneously have two properties one of which (Regular Minimality) is an intuitively plausible and seemingly innocuous constraint contradicted by no available empirical evidence, and the second (nonconstancy of minima) can be viewed as a well-established empirical fact.*

Regular Minimality is about the relation of being "least discriminable from". It says that in same–different judgments of ordered stimulus pairs (x, y), every point x has a unique least discriminable from it stimulus y (and vice versa),[3] and that, moreover, x is least discriminable from y if and only if y is least discriminable from x. Nonconstancy of minima is the observation that if y_1 and y_2 are least discriminable from, respectively, x_1 and x_2 (or vice versa), the values of $\psi(x_1, y_1)$ and $\psi(x_2, y_2)$ are generally not the same.[4] The theoretical result in question says that if $\psi(x, y)$ is generated by a well-behaved Thurstonian-type model then these two properties cannot be satisfied simultaneously (i.e., $\psi(x_1, y_1)$ must always equal $\psi(x_2, y_2)$ or else Regular Minimality is not satisfied).

This is of interest because Thurstonian-type modeling in psychophysics is routine,[5] and models constructed to fit empirical data are likely to be well-behaved in the technical sense of the present paper. On the other hand, Regular Minimality is an empirical generalization which conforms with time-honored practices of psychophysics (where the relations "y matches x" and "x matches y" are always tacitly considered equivalent),[6] and it may be viewed as a candidate for one of the most basic properties of discrimination (Dzhafarov, 2002; Dzhafarov & Colonius, 2006). At the same time, to the extent one can reliably determine the least discriminable counterparts $y_1, y_2, \ldots$ for several different stimuli $x_1, x_2, \ldots$, the nonconstancy of the discrimination levels across different pairs $(x_1, y_1), (x_2, y_2), \ldots$ seems well established (see, e.g., Dzhafarov & Colonius, 2005). While we do not address in this paper the empirical truth of well-behaved Thurstonian-type models versus Regular Minimality, the discussion of their relationship is amply motivated.

The notion of well-behavedness for a Thurstonian-type model presupposes that stimuli x and y vary continuously (e.g., both take their values on an interval of reals) and its intuitive meaning is as

[1] In this paper we use the terms "random entity" and "random variable" as synonyms, indicating a measurable function from one probability space to another. In other contexts (see, e.g., Kujala & Dzhafarov, 2008b, it may be useful to reserve the term "random variable" for random entities whose codomain is an interval of real numbers endowed with Borel sigma algebra.

[2] If the two variables are interdependent, one should still be able to selectively attribute one of them to x and the other to y: thus, the perceptual image of x is presented as $\mathbf{A}(x)$ rather than $\mathbf{A}(x, y)$. In accordance with the general theory of selective influence (Dzhafarov, 2003c; Dzhafarov & Gluhovsky, 2006; Kujala & Dzhafarov, 2008b), the perceptual representations $\mathbf{A}(x)$ and $\mathbf{B}(y)$ can be treated as conditionally independent, given some common source of randomness $\mathbf{C}$ (see Section 3.2 for details).

[3] The uniqueness here is understood "up to psychological equivalence", as explained in Section 2.1.

[4] In this formulation the property in question is independent of Regular Minimality: it is sufficient for some x-values to have least discriminable from them stimuli y or vice versa. In the formal development to follow, however, we always assume Regular Minimality when we speak of (non)constancy of minima. This leads to no loss of generality as our primary aim is to look at the (in)compatibility of the two properties in discrimination probability functions generated by Thurstonian-type models.

[5] In Luce (1977) we read that Thurstone's model is the "essence of simplicity …this conception of internal representation of signals is so simple and so intuitively compelling that no one ever really manages to escape from it. No matter how one thinks about psychophysical phenomena, one seems to come back to it".

[6] It should be noted that in the psychophysical literature the matching relation is more often defined in terms of greater–less rather than same–different comparisons. It seems natural, however, to posit that if x has any y which subjectively matches it, then this y should be judged different from x less frequently than other values of y. Available empirical evidence involving same–different comparisons is not vast (for an overview see Dzhafarov, 2002, 2003a,b; Dzhafarov & Colonius, 2005, 2006), and there is always the possibility that violations of Regular Minimality are too small to be detected empirically (see the exchange between Ennis, 2006, and Dzhafarov, 2006).

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follows: as x and y change, the joint distribution of the perceptual representations $\mathbf{A}(x)$ and $\mathbf{B}(y)$ in the model changes continuously and the rate of its change is bounded. In Dzhafarov (2003a,b), it was assumed that the stimuli belong to an open connected area of n-dimensional Euclidean space (e.g., x and y are colors represented by CIE coordinates, or they are tones represented by amplitudes and frequencies). The well-behavedness of a Thurstonian-type model was formulated in terms of continuity and directional derivatives of probabilities with which random perceptual representations fall within various subsets of the perceptual space. Thus, in reference to Fig. 1, the model it depicts will be well-behaved if, e.g., the means and variances of the two normal distributions are smooth functions of respective stimuli.

In this paper, the analysis of well-behaved Thurstonian-type models is extended from n-dimensional Euclidean spaces to a much broader class of all Hausdorff arc-connected stimulus spaces,[7] and the dependence of random perceptual representations on stimuli in well-behaved models is greatly generalized and refined. In fact, unlike in Dzhafarov (2003a,b), we do not attempt to develop a single general definition of a well-behaved Thurstonian-type model. Instead, we define this notion as a property of discrimination probability functions $\psi(x, y)$, and then present certain classes of Thurstonian-type models which generate discrimination probabilities with this property.

2. Basics of discrimination probability functions

In this section, we closely follow Kujala and Dzhafarov (2008a, 2009) to introduce notation, notions, and basic facts pertaining to discrimination probability functions on a broad class of stimulus spaces.

2.1. Discrimination probabilities and their properties

A *discrimination probability function* (or *discrimination function*, for short) is defined as

$$\psi(x, y) = \Pr[x \in X \text{ and } y \in Y \text{ are judged to be different}]. \quad (1)$$

The sets of stimuli X and Y are distinguished because they represent distinct *observation areas* (Dzhafarov, 2002, 2006): thus, if stimuli x and y are, say, auditory tones taking their values in the same sets of amplitudes and frequencies, one of these stimuli has to be presented chronologically first (X) and the other second (Y). More generally, stimuli $x \in X$ and $y \in Y$ are characterized by their variable properties (e.g., amplitude and frequency in the case of tones, or CIE coordinates in the case of colors), fixed properties common to all stimuli (e.g., duration of tones, or shape of color patches), and fixed properties which are different for x-stimuli and y-stimuli (in our examples, chronological position of a tone, first–second, or spatial location of a color, say, left–right). The latter define two distinct observation areas.[8] The variable characteristics of x and y stimuli typically form identical sets and are referred to as *stimulus values*. By abuse of language the term "stimulus" is often used interchangeably with "stimulus value". Thus, two tones x and y can be said to be physically equal, $x = y$, even though it is only their amplitudes and frequencies (and not temporal intervals containing them) that are equal. For a more detailed discussion, see Dzhafarov and Colonius (2005, 2006).

We assume in this paper that X and Y have the following properties:

1. Neither set contains two distinct stimuli which are "psychologically equivalent", i.e., indistinguishable by means of values of ψ. This means that if $\psi(x_1, y) = \psi(x_2, y)$ for all $y \in Y$, then $x_1 = x_2$, and if $\psi(x, y_1) = \psi(x, y_2)$ for all $x \in X$, then $y_1 = y_2$. This property can always be achieved by relabeling stimuli so that any two psychologically equivalent stimuli in the same observation area are assigned one and the same label (see Dzhafarov & Colonius, 2006, 2007, for details). Thus, each color represented by given values of CIE coordinates represents in fact an infinity of metameric radiometric spectra.
2. Both sets are endowed with topologies which make them Hausdorff arc-connected spaces.[9] The Hausdorff property means that any two distinct points have disjoint open neighbourhoods: it follows, in particular, that no sequence of points in a Hausdorff space can converge to more than one limit point. The arc-connectedness formalizes the intuitive notion of a "continuous" stimulus space: for any $x, x' \in X$ (the definition for $y, y' \in Y$ being analogous) one can find a *path* connecting them, a continuous mapping $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = x'$. This path is called an *arc* if its image $f([0, 1])$ consists of a single point (the point $x = x'$), or if the mapping $f : [0, 1] \to f([0, 1])$ is homeomorphic, i.e., if it has the continuous inverse $f^{-1} : f([0, 1]) \to [0, 1]$.[10] The image $f([0, 1])$ of a path (hence also of an arc), as a subspace of X with relative topology, is itself a Hausdorff arc-connected space. In particular, for any path $f : [0, 1] \to X$ one can find an arc $g : [0, 1] \to X$ such that $g([0, 1]) \subset f([0, 1])$ with $f(0) = g(0)$ and $f(1) = g(1)$ (which is the reason a path-connected Hausdorff space can be called arc-connected).

A discrimination function ψ is said to satisfy the law of *Regular Minimality* if there is a bijection $h : X \to Y$ such that

$$\psi(x, y) > \max\{\psi(x, h(x)), \psi(h^{-1}(y), y)\},$$

for all $x \in X$ and $h(x) \neq y \in Y$.

The function h is called the *PSE function* (PSE standing for *Point of Subjective Equality*). If h is a *homeomorphism* (i.e., both h and h^{-1} are continuous), then we say that ψ satisfies the law of Regular Minimality with a homeomorphic PSE function.

Assuming Regular Minimality, the function $\omega_h : X \to [0, 1]$ defined by

$$\omega_h(x) = \psi(x, h(x))$$

is called the *minimum level function*. Clearly, the function

$$\omega_{h^{-1}}(y) = \psi(h^{-1}(y), y)$$

[7] The necessity or at least desirability of considering stimuli on a more abstract level than representations by real-valued features is discussed in Dzhafarov and Colonius (2005). Put briefly, conventional descriptions of stimuli are both incomplete and non-unique, and it is desirable therefore to ensure that one's theoretical considerations do not critically depend on one's choice of such a description. The level of Hausdorff arc-connected spaces ensures that our considerations apply to any description of stimuli (by features, functions, functional decompositions, etc.) in which the stimuli can change "continuously".

[8] We confine our consideration to pairs of stimuli that belong to *two fixed* observation areas. Presentation paradigms involving multiple observation areas (e.g., pairs of colors which, in a given trial, may occupy any two of k different locations) are outside the scope of this paper.

[9] For the topological notions and facts mentioned in this paragraph see, e.g., Hocking and Young (1961).

[10] The inclusion of paths with singleton images into the class of arcs is necessary to consider every point arc-connected to itself. This inclusion can be achieved in a more natural way if one defines a path as a continuous mapping $f : [a, b] \to X$, where $[a, b]$ is an interval of reals: then $f : [a, a] \to \{f(a)\}$ is formally a homeomorphic path, hence an arc. Another (equivalent) way is to define an arc as a path $f : [0, 1] \to X$ such that for any $s \in [0, 1]$, $f^{-1}(f(s))$ is an interval (necessarily closed) in [0, 1]: then $f : [0, 1] \to \{x\}$ is an arc.

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has the same graph as $\omega_h(x)$ and can also be called the minimum level function. We need not differentiate between these two forms.

The function ψ is said to have *(non)constant minima* if its minimum level function ω_h is (non)constant on X.

2.2. Arcwise parametrization and well-behavedness of ψ

In this paper, we are exclusively interested in discrimination functions $\psi : X \times Y \to [0, 1]$ which are continuous with respect to the product topology of X and Y (which are Hausdorff arc-connected topological spaces) and have homeomorphic PSE functions h. As shown in Kujala and Dzhafarov (2008a), the continuity of ψ, if the latter satisfies Regular Minimality, does not imply the continuity of h or h^{-1} (although it is also shown there that cases with non-homeomorphic h are of an "aberrant" nature).[11]

Let Z denote either X or Y, and $p, q \in Z$ be two distinct points. It is convenient to present an arc $z : [0, 1] \to Z$ with $z(0) = p$ and $z(1) = q$ as z_p^q, to indicate its endpoints and to distinguish it from points in Z (even if the image of the arc consists of the single point $p = q$). So we will speak of arcs z_p^q with points $z(t)$, $t \in [0, 1]$ (of course, the notation z_p^q itself does not determine the arc, only its endpoints). To distinguish an arc as a mapping, $z_p^q : [0, 1] \to Z$, from its image $z_p^q([0, 1])$ in Z, we will denote the image $\left[z_p^q\right]$. Clearly, different arcs z_p^q and $\widetilde{z}_p^q$ may have the same image, $\left[z_p^q\right] = \left[\widetilde{z}_p^q\right]$.

Given any two arcs $x_u^{u'} : [0, 1] \to X$ and $y_v^{v'} : [0, 1] \to Y$ (with endpoints u, u' and v, v', respectively), the function

$$\varphi(s, t) = \psi(x(s), y(t)) \tag{2}$$

is called an *arc-parametrized facet* (*AP-facet*, for short) of ψ. Since ψ is continuous, $\varphi(s, t)$ is continuous (hence uniformly continuous) on $[0, 1] \times [0, 1]$.

