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Theoretical note

A new definition of well-behaved discrimination functions

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ABSTRACT

A discrimination function shows the probability or degree with which stimuli are discriminated from each other when presented in pairs. In a previous publication [Kujala, J.V., & Dzhafarov, E.N. (2008). On minima of discrimination functions. *Journal of Mathematical Psychology*, *52*, 116–127] we introduced a condition under which the conformity of a discrimination function with the law of Regular Minimality (which says, essentially, that "being least discriminable from" is a symmetric relation) implies the constancy of the function's minima (i.e., the same level of discriminability of every stimulus from the stimulus least discriminable from it). This condition, referred to as "well-behavedness," turns out to be unnecessarily restrictive. In this note we give a significantly more general definition of well-behavedness, applicable to all Hausdorff arc-connected stimulus spaces. The definition employs the notion of the smallest transitively and topologically closed extension of a relation. We provide a transfinite-recursive construction for this notion and illustrate it by examples.

In Kujala and Dzhafarov (2008) we studied *discrimination functions* $\psi : X \times Y \to [0, 1]$ with the codomain representing probabilities with which, or degrees to which, x and y are judged to be different. The domain components, X and Y, represent stimulus sets in two *observation areas* (e.g