Given an AP-facet φ, we use the following notation for finite differences of the first and second order. For any $s, s', t, t' \in [0, 1]$,

$$\begin{aligned} \Delta^1_{s'}\varphi(s, t) &= \varphi(s', t) - \varphi(s, t), \\ \Delta^2_{t'}\varphi(s, t) &= \varphi(s, t') - \varphi(s, t), \end{aligned} \tag{3}$$

with the superscripts referring to the position of the arguments changed. Analogously,

$$\begin{aligned} \Delta^{12}_{(s',t')}\varphi(s, t) &= \Delta^1_{s'}\Delta^2_{t'}\varphi(s, t) = \Delta^2_{t'}\Delta^1_{s'}\varphi(s, t) \\ &= \varphi(s', t') - \varphi(s', t) - \varphi(s, t') + \varphi(s, t). \end{aligned} \tag{4}$$

Another notation convention: we use double arrows $(s', t') \rightrightarrows (s, t)$ to indicate that s' and t' approach, respectively, s and t from the same side. Specifically:

$$\begin{aligned} &(s', t') \rightrightarrows (s, t)+ \text{ means } s' \to s+ \text{ and } t' \to t+, \\ &(s', t') \rightrightarrows (s, t)- \text{ means } s' \to s- \text{ and } t' \to t-, \\ &(s', t') \rightrightarrows (s, t)\pm \text{ means one of the two: } \begin{array}{l}(s', t') \rightrightarrows (s, t)+, \\ (s', t') \rightrightarrows (s, t)-,\end{array} \\ &(s', t') \rightrightarrows (s, t) \text{ means } s' \to s \text{ and } t' \to t \\ &\quad \text{and } (s' - s)(t' - t) \ge 0. \end{aligned} \tag{5}$$

Definition 2.1. Given a continuous discrimination function $\psi : X \times Y \to [0, 1]$ and a pair of arc images, $\left[x_u^{u'}\right]$ and $\left[y_v^{v'}\right]$, we say that the restriction $\psi | \left[x_u^{u'}\right] \times \left[y_v^{v'}\right]$ of ψ is *well-behaved* on $\left[x_u^{u'}\right]$ if, for some parametrizations $x_u^{u'} : [0, 1] \to \left[x_u^{u'}\right]$ and $y_v^{v'} : [0, 1] \to \left[y_v^{v'}\right]$, the resulting AP-facet φ of ψ has the following properties[12]:

($\mathcal{R}1$) for all $(s, t) \in [0, 1] \times [0, 1]$ except for an at most denumerable set,

$$\limsup_{(s',t') \rightrightarrows (s,t)} \left| \frac{\Delta^{12}_{(s',t')}\varphi(s, t)}{s' - s} \right| < \infty; \tag{6}$$

($\mathcal{R}2$) for almost all $s \in [0, 1]$ and almost all $t \in [0, 1]$,[13]

$$\lim_{(s',t') \rightrightarrows (s,t)\pm} \frac{\Delta^{12}_{(s',t')}\varphi(s, t)}{s' - s} = 0, \tag{7}$$

where the choice of $+$ or $-$ may depend on (s, t).

The definition of a restriction $\psi | \left[x_u^{u'}\right] \times \left[y_v^{v'}\right]$ well-behaved on $\left[y_v^{v'}\right]$ is obtained by replacing the quotient in (6) and (7) with

$$\frac{\Delta^{12}_{(s',t')}\varphi(s, t)}{t' - t}.$$

Definition 2.2. A continuous discrimination function $\psi : X \times Y \to [0, 1]$ is *well-behaved on a pair of points* $(u, u') \in X \times X$ *with respect to a homeomorphism* $h : X \to Y$ if, for some arc $x_u^{u'}$, the restriction $\psi | \left[x_u^{u'}\right] \times h\left(\left[x_u^{u'}\right]\right)$ is well-behaved on at least one of the two arc images, $\left[x_u^{u'}\right]$ or $h\left(\left[x_u^{u'}\right]\right)$. The well-behavedness of ψ on a pair $(v, v') \in Y \times Y$ with respect to a homeomorphism $g : Y \to X$ is defined analogously.

Clearly, ψ is well-behaved on $\left(u, u'\right) \in X \times X$ with respect to a homeomorphism h if and only if ψ is well-behaved on $\left(h(u), h(u')\right) \in Y \times Y$ with respect to the homeomorphism h^{-1}. It is also easy to see that any ψ is well-behaved on any pair (u, u) of identical points with respect to any homeomorphism (choose $x : [0, 1] \to \{u\}$, $y : [0, 1] \to \{h(u)\}$ and observe the compliance with $\mathcal{R}1$ and $\mathcal{R}2$ of Definition 2.1).

This "two-point" version of the notion of well-behavedness is sufficient for the needs of this paper. For completeness, however, we should mention the global notion of well-behaved functions, as proposed in Kujala and Dzhafarov (2008a) and generalized in Kujala and Dzhafarov (2009).

Definition 2.3. Given a continuous discrimination function $\psi : X \times Y \to [0, 1]$ and a homeomorphism $h : X \to Y$, let $E(\psi, h)$ be the set of all pairs $(u, u') \in X \times X$ on which ψ is well-behaved with respect to h. The function ψ is said to be (globally) *well-behaved with respect to* h if the only topologically and transitively closed[14] subset of $X \times X$ containing $E(\psi, h)$ is $X \times X$ itself.

[11] The continuity of h and h^{-1} had been part of the original formulation of Regular Minimality (in Dzhafarov, 2002, 2003a), but the formulation was made more general (referring to any bijective h) in subsequent publications. In most of these, later publications h is transformed into an identity function by means of a so-called canonical transformation of ψ. We adopt a "compromise" approach in which Regular Minimality is formulated in complete generality and the homeomorphic nature of the PSE functions is stipulated additionally.

[12] According to Lemma 5 of Kujala and Dzhafarov (2008a, p. 125), these conditions are insensitive to the parametrization of $y_v^{v'} : [0, 1] \to \left[y_v^{v'}\right]$; it can always be chosen arbitrarily.

[13] "Almost all" here refers to the Lebesgue measure on [0, 1].

[14] A set $S \subset X \times X$ is *transitively closed* if $(a, b), (b, c) \in S$ implies $(a, c) \in S$. As both topological and transitive closedness is preserved in intersections, the smallest subset with these properties containing $E(\psi, h)$ is well-defined.

Obviously, the definition can be equivalently given in terms of pairs (v, v') in $Y \times Y$ and the homeomorphism h^{-1}.

Theorem 2.1. *Let $\psi : X \times Y \to [0, 1]$ be a continuous discrimination function subject to Regular Minimality, with a homeomorphic PSE function h. If ψ is well-behaved on a pair $(u, u') \in X \times X$ with respect to h, then $\psi(u, h(u)) = \psi(u', h(u'))$.*

A proof of this statement is given in Kujala and Dzhafarov (2008a, Theorem 5).[15]

Theorem 2.1 makes it unnecessary for us to develop an independent definition of well-behavedness for Thurstonian-type models. Instead we can employ the following strategy. Suppose there is a property $\mathcal{P}(x_u^{u'}, y_v^{v'})$ of Thurstonian-type models (with $u, u' \in X$, $v, v' \in Y$) such that if a model T has this property, then, for the arcs $x_u^{u'}$ and $y_v^{v'}$, the model generates an AP-facet φ which satisfies Properties $\mathcal{R}1$ and $\mathcal{R}2$ of Definition 2.1. Then, by some abuse of terminology, we can call this property $\mathcal{P}$ a *well-behavedness condition* (or *constraint*), and we can say that a model T with this property is *well-behaved in the sense of this condition*. In particular, let a model T satisfy a well-behavedness condition $\mathcal{P}(x_u^{u'}, y_v^{v'})$ for arcs $x_u^{u'}$ and $y_v^{v'}$ such that $\left[y_v^{v'}\right] = h\left(\left[x_u^{u'}\right]\right)$, where h is a homeomorphism $X \to Y$; and let ψ be a continuous discrimination function subject to Regular Minimality with the PSE function h, such that $\psi(u, h(u)) \neq \psi(u', h(u'))$. Then, due to Theorem 2.1, we know that ψ cannot be generated (accounted for) by the model T.

It must be clear that this approach is of interest insofar as the property $\mathcal{P}$ from which we derive the compliance with $\mathcal{R}1$ and $\mathcal{R}2$ is shared by a broad class of models that are likely to be constructed to account for empirical data.

3. Basics of Thurstonian-type models

Here, we introduce notation and terminology pertaining to Thurstonian-type models of Det variety, with deterministic mapping of pairs of perceptual images into responses "same" and "different". We will postpone the introduction of the models of Prob variety (in which a given pair of perceptual images maps into responses "same" and "different" probabilistically) until Section 6.

3.1. Thurstonian-type models of DetInd variety

A Thurstonian-type model with *independent perceptual images and deterministic decision rule* (DetInd variety) is defined as

$\{A_x, B_y, \mathfrak{S}\}$,

or more explicitly,

$$\left\{\left\{X, \mathfrak{A}, \Sigma_A, \{A_x\}_{x\in X}\right\}, \left\{Y, \mathfrak{B}, \Sigma_B, \left\{B_y\right\}_{y\in Y}\right\}, \mathfrak{S}\right\}, \quad (8)$$

where

- X and Y are the stimulus spaces, as characterized above;
- $\mathfrak{A}$ and $\mathfrak{B}$ are the corresponding *spaces of internal representations*, or *perceptual images*, endowed with respective *sigma algebras* Σ_A and Σ_B[16];
- $\{A_x\}_{x\in X}$ and $\{B_y\}_{y\in Y}$ are *probability measures* on these sigma algebras, indexed by stimuli from respective spaces (so that the measure A on Σ_A depends on stimulus $x \in X$ and the measure B on Σ_B depends on stimulus $y \in Y$);
- finally, $\mathfrak{S}$ is the "*decision area*", a measurable subset of $\mathfrak{A} \times \mathfrak{B}$ such that any $(a, b) \in \mathfrak{S}$ is mapped into the response "different" (and any $(a, b) \in \mathfrak{A} \times \mathfrak{B} \setminus \mathfrak{S}$ is mapped into "same").[17]

Note that the sigma algebras Σ_A and Σ_B are the same for, respectively, all different measures A_x and B_y (varying with x and y). This allows one to speak unambiguously and without referring to x or y of sets and functions as being A-measurable (i.e., measurable with respect to Σ_A), B-measurable (with respect to Σ_B), and AB-measurable (with respect to the smallest sigma algebra $\Sigma_A \otimes \Sigma_B$ containing $\Sigma_A \times \Sigma_B$).

We will use bracket-free notation for measures of sets: e.g., if $\mathfrak{a} \in \Sigma_A$, then, given a stimulus x, the measure of $\mathfrak{a}$ is denoted by $A_x\mathfrak{a}$. To enable one to use the conventional probabilistic language, we associate with measure A_x a *random variable* $\mathbf{A}(x)$ (formally, the identity function on the probability space $(\mathfrak{A}, \Sigma_A, A_x)$) and interpret $A_x\mathfrak{a}$ as the probability with which $\mathbf{A}(x)$ falls in $\mathfrak{a}$. The meaning of $B_y\mathfrak{b}$ and $\mathbf{B}(y)$ is analogous.

$\mathbf{A}(x)$ and $\mathbf{B}(y)$ in the models of DetInd variety being stochastically independent, we have

$$\psi(x, y) = \Pr\left[(\mathbf{A}(x), \mathbf{B}(y)) \in \mathfrak{S}\right] = \int_{(a,b)\in\mathfrak{S}} \mathrm{d}A_x(a)\mathrm{d}B_y(b). \quad (9)$$

This formula is referred to as the *generation rule* for the model in question. The model itself can be called a Thurstonian-type *representation* for the function ψ it generates.

Other ways of writing the generation rule above, using general properties of Lebesgue integrals, are

$$\psi(x, y) = \int_{a\in\mathfrak{A}} (B_y\mathfrak{S}_a)\mathrm{d}A_x(a) = \int_{b\in\mathfrak{B}} (A_x\mathfrak{S}_b)\mathrm{d}B_y(b), \quad (10)$$

where $\mathfrak{S}_a$ and $\mathfrak{S}_b$ are *cross-sections* of the decision area $\mathfrak{S}$, defined as

$\mathfrak{S}_a = \{b \in \mathfrak{B} : (a, b) \in \mathfrak{S}\}$,
$\mathfrak{S}_b = \{a \in \mathfrak{A} : (a, b) \in \mathfrak{S}\}$.

The cross-sections $\mathfrak{S}_a$ and $\mathfrak{S}_b$ are, respectively, B-measurable and A-measurable sets.

3.2. Thurstonian-type models of DetInt variety

A Thurstonian-type model with *interdependent perceptual images and deterministic decision rule* (DetInt variety) is defined by

$\{A_{x,c}, B_{y,c}, C, \mathfrak{S}\}$,

15 There is a minor difference in the formulation of the theorem referred to: it says that if ψ is well-behaved with respect to h on *all* pairs $(u, u') \in X \times X$, then its minimum level function is constant. But the demonstration consists in proving the present Theorem 2.1 for *some* pair (u, u'), and then observing that this pair can be chosen in $X \times X$ arbitrarily. (And in Kujala & Dzhafarov, 2009, it is pointed out that for the global constancy of minima it is sufficient to assume that ψ is well-behaved with respect to its homeomorphic PSE function in the generalized meaning of Definition 2.3.)

16 It may seem excessively general to have different probability spaces for the perceptual images from the two observation areas. One might maintain that while the observation area of a stimulus may influence the mapping of this stimulus into its percept, the space of the perceptual images should be common to both observation areas. However, putting $(\mathfrak{A}, \Sigma_A) = (\mathfrak{B}, \Sigma_B)$ does not lead to any simplifications in the theory, and in fact occasionally creates notational difficulties. Besides, one should consider the possibility that the stimuli in X and Y are of different nature (e.g., X may represent graphical and Y vocal renderings of letters presented as visual-auditory pairs and judged in terms of "the same letter" vs "different letters".)

17 $\mathfrak{S}$ and $\mathfrak{A}\times\mathfrak{B}\setminus\mathfrak{S}$ are interchangeable, in the sense that all our formulations can be given symmetrically in terms of either of the two. This is important to keep in mind when we speak of the "relative" conditions for well-behavedness (in Section 4.3).

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or more explicitly,

$$\left\{\left\{X, \mathfrak{A}, \Sigma_A, \left\{A_{x,c}\right\}_{x\in X, c\in\mathfrak{C}}\right\},\right.$$
$$\left.\left\{Y, \mathfrak{B}, \Sigma_B, \left\{B_{y,c}\right\}_{y\in Y, c\in\mathfrak{C}}\right\}, \{\mathfrak{C}, \Sigma_C, C\}, \mathfrak{S}\right\}. \quad (11)$$

The meaning of the symbols $X, \mathfrak{A}, \Sigma_A, Y, \mathfrak{B}, \Sigma_B, \mathfrak{S}$ here is the same as for the DetInd variety. The main difference is that the random variables $\mathbf{A}(x)$ and $\mathbf{B}(y)$ evoked by stimuli x and y are not necessarily stochastically independent. Formally, $\mathbf{A}(x)$ and $\mathbf{B}(y)$ are the (first and second coordinate) projection maps of the measurable space $(\mathfrak{A} \times \mathfrak{B}, \Sigma_A \otimes \Sigma_B)$ endowed with a joint probability measure which depends on x and y. In the case of two perceptual images selectively attributed to two stimuli, this joint probability measure has a specific structure. In accordance with the general theory of selective influence (Dzhafarov, 2003c; Dzhafarov & Gluhovsky, 2006; Kujala & Dzhafarov, 2008b), the selective attribution of $\mathbf{A}(x)$ and $\mathbf{B}(y)$ to x and y, respectively, requires a common source of randomness $\mathbf{C}$ taking its values on a set $\mathfrak{C}$ endowed with a sigma algebra Σ_C and associated with a probability measure C (independent of both x and y). Given any value c of $\mathbf{C}$, the conditional random variables $\mathbf{A}(x)|c$ and $\mathbf{B}(y)|c$ are stochastically independent, with respective (conditional) probability measures $A_{x,c}$ and $B_{y,c}$. The generation rule in such a model is given by

$$\psi(x, y) = \Pr\left[(\mathbf{A}(x), \mathbf{B}(y)) \in \mathfrak{S}\right]$$
$$= \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{S}} \mathrm{d}A_{x,c}(a)\mathrm{d}B_{y,c}(b)\mathrm{d}C(c), \quad (12)$$

which can also be written as

$$\psi(x, y) = \int_{c\in\mathfrak{C}} \int_{a\in\mathfrak{A}} (B_{y,c}\mathfrak{S}_a)\mathrm{d}A_{x,c}(a)\mathrm{d}C(c)$$
$$= \int_{c\in\mathfrak{C}} \int_{b\in\mathfrak{B}} (A_{x,c}\mathfrak{S}_b)\mathrm{d}B_{y,c}(b)\mathrm{d}C(c). \quad (13)$$

The models of DetInd variety can be viewed as a special case of those of DetInt variety, obtained by assuming that C is concentrated at a single point. In the subsequent sections we never formulate our propositions for DetInd models, merely mentioning instead how the formulations for DetInt models specialize (and simplify) when C is concentrated at a point.

Remark 3.1. In dealing with Thurstonian-type models of Det variety, the analysis of a model can sometimes be simplified by considering generalized ("*defective*" using Feller, 1968, p. 309, term) probability measures. A measure M defined on a measurable space $\{\mathfrak{M}, \Sigma\}$ is called a defective probability measure if $M\mathfrak{M} < 1$. With regard to our models (8) and (11), we can profitably use this notion if we consider the *projections* of $\mathfrak{S}$ into the sets $\mathfrak{A}$ and $\mathfrak{B}$,

$$\mathfrak{S}_{\mathfrak{A}} = \{a \in \mathfrak{A} : (a, b) \in \mathfrak{S} \quad \text{for some } b \in \mathfrak{B}\},$$
$$\mathfrak{S}_{\mathfrak{B}} = \{b \in \mathfrak{B} : (a, b) \in \mathfrak{S} \quad \text{for some } a \in \mathfrak{A}\},$$

and agree to view them as substitutes for the sets $\mathfrak{A}$ and $\mathfrak{B}$ themselves, i.e., put

$$\mathfrak{A} = \mathfrak{S}_{\mathfrak{A}}, \qquad \mathfrak{B} = \mathfrak{S}_{\mathfrak{B}}.$$

The sigma algebras Σ_A and Σ_B can then be redefined as the smallest sigma algebras which contain, respectively, the cross-sections $\{\mathfrak{S}_b\}_{b\in\mathfrak{B}}$ and $\{\mathfrak{S}_a\}_{a\in\mathfrak{A}}$, which can sometimes be much smaller collections of sets than the original sigma algebras.[18] This simplification leads to no confusions if one remembers that $A_{x,c}\mathfrak{A}$ and $B_{y,c}\mathfrak{B}$ can now have values less than 1 and varying with c, x, and y.

[18] And if not, we may achieve more "economic" sigma algebras by switching the definition of the response area in accordance with footnote 17 and replacing $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \setminus \mathfrak{S}_{\mathfrak{A}}$, $\mathfrak{B} \setminus \mathfrak{S}_{\mathfrak{B}}$, respectively, and $\mathfrak{S}$ with $\mathfrak{S}_{\mathfrak{A}} \times \mathfrak{S}_{\mathfrak{B}} \setminus \mathfrak{S}$.

4. Well-behaved Thurstonian-type models

Given a Thurstonian-type model $\{A_{x,c}, B_{y,c}, C, \mathfrak{S}\}$, the arc-parametrized restriction (AP-restriction) of this model to arcs $x_u^{u'}$ and $y_v^{v'}$ is the Thurstonian-type model $\{A^*_{s,c}, B^*_{t,c}, C, \mathfrak{S}\}$ obtained by putting $A^*_{s,c} = A_{x(s),c}$ and $B^*_{t,c} = B_{y(t),c}$ for $s, t \in [0, 1]$. Clearly, if the original model represents (generates) a discrimination function ψ, the AP-restriction of this model to arcs $x_u^{u'}$ and $y_v^{v'}$ represents (generates) the AP-facet $\varphi(s, t) = \psi(x(s), y(t))$ of ψ by means of the generation rule

$$\varphi(s, t) = \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{S}} \mathrm{d}A^*_{s,c}(a)\mathrm{d}B^*_{t,c}(b)\mathrm{d}C(c).$$

In the following, we will omit the asterisks for simplicity and denote the arc-parametrized conditional measures by $A_{s,c}$ and $B_{t,c}$.

In Sections 4.2 and 4.3, we will present sufficient conditions for an AP-restriction of a Thurstonian-type model to generate an AP-facet $\varphi(s, t)$ which satisfies Properties $\mathcal{R}1$ and $\mathcal{R}2$ of Definition 2.1. We shall make use of the fact that, for a Thurstonian-type representation, the quotient in $\mathcal{R}1$ and $\mathcal{R}2$ can be written in the special form

$$\frac{\Delta^{12}_{(s',t')}\varphi(s, t)}{s' - s}$$
$$= \frac{1}{s' - s} \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{S}} \mathrm{d}[A_{s',c} - A_{s,c}](a)\mathrm{d}[B_{t',c} - B_{t,c}](b)\mathrm{d}C(c),$$

which is easy to verify using the linearity property of integrals. Here, $A_{s',c} - A_{s,c}$ (as well as $B_{t',c} - B_{t,c}$) is a *signed measure*, in this case a countably additive set function $\Sigma_A \to [-1, 1]$ (respectively, $\Sigma_B \to [-1, 1]$). The codomain is $[-1, 1]$ because both $A_{s,c}$ and $B_{t,c}$ are bounded between 0 and 1.

4.1. Continuity of measures

Let Σ be a sigma algebra on some space $\mathfrak{M}$, and let Z be any Hausdorff space. It can be thought of, e.g., as representing the unit interval $[0, 1]$.

Let $\{M_z\}_{z\in Z}$ and M be some probability measures on Σ. As $z \to z_0$, we say that $M_z \to M$ *setwise* if $M_z\mathfrak{s} \to M\mathfrak{s}$ for every $\mathfrak{s} \in \Sigma$. If $M_{z'} \to M_z$ setwise as $z' \to z$ we say that M_z is *setwise continuous* (at z, or globally if at any $z \in Z$).

As $z \to z_0$, we say that $M_z \to M$ *in total variation* if $\sup_{\mathfrak{s}\in\Sigma} |M_z\mathfrak{s} - M\mathfrak{s}| \to 0$. If $M_{z'} \to M_z$ in total variation as $z' \to z$ we say that M_z is *V-continuous* (at z, or globally if at any $z \in Z$). Clearly, a V-continuous measure is setwise continuous. Fig. 2 illustrates the fact that the reverse of this statement is not true.

If the indexing set Z is the unit interval, as it will be whenever we consider the arc-parametrized measures $A_{s,c}$ and $B_{t,c}$ ($s, t \in [0, 1]$), the continuity of functions on Z can be defined in terms of sequences: a function f defined on Z is continuous at z if and only if $z_n \to z$ implies $f(z_n) \to f(z)$ for all sequences $\{z_n\}$. Thus, we can apply the lemmas of Appendix A, which are formulated for sequences.

Although in this paper we do not make use of this fact, it is worthwhile to observe that the setwise continuity of the probability measures in an AP-restriction of a Thurstonian-type model is related to the continuity of the AP-facet φ it generates as shown in the lemma below. Recall that, given some arcs $x_u^{u'}$ and $y_v^{v'}$, the arc-parametrized measures $A_{s,c}$ and $B_{t,c}$ stand, with some abuse of notation, for $A_{x(s),c}$ and $A_{y(t),c}$, respectively.

Lemma 4.1. *Let $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ be an AP-restriction of a Thurstonian-type model (of DetInt variety) to some arcs $x_u^{u'}$ and $y_v^{v'}$. If $A_{s,c}$ and $B_{t,c}$ are setwise continuous at, respectively, $s = s_0$ and $t = t_0$ for C-almost all c, then the AP-facet φ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ is continuous in (s, t) at $(s, t) = (s_0, t_0)$.*

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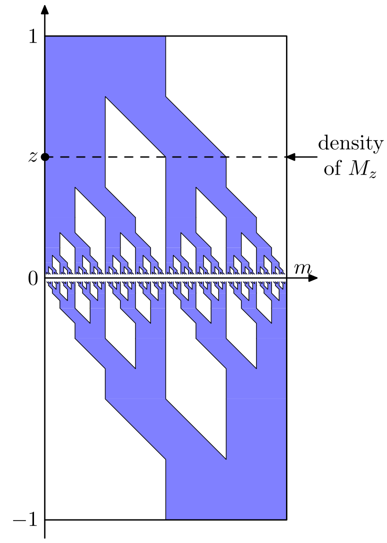

Fig. 2. An example of a measure M_z that is setwise continuous on $z \in [-1, 1]$ but not V-continuous at $z = 0$. For each value of z, the density of the measure is uniform over the shaded intervals shown in the figure. This pattern continues to arbitrarily small scale. At $z = 0$, the measure is defined as $M_0\,\mathfrak{s} = \frac{1}{2}\lambda_{[0,1]}\mathfrak{s}$, where $\lambda_{[0,1]}$ is the uniform measure on [0, 1]; this makes the measure setwise continuous at $z = 0$. However, $\sup_{\mathfrak{s}\in\Sigma_{[0,1]}} |M_0\mathfrak{s} - M_z\mathfrak{s}| = \frac{1}{2}$ for any $z \neq 0$ and so M_z is not V-continuous at $z = 0$.

Proof. We have to prove that, as $(s, t) \to (s_0, t_0)$,

$$\varphi(s,t) = \int_{c\in\mathfrak{C}} \int_{b\in\mathfrak{B}} A_{s,c}\mathfrak{S}_b \,\mathrm{d}B_{t,c}(b)\,\mathrm{d}C(c)$$

$$\to \int_{c\in\mathfrak{C}} \int_{b\in\mathfrak{B}} A_{s_0,c}\mathfrak{S}_b \mathrm{d}B_{t_0,c}(b) = \varphi(s_0, t_0). \tag{14}$$

For any c outside the exceptional null subset of $\mathfrak{C}$, as $(s, t) \to (s_0, t_0)$, we have both $A_{s,c}\mathfrak{S}_b \to A_{s_0,c}\mathfrak{S}_b$ pointwise on $\mathfrak{B}$ (with the functions being uniformly bounded, $A_{s,c}\mathfrak{S}_b \in [0, 1]$) and $B_{t,c} \to B_{t_0,c}$ setwise. Then, by Lemma A.1 (Statement 3),

$$\int_{b\in\mathfrak{B}} A_{s,c}\mathfrak{S}_b \mathrm{d}B_{t,c}(b) \to \int_{b\in\mathfrak{B}} A_{s_0,c}\mathfrak{S}_b \mathrm{d}B_{t_0,c}(b).$$

As the value of the left-hand side integral is within [0,1] for all $c \in \mathfrak{C}$, (14) follows by the dominated convergence theorem. □

4.2. A symmetric "absolute" sufficient condition

The condition stipulated in the theorem below is called *symmetric* because the constraints it imposes on the two arc-parametrized probability measures $A_{s,c}$ and $B_{t,c}$ are of the same kind. Following the terminology adopted in Dzhafarov (2003a), the condition in question is called *absolute* because it is formulated entirely in terms of the probability measures, so its truth value does not depend on one's choice of the decision area $\mathfrak{S}$.

Theorem 4.2. *Let $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ be an AP-restriction of a Thurstonian-type model (of DetInt variety) to some arcs $x_u^{u'}$ and $y_v^{v'}$. Suppose that for all $(s, t) \in [0, 1] \times [0, 1]$, except on an at most denumerable set, and for all $\mathfrak{a} \in \Sigma_A$, $\mathfrak{b} \in \Sigma_B$,*

$$\sup_{c\in\mathfrak{C}\setminus\mathfrak{e}_{s,t}} \sup_{s'\neq s} \left|\frac{A_{s',c}\mathfrak{a} - A_{s,c}\mathfrak{a}}{s'-s}\right| = L_1(s, \mathfrak{a}) < \infty,$$

$$\sup_{c\in\mathfrak{C}\setminus\mathfrak{e}_{s,t}} \sup_{t'\neq t} \left|\frac{B_{t',c}\mathfrak{b} - B_{t,c}\mathfrak{b}}{t'-t}\right| = L_2(t, \mathfrak{b}) < \infty,$$

where $\mathfrak{e}_{s,t}$ are subsets of $\mathfrak{C}$ of C-measure zero (generally different for different s, t). Then, the AP-facet $\varphi(s, t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}1$ and $\mathcal{R}2$.

Remark 4.1. If C is concentrated at a point (i.e., we deal with a Thurstonian-type model of DetInd variety), then the conditions become: for all $(s, t) \in [0, 1] \times [0, 1]$, except on an at most denumerable set, and for all $\mathfrak{a} \in \Sigma_A$, $\mathfrak{b} \in \Sigma_B$,

$$\sup_{s'\neq s} \left|\frac{A_{s'}\mathfrak{a} - A_s\mathfrak{a}}{s'-s}\right| = L_1(s, \mathfrak{a}) < \infty,$$

$$\sup_{t'\neq t} \left|\frac{B_{t'}\mathfrak{b} - B_t\mathfrak{b}}{t'-t}\right| = L_2(t, \mathfrak{b}) < \infty.$$

Proof. Let $(s, t) \in [0, 1] \times [0, 1]$ be chosen outside the countable exceptional set. We begin by observing that for any $c \in \mathfrak{C} \setminus \mathfrak{e}_{s,t}$ and any $s' \neq s$ and $t' \neq t$ in [0, 1], the functions $\mathfrak{a} \mapsto (A_{s',c}\mathfrak{a} - A_{s,c}\mathfrak{a})/(s'-s)$ and $\mathfrak{b} \mapsto (A_{t',c}\mathfrak{b} - A_{t,c}\mathfrak{b})/(t'-t)$ are bounded signed measures on Σ_A and Σ_B, respectively:

$$\sup_{\mathfrak{a}\in\Sigma_A} \left|\frac{A_{s',c}\mathfrak{a} - A_{s,c}\mathfrak{a}}{s'-s}\right| \le \frac{1}{|s'-s|},$$

$$\sup_{\mathfrak{b}\in\Sigma_B} \left|\frac{A_{t',c}\mathfrak{b} - A_{t,c}\mathfrak{b}}{t'-t}\right| \le \frac{1}{|t'-t|}.$$

As these families of bounded signed measures are setwise bounded by $L_1(s, \mathfrak{a})$ and $L_2(t, \mathfrak{b})$, respectively, the Nikodým boundedness theorem (Lemma A.4) implies the existence of uniform bounds

$$\sup_{\mathfrak{a}\in\Sigma_A} \sup_{c\in\mathfrak{C}\setminus\mathfrak{e}_{s,t}} \sup_{s'\neq s} \left|\frac{A_{s',c}\mathfrak{a} - A_{s,c}\mathfrak{a}}{s'-s}\right| = L_A(s) < \infty \tag{15}$$

and

$$\sup_{\mathfrak{b}\in\Sigma_B} \sup_{c\in\mathfrak{C}\setminus\mathfrak{e}_{s,t}} \sup_{t'\neq t} \left|\frac{B_{t',c}\mathfrak{b} - B_{t,c}\mathfrak{b}}{t'-t}\right| = L_B(t) < \infty. \tag{16}$$

It follows from (15) that for all $c \in \mathfrak{C} \setminus \mathfrak{e}_{s,t}$ and $s' \neq s$ in [0, 1],

$$\begin{aligned}
&\left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}[B_{t',c} - B_{t,c}](b)\right| \\
&\le \left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}B_{t',c}(b)\right| \\
&\quad + \left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}B_{t,c}(b)\right| \le 2L_A(s),
\end{aligned} \tag{17}$$

whence,

$$\begin{aligned}
&\limsup_{(s',t')\to(s,t)} \left|\frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s}\right| \\
&= \limsup_{(s',t')\to(s,t)} \left|\int_{c\in\mathfrak{C}}\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}[B_{t',c} - B_{t',c}](b)\mathrm{d}C(c)\right| \\
&\le \limsup_{(s',t')\to(s,t)} \int_{c\in\mathfrak{C}} \left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}[B_{t',c} - B_{t',c}](b)\right| \mathrm{d}C(c) \\
&\le 2L_A(s) < \infty.
\end{aligned}$$

Thus, $\varphi(s, t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}1$.

Now, it follows from (16) that $B_{t',c}$ is V-continuous in t' at $t' = t$ for all $c \in \mathfrak{C} \setminus \mathfrak{e}_{s,t}$: as $t' \to t$ in [0, 1],

$$\sup_{\mathfrak{b}\in\Sigma_B} |B_{t',c}\mathfrak{b} - B_{t,c}\mathfrak{b}| \le L_B(t)\left|t'-t\right| \to 0.$$

For the same c then, by (15) and Lemma A.3,

$$\lim_{(s',t')\to(s,t)} \int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}[B_{t',c} - B_{t,c}](b) = 0.$$

Since we also have (17), by the dominated convergence theorem,

$$\lim_{(s',t')\to(s,t)} \frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s} = \lim_{(s',t')\to(s,t)} \int_{c\in\mathfrak{C}}\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t,c}](b)\mathrm{d}C(c) = 0.$$

Thus, $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}2$. □

When applied to arcs $x_u^{u'}$ and $y_v^{v'}$ with $\left[y_v^{v'}\right] = h\left(\left[x_u^{u'}\right]\right)$ (where h is a homeomorphism $X \to Y$) and combined with Theorem 2.1, this result has obvious consequences for the conjunction of the PSE function h and (non)constancy of minima in discrimination functions generated by the Thurstonian-type models the result entails. We will postpone formulating these consequences until after we have presented similar results for two other well-behavedness conditions.

4.3. Two asymmetric "relative" sufficient conditions

In the theorems below, the arc-parametrized probability measures $A_{s,c}$ and $B_{t,c}$ are treated asymmetrically (made different assumptions about), and the formulations are *relative* because their truth value may depend on the decision area $\mathfrak{S}$.

Theorem 4.3. *Let $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ be an AP-restriction of a Thurstonian-type model (of DetInt variety) to some arcs $x_u^{u'}$ and $y_v^{v'}$. Assume that*

1. *for almost all $t \in [0,1]$, $B_{t,c}$ is V-continuous in t for C-almost all c;*
2. *for all $s \in [0,1]$, except on an at most denumerable set,*

$$L(s,c) = \sup_{b\in\mathfrak{B}}\sup_{s'\neq s}\left|\frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\right|$$

(possibly ∞ at some s, c) is C-integrable over $\mathfrak{C}$.

Then, the AP-facet $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}1$ and $\mathcal{R}2$.

Remark 4.2. If C is concentrated at a point, then the conditions become: (i) B_t is V-continuous at almost all $t \in [0,1]$; (ii) for all $s \in [0,1]$, except on an at most denumerable set,

$$L(s) = \sup_{b\in\mathfrak{B}}\sup_{s'\neq s}\left|\frac{A_{s'}\mathfrak{S}_b - A_s\mathfrak{S}_b}{s'-s}\right| < \infty.$$

Proof. Let $s \in [0,1]\setminus D$ and $t \in [0,1]$ (where D denotes the countable exceptional set in Condition 2). For any $c \in \mathfrak{C}$ and any $s' \neq s$ and t' in $[0,1]$, Condition 2 implies

$$\left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t,c}](b)\right| \le 2L(s,c), \quad (18)$$

whence

$$\limsup_{(s',t')\to(s,t)}\left|\frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s}\right| = \limsup_{(s',t')\to(s,t)}\left|\int_{c\in\mathfrak{C}}\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t',c}](b)\mathrm{d}C(c)\right|$$
$$\le \limsup_{(s',t')\to(s,t)}\int_{c\in\mathfrak{C}}\left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t',c}](b)\right|\mathrm{d}C(c) \le \int_{c\in\mathfrak{C}} 2L(s,c)\mathrm{d}C(c) < \infty.$$

Thus, $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}1$.

For the same $s \in [0,1]\setminus D$, let now t be chosen in $[0,1]\setminus E$ (where E is the exceptional null subset of $[0,1]$ in Condition 1). By Condition 2, the functions $b \mapsto (A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b)/(s'-s)$ on $\mathfrak{B}$ are uniformly bounded across all $s' \neq s$ in $[0,1]$ for any $c \in \mathfrak{C}\setminus\mathfrak{e}_s$ (where $\mathfrak{e}_s$ is the necessarily C-null subset of $\mathfrak{C}$ on which $L(s,c) = \infty$).[19] By Lemma A.3 then, denoting the exceptional C-null subset of $\mathfrak{C}$ in Condition 1 by $\mathfrak{e}_t$, we have, for any $c \in \mathfrak{C}\setminus(\mathfrak{e}_t\cup\mathfrak{e}_s)$,

$$\lim_{(s',t')\to(s,t)}\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t,c}](b) = 0.$$

Since we also have (18), by the dominated convergence theorem,

$$\lim_{(s',t')\to(s,t)} \frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s} = \lim_{(s',t')\to(s,t)} \int_{c\in\mathfrak{C}}\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t',c}](b)\mathrm{d}C(c) = 0.$$

Thus, $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}2$. □

Theorem 4.4. *Let $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ be an AP-restriction of a Thurstonian-type model (of DetInt variety) to some arcs $x_u^{u'}$ and $y_v^{v'}$. Assume that*

1. *for almost all $t \in [0,1]$, $B_{t,c}$ is setwise continuous in t for C-almost all c;*
2. *for all $s \in [0,1]$, except on an at most denumerable set,*

$$L(s,c) = \sup_{b\in\mathfrak{B}}\sup_{s'\neq s}\left|\frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\right|$$

(possibly ∞ at some s, c) is C-integrable over $\mathfrak{C}$;

3. *for almost all $s \in [0,1]$,*

$$d(s,c,b) = \lim_{s'\to s\pm} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}$$

(possibly infinite at some s, c, b) exists for C-almost all c and all $b \in \mathfrak{B}$ (where $\pm$ should be read as "either $+$ or $-$", and the choice may depend on s).

Then, the AP-facet $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}1$ and $\mathcal{R}2$.

Remark 4.3. If C is concentrated at a point, then the conditions become: (i) B_t is setwise continuous at almost all $t \in [0,1]$; (ii) for all $s \in [0,1]$, except on an at most denumerable set,

$$L(s) = \sup_{b\in\mathfrak{B}}\sup_{s'\neq s}\left|\frac{A_{s'}\mathfrak{S}_b - A_s\mathfrak{S}_b}{s'-s}\right| < \infty;$$

(iii) for almost all $s \in [0,1]$,

$$d(s,b) = \lim_{s'\to s\pm} \frac{A_{s'}\mathfrak{S}_b - A_s\mathfrak{S}_b}{s'-s}$$

(possibly infinite at some s, b) exists for all $b \in \mathfrak{B}$ (where $\pm$ should be read as "either $+$ or $-$", and the choice may depend on s).

Proof. Let $s \in [0,1]\setminus D$ and $t \in [0,1]$ (where D denotes the countable exceptional set in Condition 2). By the same argument as in the previous theorem, for any $c \in \mathfrak{C}$ and any $s' \neq s$ and t' in $[0,1]$,

$$\left|\int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s}\mathrm{d}[B_{t',c}-B_{t,c}](b)\right| \le 2L(s,c), \quad (19)$$

[19] Here and in the next theorem, an index at a set, $\mathfrak{e}_s$, $\mathfrak{e}_t$, etc., indicates that the choice of the set may depend on the index value.

whence

$$\limsup_{(s',t')\to(s,t)} \left| \frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s} \right| < \infty.$$

So $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}1$.

Let now $s \in [0,1] \setminus (D \cup F)$ and $t \in [0,1] \setminus E$ (where F and E are the exceptional null subsets of $[0,1]$ in, respectively, Condition 1 and Condition 3). As in the previous theorem, the functions $b \mapsto (A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b)/(s'-s)$ on $\mathfrak{B}$ are uniformly bounded across all $s' \neq s$ in $[0,1]$ for any $c \in \mathfrak{C}\backslash\mathfrak{e}_s$ (where $\mathfrak{e}_s$ is the C-null subset of $\mathfrak{C}$ on which $L(s,c) = \infty$). Conditions 2 and 3 imply that as $s' \to s\pm$, the pointwise limits $d(s,c,b)$ of these functions exist as finite numbers for all $c \in \mathfrak{C} \setminus (\mathfrak{e}_s \cup \mathfrak{f}_s)$ (where $\mathfrak{f}_s$ is the exceptional C-null subset of $\mathfrak{C}$ in Condition 3). Then, it follows from Statement 4 of Lemma A.1 that for any $c \in \mathfrak{C} \setminus (\mathfrak{e}_t \cup \mathfrak{e}_s \cup \mathfrak{f}_s)$ (where $\mathfrak{e}_t$ is the exceptional C-null subset of $\mathfrak{C}$ in Condition 1), with the same choice of $+$ or $-$ as above,

$$\lim_{(s',t')\rightrightarrows(s,t)\pm} \int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}[B_{t',c} - B_{t,c}](b) = 0.$$

Since we also have (19), by the dominated convergence theorem,

$$\lim_{(s',t')\rightrightarrows(s,t)\pm} \frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s} = \lim_{(s',t')\rightrightarrows(s,t)\pm} \int_{c\in\mathfrak{C}} \int_{b\in\mathfrak{B}} \frac{A_{s',c}\mathfrak{S}_b - A_{s,c}\mathfrak{S}_b}{s'-s} \mathrm{d}[B_{t',c} - B_{t,c}](b)\mathrm{d}C(c) = 0.$$

Thus, $\varphi(s,t)$ generated by $\{A_{s,c}, B_{t,c}, C, \mathfrak{S}\}$ satisfies $\mathcal{R}2$. □

Remark 4.4. The following simple observation may be useful in some applications, especially when X and Y are intervals of real numbers. If the AP-restriction of a Thurstonian-type model T to some arcs $x_u^{u'}$ and $y_v^{v'}$ satisfies the conditions of one of Theorems 4.2–4.4, then the same conditions are satisfied for the AP-restrictions of T to subarcs $x|[s_1,s_2]$ and $y|[t_1,t_2]$, for any $[s_1,s_2] \subset [0,1]$ and $[t_1,t_2] \subset [0,1]$. If convenient, these subarcs can be linearly rescaled and presented as arcs

$$\lambda \mapsto x(s_1 + \lambda(s_2 - s_1)), \quad \lambda \in [0,1],$$
$$\mu \mapsto y(t_1 + \mu(t_2 - t_1)), \quad \mu \in [0,1].$$

4.4. Consequences for Regular Minimality with nonconstancy of minima

Let $\psi : X \times Y \to [0,1]$ be a continuous discrimination function which satisfies Regular Minimality with a homeomorphic PSE function $h : X \to Y$ and a nonconstant minimum level function $x \mapsto \psi(x, h(x))$, $x \in X$. Let $T = \{A_{x,c}, B_{y,c}, C, \mathfrak{S}\}$ ($x \in X, y \in Y$) be a Thurstonian-type model, and let $E(T,h)$ be the set of all (u,u') in $X \times X$ with the following property: the AP-restriction of T to some arcs $x_u^{u'}$ and $y_v^{v'}$ with $\left[y_v^{v'}\right] = h\left(\left[x_u^{u'}\right]\right)$ satisfies the conditions of (at least) one of Theorems 4.2–4.4. Combined with Theorem 2.1, each of these theorems tells us that if ψ is generated by T, then $\psi(u,h(u)) = \psi(u',h(u'))$ for all $(u,u') \in E(T,h)$. This has the following immediate implication.

Corollary 4.5. *If $(u,u') \in E(T,h)$ for some (u,u') such that $\psi(u,h(u)) \neq \psi(u',h(u'))$, then ψ is not generated by T.*

The same argument yields another implication of interest.

Corollary 4.6. *If the only topologically and transitively closed subset of $X \times X$ containing $E(T,h)$ is $X \times X$ itself, then ψ (with nonconstant minima) is not generated by T.*

Proof. Deny this and assume that ψ is generated by T. Then,

$$E(T,h) \subset K = \{(u,u') \in X \times X : \psi(u,h(u)) = \psi(u',h(u'))\}.$$

Obviously K is transitive. It is also topologically closed since given any $(u,u') \in X \times X$ such that $\psi(u,h(u)) \neq \psi(u',h(u'))$, the continuity of $x \mapsto \psi(x,h(x))$ implies that the inequality should hold in some open neighborhood of (u,u'). It follows that $K = X \times X$ (as we have assumed that there are no other topologically closed, transitive subsets of $X \times X$ containing $E(T,h)$). But then ψ should have constant minima, contradicting the premise of the corollary. □

One can formulate a multitude of other consequences by considering various special cases of ψ and T. We will mention one. In psychophysical theorizing and, especially, experimental practice, one often deals with the case when both X and Y are represented by intervals of real numbers. In this case, the image $\left[x_u^{u'}\right]$ ($u \le u'$) of any arc is the interval $[u,u']$. Let us call a function $f : X \to \mathbb{R}$ *strictly nonconstant* if it is nonconstant in any interval within its domain. It is safe to assume that in all cases involving ψ with nonconstant minima constructed from empirical data, ψ will be strictly nonconstant at least on some interval $X' \subset X$, making the corollary below applicable to $\psi|X' \times h(X')$.

Corollary 4.7. *With X and Y being intervals of real numbers (finite or infinite), if the minimum level function $x \mapsto \psi(x,h(x))$ is strictly nonconstant, and if $E(T,h)$ contains a pair (u,u') with $u < u'$, then ψ is not generated by T.*

The proof obtains by observing (based on Remark 4.4) that the function generated by T will have a plateau on the interval $[u,u']$.

Clearly, instead of defining $E(T,h)$ in terms of Theorems 4.2–4.4, one could define it more generally, by referring to any condition that guarantees the compliance of the corresponding AP-facet with Properties $\mathcal{R}1$ and $\mathcal{R}2$. But the significance of our corollaries is not determined by the generality of their premises (e.g., the general statement "if T predicts $\psi(u,h(u)) = \psi(u',h(u'))$ while de facto $\psi(u,h(u)) \neq \psi(u',h(u'))$, then ψ is not generated by T" has no value). Rather the significance of the three corollaries just presented is determined by "extra-mathematical" considerations: by how "natural" the conditions stated in Theorems 4.2–4.4 are, and by how "unusual" it would be to construct a model which does not comply with these conditions. These considerations cannot be proved, appealing instead to one's experience and intuition.

One way of supporting these considerations is to observe that in all published Thurstonian-type models the stimulus spaces X and Y are regions of $\mathbb{R}^n$, the spaces of perceptual representations $\mathfrak{A}$ and $\mathfrak{B}$ are regions of $\mathbb{R}^m$, and the random variables $\mathbf{A}(x)$ and $\mathbf{B}(y)$ in a DetInd model have conventional finite multivariate density functions (with respect to Lebesgue measure). If parameters of these densities are assumed to piecewise smoothly depend on the stimuli, then it can usually be shown that, depending on details, the conditions of some or all of Theorems 4.2–4.4 apply to all pairs (u,u'), (v,v') in $X \times X$ and $Y \times Y$. It is safe to assume that most psychophysicists would not hesitate to postulate such dependences (although they usually do not have to, as the models are used to fit finite sets of $\psi(x,y)$-values, without much consideration given to the behavior of ψ "in between").

Another way of supporting these considerations is to contemplate toy examples of Thurstonian-type models constructed to account for both Regular Minimality and nonconstant minima. We take on this demonstration in the next section.

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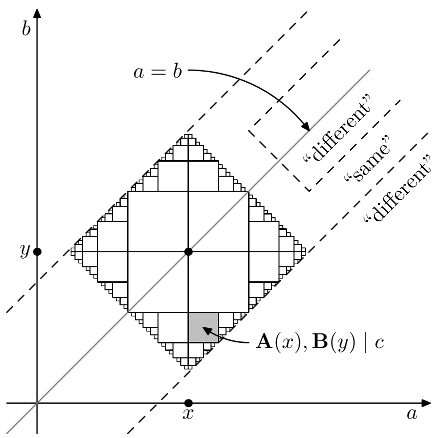

Fig. 3. An example of a non-well-behaved Thurstonian-type model of DetInt variety satisfying Regular Minimality with a nonconstant minimum level function. The joint distribution of the perceptual images $\mathbf{A}(x), \mathbf{B}(y)$ is uniform over the diamond-shaped area centered at $(a, b) = (x, y)$. The figure illustrates how this joint distribution arises from the conditional random variables $\mathbf{A}(x) \mid c$ and $\mathbf{B}(y) \mid c$ that are independent of each other for each value c of the common source of randomness (see the text for details).

5. Examples of non-well-behaved models

We now present two examples of non-well-behaved Thurstonian-type models which generate discrimination functions that both satisfy Regular Minimality and have nonconstant minima. The main purpose of these examples is to demonstrate that such representations can indeed be constructed but that they may be considerably more artificial and technically involved than the well-behaved models of the type considered at the opening of this paper. By the virtue of their artificiality, these examples are unrelated to any empirical problem or existing theoretical model. The examples also serve the purpose of demonstrating that violations of the sufficient conditions for well-behavedness which are stipulated in Theorems 4.2–4.4 may indeed lead to non-well-behaved Thurstonian-type models.

5.1. Example 1

We begin with an example of ψ generated by a non-well-behaved Thurstonian-type model of DetInt variety (Figs. 3 and 4). In this example, x and y are real numbers, and the representations of the respective perceptual images a and b are real numbers too. In Fig. 3, each square inside the diamond-shaped area is characterized by its width w and location $(x + a^i_w, y + b^i_w)$, where i is an arbitrary index counting the squares of a given width. Assuming that $w = 1$ for the four largest central squares $i = 1, \ldots, 4$, there are eight squares $i = 1, \ldots, 8$ of width $w = \frac{1}{2}$ adjacent to the four central squares, and in general, for w any nonpositive whole power of 2, the squares of width w are indexed by $i = 1, 2, \ldots, 4/w$. We assume that the common source of randomness $\mathbf{C}$ varies on the discrete set of values $c = (w, i)$ and is distributed as

$$C\{c\} = C\{(w, i)\} = \frac{w^2}{8}, i = 1, 2, \ldots, \frac{4}{w}, w = 1, \frac{1}{2}, \frac{1}{4}, \ldots \quad (20)$$

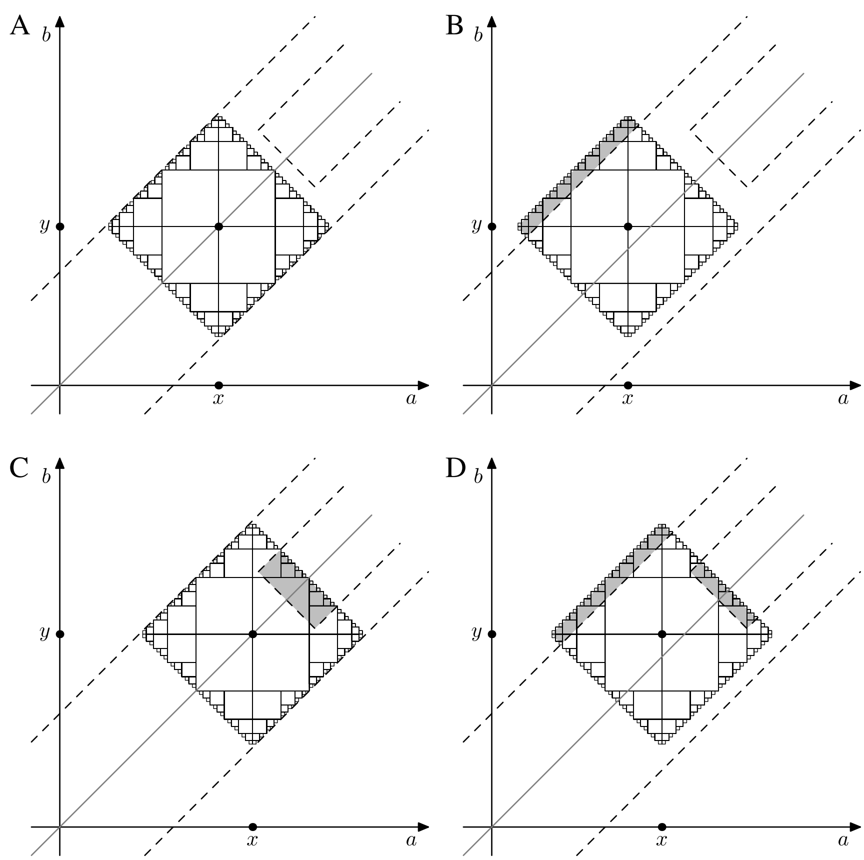

Fig. 4. An illustration of how the distribution of the perceptual images in Fig. 3 moves as the stimuli x and y change. The shaded areas indicate where the support of the perceptual distribution overlaps the decision area; these areas comprise the discrimination probability $\psi(x, y)$. (A) For small values of $x = y$, the discrimination probability is zero, and (B) any horizontal or vertical deviation (here $x \to x - \varepsilon$) from the diagonal shifts part of the distribution into the decision area, thereby increasing the discrimination probability; (C) for larger values of $x = y$, the discrimination probability increases due to the middle piece of the decision area, but still (D) any horizontal or vertical deviation (here $x \to x - \varepsilon$) from the diagonal will increase the discrimination probability as any decrease of the contribution of the middle piece will be more than offset by the larger contribution of the side pieces. Thus, Regular Minimality is satisfied even though the minimum level is nonconstant.

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For any given $c = (w, i)$, the conditional distributions of the perceptual images $\mathbf{A}(x)$, $\mathbf{B}(y)$ are independent and uniform,

$$[\mathbf{A}(x) \mid c] \sim \text{Uniform}\left[x + a_w^i - \frac{w}{2}, x + a_w^i + \frac{w}{2}\right],$$
$$[\mathbf{B}(y) \mid c] \sim \text{Uniform}\left[y + b_w^i - \frac{w}{2}, y + b_w^i + \frac{w}{2}\right],$$

whence their joint distribution is uniform over the respective square,

$$[\mathbf{A}(x), \mathbf{B}(y) \mid c] \sim \text{Uniform}\left(\left[x + a_w^i - \frac{w}{2}, x + a_w^i + \frac{w}{2}\right] \times \left[y + b_w^i - \frac{w}{2}, y + b_w^i + \frac{w}{2}\right]\right).$$

The whole diamond-shaped union of the squares has an area of 8 units, and it is easy to compute that the unconditional joint distribution of $\mathbf{A}(x)$, $\mathbf{B}(y)$ is uniform over this area. Fig. 4 shows how this distribution moves as x and y change; it is obvious that Regular Minimality is satisfied (with PSE function $y = x$) and that the minimum level function is not constant. Thus, the discrimination function generated by this model cannot be well-behaved—that would contradict Theorem 2.1.

It is instructive to see how the sufficient conditions of Theorems 4.2–4.4 fail in this case. Choosing an arc $x_u^{u'}$ (with $u' > u$, the case $u' < u$ being considered analogously), the ratio appearing in the conditions of these theorems can be expanded as

$$\left|\frac{A_{s',c}\mathfrak{a} - A_{s,c}\mathfrak{a}}{s' - s}\right| = \left|\frac{A_{x(s'),(w,i)}\mathfrak{a} - A_{x(s),(w,i)}\mathfrak{a}}{s' - s}\right|$$
$$= \frac{1}{|s' - s|}\left|\frac{\lambda\left(\mathfrak{a} \cap \left[x(s') + a_w^i - \frac{w}{2}, x(s') + a_w^i + \frac{w}{2}\right]\right)}{w} - \frac{\lambda\left(\mathfrak{a} \cap \left[x(s) + a_w^i - \frac{w}{2}, x(s) + a_w^i + \frac{w}{2}\right]\right)}{w}\right|,$$

where λ denotes the Lebesgue measure and $\mathfrak{a}$ is a Borel subset of reals. Choose an arbitrary s in $]0, 1[$ such that

$$\limsup_{s' \to s+} \frac{x(s') - x(s)}{s' - s} > 0$$

(there exists an uncountable number of such s as $x(s)$ is increasing), and consider all values of $c = (w, i)$ for which $a_w^i = -\frac{w}{2}$ (we have an infinity of such c, with $w \to 0$ as we count them up or down from the center). Then, for $\mathfrak{a} = [x(s), \infty[$ and any sufficiently small $s' > s$, the expression above becomes

$$\frac{1}{s' - s}\frac{\lambda\left(\left[x(s), x(s')\right]\right)}{w} = w^{-1}\frac{x(s') - x(s)}{s' - s}.$$

As this expression is not bounded across all values of $c = (w, i)$ and s' under consideration, the model violates the conditions of Theorem 4.2. Theorems 4.3 and 4.4 do not apply either, as the upper bound $L(s, c)$ can be shown to satisfy

$$L(s, c) = L(s, (w, i)) \geq \limsup_{s' \to s+} w^{-1}\frac{x(s') - x(s)}{s' - s},$$

and so $L(s, c)$ is not integrable over the set $\mathfrak{C}$ of the (w, i)-values: in accordance with (20),

$$\int_{(w,i) \in \mathfrak{C}} L(s, (w, i))\mathrm{d}C(w, i)$$
$$\geq \limsup_{s' \to s+} \frac{x(s') - x(s)}{s' - s} \sum_{w = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots} \sum_{i=1}^{4/w} w^{-1}\frac{w^2}{8} = \infty.$$

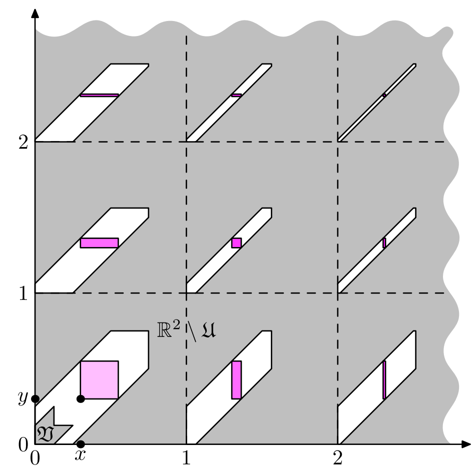

Fig. 5. An example of a non-well-behaved Thurstonian-type model of DetInd variety satisfying Regular Minimality with a nonconstant minimum level function. The joint density of $\mathbf{A}(x)$, $\mathbf{B}(y)$ is shown in magenta. The density in each rectangle doubles and the area reduces to one fourth on each step to the right or up; the pattern continues *ad infinitum*. The decision area is shaded in gray. See the text for details.

5.2. Example 2

In our second example, ψ is generated by a non-well-behaved Thurstonian-type model of DetInd variety with V-continuous measures (Figs. 5 and 6). In this example, x and y vary on the open interval $(0, \frac{1}{2})$, and the perceptual images are represented by positive reals. The distribution of the perceptual image $\mathbf{A}(x)$ is given by the measure

$$A_x\mathfrak{a} = \sum_{k=1}^{\infty} 2^k \lambda(\mathfrak{a} \cap \mathfrak{s}_k(x))$$

where $\mathfrak{a}$ is a Borel set, λ is the Lebesgue measure, and

$$\mathfrak{s}_k(x) = [k + x - 1, k + x - 1 + 4^{-k}].$$

The mapping $x \mapsto A_x$ is V-continuous because each of its terms converges in total variation,

$$\sup_{\mathfrak{a}} |2^k\lambda(\mathfrak{a} \cap \mathfrak{s}_k(x')) - 2^k\lambda(\mathfrak{a} \cap \mathfrak{s}_k(x))| \leq 2^k|x' - x| \to 0,$$

and these terms are dominated by

$$\sup_{\mathfrak{a}} |2^k\lambda(\mathfrak{a} \cap \mathfrak{s}_k(x))| = 2^k \cdot 4^{-k} = 2^{-k},$$

which forms a converging series (the total variation of the sum is at most the sum of the total variations of its terms, which tends to zero by the dominated convergence theorem applied to the summation over k). The distribution of the perceptual image $\mathbf{B}(y)$ is defined identically: $B_y = A_y$ for all $y \in (0, \frac{1}{2})$.

If we define the decision area by $\mathfrak{S} = (\mathbb{R}^2 \setminus \mathfrak{U}) \cup \mathfrak{V}$, where

$$\mathfrak{U} = \bigcup_{a,b \in [0, \frac{1}{2}]} \bigcup_{i,j=1}^{\infty} \mathfrak{s}_i(a) \times \mathfrak{s}_j(b),$$

$$\mathfrak{V} = \bigcup_{a \in \left[0, \frac{1}{8}\right]} \left(\left[a, a + \frac{1}{8}\right] \times \{a\}\right) \cup \left(\{a\} \times \left[a, a + \frac{1}{8}\right]\right)$$

(see Fig. 5), then the generated discrimination function will satisfy Regular Minimality with $h(x) = x$ and with the minimum level changing as $\psi(x, h(x)) = 1/8 - x$ for $x \in (0, 1/8)$.

To show that indeed $h(x) = x$, observe the following. Any horizontal deviation $x \to x + \varepsilon$ from the main diagonal $x = y$ may decrease the contribution of the part $\mathfrak{S} \cap (0, 1)^2$ of the decision

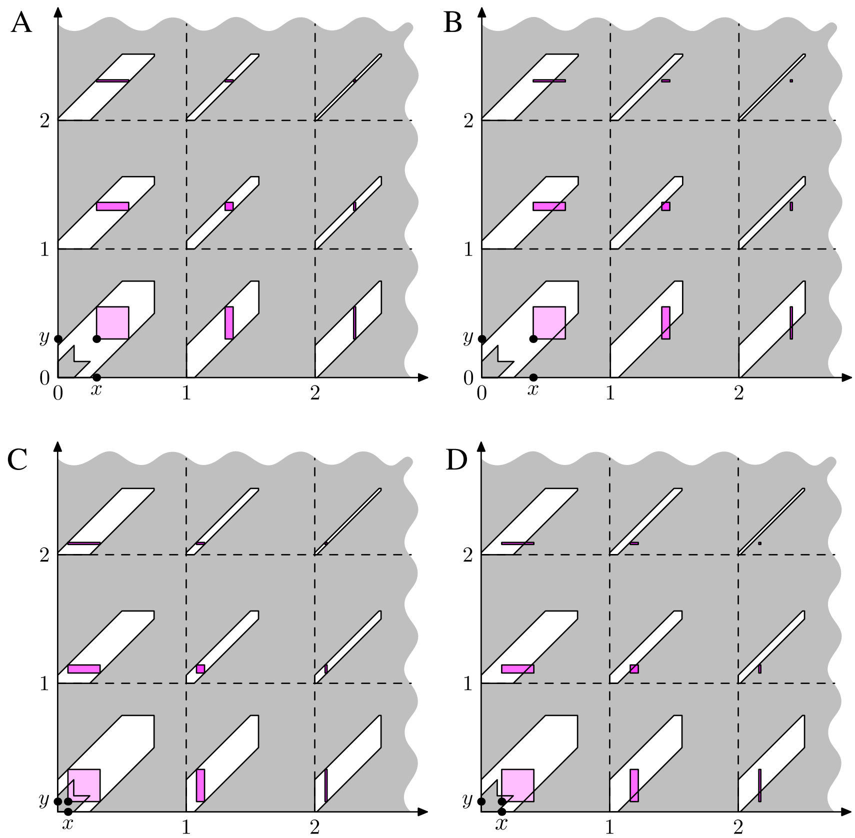

Fig. 6. An illustration of how the perceptual distribution $\mathbf{A}(x)$, $\mathbf{B}(y)$ moves in Fig. 5 as the stimuli x and y change. (A) For large values of $x = y$, the discrimination probability is zero, and (B) any horizontal or vertical deviation (here $x \to x + \varepsilon$) from the diagonal will result in an increase of the discrimination probability; (C) for smaller values of $x = y$, the discrimination probability increases due to the additional piece $\mathfrak{V}$ of decision area in the lower left square, but still (D) it can be shown that any horizontal or vertical deviation (here $x \to x + \varepsilon$) from the diagonal will result in a net increase of the discrimination probability (see the text for details) and so Regular Minimality is satisfied (with PSE function $y = x$) even though the minimum level is nonconstant.

area to the discrimination probability by at most $\varepsilon/2$. However, that deviation will also shift some of the smaller rectangular pieces $\mathfrak{s}_i(x) \times \mathfrak{s}_j(y)$, $i, j \geq 2$, of the support of the perceptual images into the decision area (see, Fig. 6). Given a deviation of size $\varepsilon \in [2 \cdot 4^{-k}, 8 \cdot 4^{-k})$ for $k \geq 2$, all the pieces $\mathfrak{s}_i(x + \varepsilon) \times \mathfrak{s}_j(x)$ for $i, j \geq k$ will shift into the decision area. The total probability mass of these pieces is $\sum_{i,j \geq k} 2^{-i}2^{-j} = 4 \cdot 4^{-k} > \varepsilon/2$. The net effect of any deviation $x \to x + \varepsilon$ from $x = y$ will therefore always be an increase of the discrimination probability. Thus, we have $h(x) = x$, and Regular Minimality follows by symmetry.

In this example, A_x and B_y do have well-defined finite densities with respect to the Lebesgue measure. Still, these measures are not particularly well-behaved: choosing any arc $x_u^{u'}$ ($u' \neq u$) and the set $\mathfrak{a} = \bigcup_k \mathfrak{s}_k(x)$,

$$\left| \frac{A_{x(s')}\mathfrak{a} - A_{x(s)}\mathfrak{a}}{s' - s} \right| = \left| \frac{A_{x(s')}\mathfrak{a} - A_{x(s)}\mathfrak{a}}{x(s') - x(s)} \right| \cdot \left| \frac{x(s') - x(s)}{s' - s} \right|,$$

and as $s' \to s$, the first of the two right-hand factors can be shown to tend to infinity while the second factor does not tend to zero for an uncountable number of $s \in [0, 1]$. This unbounded speed of change is essentially the reason why the conditions of Theorems 4.2–4.4 cannot be satisfied in this example.

Remark 5.1. This example establishes that even with independent perceptual images, the continuity (setwise or V-continuity) of A_s and B_t alone is not sufficient for the statement of Theorems 4.2–4.4 to follow. So we cannot hope to completely dispense with the conditions that make use of the parametrization of the AP-facet (such as Condition 2 of Theorem 4.3 or Conditions 2 and 3 of Theorem 4.4). Weaker forms of these conditions might, of course, still be possible.

6. Prob-Det equivalence and DetInd universality

In this section, we define the probabilistic varieties of Thurstonian-type models and prove that any such model is equivalent to a deterministic model. We use this equivalence to define well-behaved ProbInt and ProbInd models, as those whose equivalent deterministic representations are well-behaved in the sense of Theorem 4.2, Theorem 4.3, or Theorem 4.4. We also use the Prob-Det equivalence to obtain a simple proof of the universality of Thurstonian-type models of the simplest, DetInd variety, as well as a demonstration that well-behaved DetInd models can approximate certain types of non-well-behaved discrimination functions arbitrarily closely.

6.1. Thurstonian-type models of ProbInt and ProbInd varieties

A Thurstonian-type model with *interdependent perceptual images and probabilistic decision rule* (ProbInt variety) is defined as

$\{A_{x,c}, B_{y,c}, C, \sigma\}$,

or more explicitly

$$\left\{ \left\{X, \mathfrak{A}, \Sigma_A, \left\{A_{x,c}\right\}_{x \in X, c \in \mathfrak{C}}\right\}, \left\{Y, \mathfrak{B}, \Sigma_B, \left\{B_{y,c}\right\}_{y \in Y, c \in \mathfrak{C}}\right\}, \{\mathfrak{C}, \Sigma_C, C\}, \sigma : \mathfrak{A} \times \mathfrak{B} \to [0, 1] \right\},$$

where all components are the same as in the models of DetInt variety except we now have no decision area $\mathfrak{S}$ containing pairs

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$(a, b) \in \mathfrak{A} \times \mathfrak{B}$ mapped into the response "different". Instead, every $(a, b) \in \mathfrak{A} \times \mathfrak{B}$ maps into this response with probability $\sigma(a, b)$, and with probability $1 - \sigma(a, b)$ the same pair maps into the response "same".

The generation rule for this model is

$$\psi(x, y) = \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{A}\times\mathfrak{B}} \sigma(a, b)\mathrm{d}A_{x,c}(a)\mathrm{d}B_{y,c}(b)\mathrm{d}C(c),$$

which can also be written as

$$\psi(x, y) = \int_{c\in\mathfrak{C}} \int_{a\in\mathfrak{A}} \left[\int_{b\in\mathfrak{B}} \sigma(a, b)\mathrm{d}B_{y,c}(b)\right] \mathrm{d}A_{x,c}(a)\mathrm{d}C(c)$$
$$= \int_{c\in\mathfrak{C}} \int_{a\in\mathfrak{A}} B^*_{y,c}(a)\mathrm{d}A_{x,c}(a)\mathrm{d}C(c)$$

or

$$\psi(x, y) = \int_{c\in\mathfrak{C}} \int_{b\in\mathfrak{B}} \left[\int_{a\in\mathfrak{A}} \sigma(a, b)\mathrm{d}A_{x,c}(a)\right] \mathrm{d}B_{y,c}(b)\mathrm{d}C(c)$$
$$= \int_{c\in\mathfrak{C}} \int_{b\in\mathfrak{B}} A^*_{x,c}(b)\mathrm{d}B_{y,c}(b)\mathrm{d}C(c).$$

The ProbInd variety (probabilistic decisions, independent images) is obtained as a special case, by assuming that C is a measure concentrated at a point.

6.2. Equivalence of models with probabilistic and deterministic decisions

Two Thurstonian-type models are considered *equivalent* if they generate the same ψ. As it turns out, it is not necessary to deal with models of ProbInt (or ProbInd) variety separately because any such a model is equivalent to an appropriately chosen model of DetInt (respectively, DetInd) variety.

Theorem 6.1. *A ProbInt Thurstonian-type model*

$$\left\{\left\{X, \mathfrak{A}, \Sigma_A, \{A_{x,c}\}_{x\in X, c\in\mathfrak{C}}\right\}, \left\{Y, \mathfrak{B}, \Sigma_B, \{B_{y,c}\}_{y\in Y, c\in\mathfrak{C}}\right\}, \{\mathfrak{C}, \Sigma_C, C\}, \sigma\right\}$$

is equivalent to the DetInt Thurstonian-type model

$$\left\{\begin{array}{c}\left\{X, \mathfrak{A}\times[0,1], \Sigma_A\otimes\Sigma_{[0,1]}, \{A_{x,c}\times\lambda_{[0,1]}\}_{x\in X,c\in\mathfrak{C}}\right\},\\ \left\{Y, \mathfrak{B}\times[0,1], \Sigma_B\otimes\Sigma_{[0,1]}, \{B_{y,c}\times\lambda_{[0,1]}\}_{y\in Y,c\in\mathfrak{C}}\right\},\\ \{\mathfrak{C}, \Sigma_C, C\}, \mathfrak{S}\end{array}\right\},$$

where $\Sigma_{[0,1]}$ is the Borel sigma algebra on $[0, 1]$, $\lambda_{[0,1]}$ is the Lebesgue (i.e., uniform) measure on $[0, 1]$, and[20]

$$\mathfrak{S} = \{(a, \alpha, b, \beta) \in \mathfrak{A} \times [0, 1] \times \mathfrak{B} \times [0, 1] : \alpha \in [0, \sigma(a, b)]\}.$$

(If C is concentrated at a point, the ProbInt model becomes ProbInd and the equivalent DetInt model becomes DetInd.)

Proof. In the deterministic model,

$$\psi(x, y) = \int_{c\in\mathfrak{C}} \left[\int_{(a,\alpha,b,\beta)\in\mathfrak{S}} \mathrm{d}A_{x,c}(a)\mathrm{d}\alpha\mathrm{d}B_{y,c}(b)\mathrm{d}\beta\right] \mathrm{d}C(c)$$
$$= \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{A}\times\mathfrak{B}} \left[\int_{\beta\in[0,1]} \int_{\alpha\in[0,\sigma(a,b)]} \mathrm{d}\alpha\mathrm{d}\beta\right] \mathrm{d}A_{x,c}(a)\mathrm{d}B_{y,c}(b)\mathrm{d}C(c)$$
$$= \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{A}\times\mathfrak{B}} \sigma(a, b)\mathrm{d}A_{x,c}(a)\mathrm{d}B_{y,c}(b)\mathrm{d}C(c),$$

which is the generation rule for the probabilistic model. □

Note that this result makes no assumptions about the stimulus spaces X and Y.

6.3. Well-behaved ProbInt (and ProbInd) models

Returning to arc-connected Hausdorff spaces X, Y, let $\{A_{x,c}, B_{y,c}, C, \sigma\}$ be a Thurstonian-type model of ProbInt variety. Following the logic of Sections 4.2 and 4.3 our goal now is to impose certain ("well-behavedness") properties on AP-restrictions $\{A_{s,c}, B_{t,c}, C, \sigma\}$ of this model to arcs $x_u^{u'}$ and $y_v^{v'}$, such that the corresponding AP-facet

$$\frac{\Delta^{12}_{(s',t')}\varphi(s, t)}{s' - s} = \frac{1}{s' - s}$$
$$\times \int_{c\in\mathfrak{C}} \int_{(a,b)\in\mathfrak{A}\times\mathfrak{B}} \sigma(a, b)\mathrm{d}[A_{s',c}(a) - A_{s,c}(a)]\mathrm{d}[B_{t',c} - B_{t,c}](b)\mathrm{d}C(c)$$

of the discrimination function generated by the model satisfies $\mathcal{R}1$ and $\mathcal{R}2$. Then, following the logic of Section 4.4, one would conclude that if the properties in question are present in an AP-restriction to some arcs $x_u^{u'}$ and $y_v^{v'}$ with $\left[y_v^{v'}\right] = h\left(\left[x_u^{u'}\right]\right)$, where h is a homeomorphism $X \to Y$, then the model $\{A_{x,c}, B_{y,c}, C, \sigma\}$ cannot account for any continuous discrimination function ψ with the PSE function h and unequal minima $\psi(u, h(u))$ and $\psi(u', h(u'))$.

Theorem 6.1 suggests a simple way for specifying these well-behavedness properties: by using the conditions already stipulated for deterministic models in Theorems 4.2–4.4, and applying them to the equivalent deterministic representation $\{A_{s,c} \times \lambda_{[0,1]}, B_{t,c} \times \lambda_{[0,1]}, C, \mathfrak{S}\}$ of the AP-restriction $\{A_{s,c}, B_{t,c}, C, \sigma\}$. In other words, we investigate the properties of the AP-restrictions $\{A_{s,c}, B_{t,c}, C, \sigma\}$ whose equivalent DetInt representations obtained from $\{A_{s,c}, B_{t,c}, C, \sigma\}$ by the construction of Theorem 6.1[21] satisfy the conditions of one of Theorems 4.2–4.4. In particular, for the absolute conditions of Theorem 4.2, we can show the following.

Theorem 6.2. *The measures $A_{s,c} \times \lambda_{[0,1]}$ and $B_{t,c} \times \lambda_{[0,1]}$ of the deterministic representation constructed in Theorem 6.1 satisfy the conditions of Theorem 4.2 if and only if the measures $A_{s,c}$ and $B_{s,c}$ of the original probabilistic representation satisfy them.*

Proof. The "only if" direction is obvious (replace $\mathfrak{a}, \mathfrak{b}$ in Theorem 4.2 with $\mathfrak{a} \times [0, 1]$, $\mathfrak{b} \times [0, 1]$, respectively). We shall prove the "if" direction. Using (15) from the proof of Theorem 4.2 applied to the original measure $A_{s,c}$, we obtain

$$\sup_{c\in\mathfrak{C}\setminus\mathfrak{c}} \sup_{s'\neq s} \left|\frac{(A_{s',c} \times \lambda_{[0,1]})\mathfrak{s} - (A_{s,c} \times \lambda_{[0,1]})\mathfrak{s}}{s' - s}\right|$$
$$= \sup_{c\in\mathfrak{C}\setminus\mathfrak{c}} \sup_{s'\neq s} \left|\int_0^1 \frac{A_{s',c}\mathfrak{s}_u - A_{s,c}\mathfrak{s}_u}{s' - s}\mathrm{d}u\right|$$
$$\leq \int_0^1 \left(\sup_{c\in\mathfrak{C}\setminus\mathfrak{c}} \sup_{s'\neq s} \left|\frac{A_{s',c}\mathfrak{s}_u - A_{s,c}\mathfrak{s}_u}{s' - s}\right|\right) \mathrm{d}u$$
$$\leq \int_0^1 L_A(s)\mathrm{d}u = L_A(s)$$

for any $\mathfrak{s} \in \Sigma_A \otimes \Sigma_{[0,1]}$, where $\mathfrak{s}_u \in \Sigma_A$ are the cross-sections of $\mathfrak{s}$ at various $u \in [0, 1]$. The analogous result holds for $B_{t,c} \times \lambda_{[0,1]}$. □

[20] In the formula to follow $\{\ldots : \alpha \in [0, \sigma(a, b)]\}$ can be replaced with $\{\ldots : \beta \in [0, \sigma(a, b)]\}$ or $\{\ldots : \alpha \in [0, \sigma_1(a, b)], \beta \in [0, \sigma_2(a, b)]\}$ where $\sigma_1\sigma_2 \equiv \sigma$ (both functions mapping into $[0, 1]$).

[21] Clearly, the DetInt representation $\{A_{s,c} \times \lambda_{[0,1]}, B_{t,c} \times \lambda_{[0,1]}, C, \mathfrak{S}\}$ which is equivalent to an AP-restriction $\{A_{s,c}, B_{t,c}, C, \sigma\}$ of a ProbInt model $\{A_{x,c}, B_{y,c}, C, \sigma\}$ to arcs $x_u^{u'}$ and $y_v^{v'}$ is the AP-restriction to the same arcs of the DetInt model $\{A_{x,c} \times \lambda_{[0,1]}, B_{y,c} \times \lambda_{[0,1]}, C, \mathfrak{S}\}$ equivalent to the ProbInt model $\{A_{x,c}, B_{y,c}, C, \sigma\}$.

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Thus, the conditions of Theorem 4.2 are absolute in the broadest sense: the decision area does not matter and can very well be replaced by a probabilistic decision function (whose choice does not matter either). By contrast, the relative conditions of Theorems 4.3 and 4.4 cannot be formulated for the measures $A_{s,c}$ and $B_{s,c}$ of the probabilistic model as these conditions refer to a decision area. We have therefore to use the probability function σ to construct an analogue of a decision area and relate to it the measures $A_{s,c}$ and $B_{s,c}$. One possible formulation is given by the following theorem.

Theorem 6.3. *Given the measures $A_{s,c} \times \lambda_{[0,1]}$ and $B_{t,c} \times \lambda_{[0,1]}$ and the decision area*

$$\mathfrak{S} = \{(a, \alpha, b, \beta) \in \mathfrak{A} \times [0, 1] \times \mathfrak{B} \times [0, 1] : \beta \in [0, \sigma(a, b)]\}$$

of the deterministic equivalent constructed in Theorem 6.1 for a ProbInt model,

(i) *Condition 1 of Theorem 4.3 or Theorem 4.4 is satisfied if and only if it is satisfied by the measures $A_{s,c}$ and $B_{t,c}$ of the original ProbInt model;*

(ii) *Condition 2 of Theorem 4.3 or Conditions 2 and 3 of Theorem 4.4 are satisfied if and only if the original measure $A_{s,c}$ satisfies them with the set of cross-sections $\{\mathfrak{S}_b\}_{b\in\mathfrak{B}}$ replaced by the sets*

$$\{a \in \mathfrak{A} : \sigma(a, b) \geq \beta\}_{(b,\beta)\in\mathfrak{B}\times[0,1]}.$$

Proof. (i) We shall show that setwise or V-continuity of $B_{t,c}$ at a point implies that of $B_{t,c} \times \lambda_{[0,1]}$ so that Condition 1 of Theorem 4.3 or Theorem 4.4 is satisfied (the reverse implication is obvious). For any $\mathfrak{s} \in \Sigma_B \otimes \Sigma_{[0,1]}$, we have

$$(B_{t',c} \times \lambda_{[0,1]})\mathfrak{s} - (B_{t,c} \times \lambda_{[0,1]})\mathfrak{s} = \int_{\mathfrak{B}} \lambda_{[0,1]}\mathfrak{s}_b d[B_{t',c} - B_{t,c}](b),$$

where $\mathfrak{s}_b \in \Sigma_{[0,1]}$ are the cross-sections of $\mathfrak{s}$ at various $b \in \mathfrak{B}$. Thus, the setwise continuity statement follows from Lemma A.1 (Statement 2) and the V-continuity statement follows from Lemma A.3.

(ii) Denoting the cross-sections of the decision area $\mathfrak{S}$ of the equivalent deterministic model by

$$\mathfrak{S}_{b,\beta} = \{ (a, \alpha) \in \mathfrak{A} \times [0, 1] : \sigma(a, b) \geq \beta\}$$

for all $(b, \beta) \in \mathfrak{B} \times [0, 1]$, the ratio used in Conditions 2 and 3 of these theorems for the augmented measure $A_{s',c} \times \lambda_{[0,1]}$ can be written as

$$\begin{aligned} &\frac{(A_{s',c} \times \lambda_{[0,1]})\mathfrak{S}_{b,\beta} - (A_{s,c} \times \lambda_{[0,1]})\mathfrak{S}_{b,\beta}}{s' - s} \\ &= \frac{((A_{s',c} - A_{s,c}) \times \lambda_{[0,1]})\{(a, \alpha) \in \mathfrak{A} \times [0, 1] : \sigma(a, b) \geq \beta\}}{s' - s} \\ &= \frac{(A_{s',c} - A_{s,c})\{a \in \mathfrak{A} : \sigma(a, b) \geq \beta\}}{s' - s} \end{aligned}$$

for all $(b, \beta) \in \mathfrak{B} \times [0, 1]$. □

6.4. The universality of DetInd models

The equivalence result of the previous section (Theorem 6.1) allows for a surprisingly simple proof of the universality statement of Dzhafarov (2003a).

Theorem 6.4. *Every discrimination function ψ can be generated by an appropriately chosen Thurstonian-type model of DetInd variety.*

Proof. Take a ProbInd model with

$$\mathfrak{A} = X, \quad \mathfrak{B} = Y, \quad A_x = \delta_x, \quad B_y = \delta_y,$$
$$\sigma(a, b) = \psi(a, b),$$

where δ_x and δ_y represent measures concentrated at points x and y, respectively, and sigma algebras are arbitrary insofar as they include singletons. Clearly, this model generates ψ. Due to Theorem 6.1, this model is equivalent to the DetInd model with

$$\mathfrak{A} = X \times [0, 1], \quad \mathfrak{B} = Y \times [0, 1],$$
$$A_x = \delta_x \times \lambda_{[0,1]}, \quad B_y = \delta_y \times \lambda_{[0,1]}$$

(where $\lambda_{[0,1]}$ is the uniform measure on [0, 1]) and

$$\mathfrak{S} = \{(a, \alpha, b, \beta) \in \mathfrak{A} \times [0, 1] \times \mathfrak{B} \times [0, 1] : \alpha \in [0, \psi(a, b)]\}.$$

□

Note that, like Theorem 6.1 on which it is based, this result makes no assumptions about the stimulus spaces X and Y. Of course, the singular measures of these universal constructions are not well-behaved by any reasonable definition, including those adopted in the present paper.

6.5. Approximations by well-behaved models

We know from Theorem 2.1 that no well-behaved discrimination function can satisfy Regular Minimality and have a nonconstant minimum level function. However, well-behaved Thurstonian-type representation can still generate discrimination functions which arbitrarily closely *approximate* a function with these properties.

To provide an illustrating example we show that any *uniformly continuous* discrimination function $\psi : \mathbb{R} \times \mathbb{R} \to [0, 1]$ can be approximated arbitrarily well by a DetInd representation satisfying the well-behavedness conditions of Theorem 4.2. For any $\varepsilon > 0$, the uniform continuity of ψ yields a $\delta > 0$ such that $|\psi(x', y') - \psi(x, y)| < \varepsilon$ whenever $\max\{|x' - x|, |y' - y|\} < \delta$. The ProbInd representation given by

$$\mathfrak{A} = \mathbb{R}, \quad \mathfrak{B} = \mathbb{R}, \quad \mathbf{A}(x) \sim \text{Uniform}\,[x - \delta, x + \delta],$$
$$\mathbf{B}(y) \sim \text{Uniform}[y - \delta, y + \delta], \quad \sigma(a, b) = \psi(a, b)$$

generates the function

$$\psi_\delta(x, y) = \left(\frac{1}{2\delta}\right)^2 \int_{x-\delta}^{x+\delta} \int_{y-\delta}^{y+\delta} \psi(a, b) \mathrm{d}a\mathrm{d}b,$$

which, by the mean value theorem, equals $\psi_\delta(x', y')$ for some $(x', y') \in [x - \delta, x + \delta] \times [y - \delta, y + \delta]$. It follows that $\sup_{x,y} |\psi_\delta(x, y) - \psi(x, y)| < \varepsilon$. We show now that our Thurstonian-type representation for ψ_δ satisfies the conditions of Theorem 4.2 for any two arc images $\left[x_u^{u'}\right]$ and $\left[y_v^{v'}\right]$ chosen in $\mathbb{R}$. Let the arcs be parametrized as $x_u^{u'}(s) = (1 - s)u + su'$ and $y_v^{v'}(t) = (1-t)v + tv'$. It is easy to see that for any intervals (hence for any Borel sets) $\mathfrak{a}$ and $\mathfrak{b}$,

$$\limsup_{s'\to s} \left|\frac{A_{s'}\mathfrak{a} - A_s\mathfrak{a}}{s' - s}\right| \leq \frac{|u' - u|}{2\delta} < \infty,$$
$$\limsup_{t'\to t} \left|\frac{B_{t'}\mathfrak{b} - B_t\mathfrak{b}}{t' - t}\right| \leq \frac{|v' - v|}{2\delta} < \infty,$$

whence the quantities $L_1(s, \mathfrak{a})$ and $L_2(t, \mathfrak{b})$ as defined in Theorem 4.2 (Remark 4.1) are finite too. An equivalent DetInd representation is obtained by Theorem 6.1, and the well-behavedness of its AP-restriction to the arcs $x_u^{u'}$ and $y_v^{v'}$ is preserved due to Theorem 6.2.

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7. Conclusion

We have established the following results.

- Every discrimination probability function $\psi : X \times Y \to [0, 1]$ can be generated by an appropriately chosen Thurstonian-type model of DetInd variety (deterministic decision rule, independent perceptual images).
- Every Thurstonian-type model with probabilistic decisions (i.e., of ProbInt or ProbInd variety) is equivalent, in the sense of generating the same function ψ, to a model with deterministic decisions (of DetInt or DetInd variety, respectively).

These statements (with very simple proofs) hold for stimulus spaces X, Y of completely arbitrary nature.

Assuming that X and Y are Hausdorff arc-connected topological spaces, we imposed certain restrictions ("well-behavedness constraints") on the components of Thurstonian-type models and established that

- a Thurstonian-type model subject to these constraints cannot generate a continuous discrimination probability functions ψ which simultaneously (a) satisfies Regular Minimality with a homeomorphic PSE function and (b) has nonconstant minima;
- a model with probabilistic decisions has an equivalent to it model with deterministic decisions such that either of them satisfies well-behavedness constraints only if the other one does too.

We have also demonstrated that

- in certain cases, even if a continuous discrimination function satisfies the properties (a) and (b) above, it can be approximated to an arbitrary degree of accuracy by an appropriately chosen Thurstonian model subject to well-behavedness constraints.

These results greatly expand and refine the analogous results obtained in Dzhafarov (2003a,b). In particular, in our present formulations the well-behavedness restrictions imposed on the Thurstonian-type model are purely topological in the sense that they are invariant under homeomorphic reparametrizations of the stimulus spaces X and Y.

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Appendix. Auxiliary results

The following construction is useful for many purposes. Let $0 \le f(m) \le 1$ be a measurable function on a probability space $(\mathfrak{M}, \Sigma, M)$. Define

$$\mathfrak{e}_1 = \left\{ m : f_1(m) \ge \frac{1}{2} \right\}, \quad \text{where } f_1(m) = f(m),$$

$$\mathfrak{e}_2 = \left\{ m : f_2(m) \ge \frac{1}{4} \right\},$$

$$\text{where } f_2(m) = \begin{cases} f_1(m) - \frac{1}{2}, & \text{if } f_1(m) \ge \frac{1}{2}, \\ f_1(m), & \text{if } f_1(m) < \frac{1}{2}, \end{cases}$$

...

$$\mathfrak{e}_k = \left\{ m : f_k(m) \ge 2^{-k} \right\},$$

$$\text{where } f_k(m) = \begin{cases} f_{k-1}(m) - 2^{-k+1}, & \text{if } f_{k-1}(m) \ge 2^{-k+1}, \\ f_{k-1}(m), & \text{if } f_{k-1}(m) < 2^{-k+1}, \end{cases}$$

...

All these sets are clearly M-measurable if so is $f(m)$. We have

$$f(m) = \sum_{k=1}^{\infty} 2^{-k} \chi_{\mathfrak{e}_k}(m),$$

where $\chi_{\mathfrak{e}}(m)$ denotes the truth value (0 or 1) of the statement $m \in \mathfrak{e}$. Indeed, any combination of the truth values of $m \in \mathfrak{e}_1$, $m \in \mathfrak{e}_2, \ldots$ defines a binary representation of $f(m)$ and vice versa.

It follows now that in general a bounded measurable function $f : \mathfrak{M} \mapsto \mathbb{R}$ can be written as

$$f(m) = l + (L - l) \sum_{k=1}^{\infty} 2^{-k} \chi_{\mathfrak{e}_k}(m), \tag{21}$$

where l and L are any numbers such that

$$l \le \inf_{m \in \mathfrak{M}} f(m) \le \sup_{m \in \mathfrak{M}} f(m) \le L.$$

Lemma A.1. *Let finite measures $\{M_n\}_{n \in \mathbb{N}}$, M be defined on a measurable space $(\mathfrak{M}, \Sigma)$. Then, the following statements are equivalent: as $n \to \infty$,*

1. *$M_n \to M$ setwise (i.e., $M_n \mathfrak{s} \to M\mathfrak{s}$ for all $\mathfrak{s} \in \Sigma$);*
2. *$\int f(m) \mathrm{d}M_n(m) - \int f(m) \mathrm{d}M(m) \to 0$ for all bounded measurable $f : \mathfrak{M} \to [l, L]$;*
3. *$\int f_n(m) \mathrm{d}M_n(m) - \int f(m) \mathrm{d}M(m) \to 0$ for all uniformly bounded measurable f, $\{f_n\}_{n \in \mathbb{N}} : \mathfrak{M} \to [l, L]$ such that $f_n \to f$ pointwise;*
4. *$\int f_n(m) \mathrm{d}M_n(m) - \int f_n(m) \mathrm{d}M(m) \to 0$ for all uniformly bounded measurable $\{f_n\}_{n \in \mathbb{N}} : \mathfrak{M} \to [l, L]$ that converge pointwise to some function f.*

Proof. To prove 1 $\Longrightarrow$ 2 use the representation (21) to obtain

$$\int_{\mathfrak{M}} f(m) \mathrm{d}M_n(m) - \int_{\mathfrak{M}} f(m) \mathrm{d}M(m) = (L - l) \sum_{k=1}^{\infty} 2^{-k} [M_n \mathfrak{e}_k - M \mathfrak{e}_k].$$

Since $M_n \mathfrak{e}_k - M \mathfrak{e}_k \to 0$ for all k and $|M_n \mathfrak{e}_k - M \mathfrak{e}_k| \le 1$, the series vanishes as $n \to \infty$.

To prove 1&2 $\Longrightarrow$ 3 we write

$$\int_{\mathfrak{M}} f_n(m) \mathrm{d}M_n(m) - \int_{\mathfrak{M}} f(m) \mathrm{d}M(m) = \int_{\mathfrak{M}} [f_n(m) - f(m)] \mathrm{d}M_n(m) + \left[\int_{\mathfrak{M}} f(m) \mathrm{d}M_n(m) - \int_{\mathfrak{M}} f(m) \mathrm{d}M(m) \right].$$

Due to the previous result, we only have to show that the first right-hand integral tends to zero. For this we use Egorov's theorem stated below (Lemma A.2): for an arbitrary $\varepsilon > 0$, there is a set $\mathfrak{s}$ such that $M\mathfrak{s} < \varepsilon$ and $\sup_{m \in \mathfrak{M} \setminus \mathfrak{s}} |f_n(m) - f(m)| \to 0$. Then,

$$\left| \int_{\mathfrak{M} \setminus \mathfrak{s}} [f_n(m) - f(m)] \mathrm{d}M_n \right| \le \sup_{m \in \mathfrak{M} \setminus \mathfrak{s}} |f_n(m) - f(m)| M_n(\mathfrak{M} \setminus \mathfrak{s}) \to 0 \cdot M(\mathfrak{M} \setminus \mathfrak{s}) = 0,$$

$$\left| \int_{\mathfrak{s}} [f_n(m) - f(m)] \mathrm{d}M_n \right| \le \sup_{m \in \mathfrak{s}} |f_n(m) - f(m)| M_n \mathfrak{s} \le (L - l) M_n \mathfrak{s} \to (L - l) M\mathfrak{s} \le (L - l)\varepsilon,$$

and it follows that

$$\limsup_{n\to\infty}\left|\int_{\mathfrak{M}}[f_n(m)-f(m)]\mathrm{d}M_n\right| \le (L-l)\varepsilon.$$

As $\varepsilon > 0$ can be chosen arbitrarily small, the proof is completed.

To prove 3 $\Longrightarrow$ 4 we write

$$\int_{\mathfrak{M}} f_n(m)\mathrm{d}M_n(m) - \int_{\mathfrak{M}} f_n(m)\mathrm{d}M(m) = \int_{\mathfrak{M}}[f(m)-f_n(m)]\,\mathrm{d}M(m) + \left[\int_{\mathfrak{M}} f_n(m)\mathrm{d}M_n(m) - \int_{\mathfrak{M}} f(m)\mathrm{d}M(m)\right].$$

As the range of f is clearly within $[l, L]$, the first right-hand term tends to zero by the dominated convergence theorem and the second one by Statement 3.

Finally, to prove 4 $\Longrightarrow$ 1 put $f_n \equiv f$ to be the characteristic function of a measurable set $\mathfrak{s}$. □

Lemma A.2 (*Egorov's Theorem*). *Let $(\mathfrak{M}, \Sigma, M)$ be a finite measure space. If a sequence $\{f_n\}$ of almost everywhere finite measurable functions converges almost everywhere to an almost everywhere finite measurable function f, then the convergence is almost uniform, i.e., for each $\varepsilon > 0$, there exists a measurable set $\mathfrak{s} \in \Sigma$ such that $M\mathfrak{s} < \varepsilon$ and $\sup_{m\in\mathfrak{M}\setminus\mathfrak{s}} |f_n(m) - f(m)| \to 0$ as $n \to \infty$.*

Proof. See, e.g., Hewitt and Stromberg (1965, p. 158). □

Lemma A.3. *Let $\{M_n\}_{n\in\mathbb{N}}$, M be as in Lemma A.1. The following statements are equivalent: as $n \to \infty$,*

1. *$M_n \to M$ in total variation (i.e., $\sup_{\mathfrak{s}\in\Sigma} |M_n\mathfrak{s} - M\mathfrak{s}| \to 0$);*
2. *$\int f_n(m)\mathrm{d}M_n(m) - \int f_n(m)\mathrm{d}M(m) \to 0$ for every sequence $\{f_n\}$ of uniformly bounded measurable functions $\mathfrak{M} \to [l, L]$.*

Proof. To prove 1 $\Longrightarrow$ 2 we use (21) to obtain

$$\left|\int f_n(m)\mathrm{d}M_n(m) - \int f_n(m)\mathrm{d}M(m)\right| = (L-l)\left|\sum_{k=1}^{\infty} 2^{-k}[M_n\mathfrak{e}_{k,n} - M\mathfrak{e}_{k,n}]\right| \le (L-l)\sum_{k=1}^{\infty} 2^{-k}|M_n\mathfrak{e}_{k,n} - M\mathfrak{e}_{k,n}| \le (L-l)\sup_{\mathfrak{e}\in\Sigma}|M_n\mathfrak{e} - M\mathfrak{e}| \to 0.$$

To prove 2 $\Longrightarrow$ 1, deny this and assume that $\sup_{\mathfrak{s}\in\Sigma} |M_n\mathfrak{s} - M\mathfrak{s}| \nrightarrow 0$. Then, for some sequence $\{\mathfrak{s}_n\}$ in Σ, $|M_n\mathfrak{s}_n - M\mathfrak{s}_n| \nrightarrow 0$. But this contradicts 2 if we choose f_n to be the characteristic functions of $\mathfrak{s}_n$. □

Lemma A.4 (*Nikodým's Boundedness Theorem*). *Let $\{M_\gamma : \Sigma \to \mathbb{R}\}_{\gamma\in\Gamma}$ be a family of bounded signed measures, i.e.,*

$$\sup_{\mathfrak{s}\in\Sigma} |M_\gamma\mathfrak{s}| < \infty \quad \text{for all } \gamma \in \Gamma.$$

If the family is setwise bounded, i.e.,

$$\sup_{\gamma\in\Gamma} |M_\gamma\mathfrak{s}| < \infty \quad \text{for all } \mathfrak{s} \in \Sigma,$$

then the family is uniformly bounded,

$$\sup_{\gamma\in\Gamma,\mathfrak{s}\in\Sigma} |M_\gamma\mathfrak{s}| < \infty.$$

Proof. See, e.g., Faires (1965), Theorem 2. □

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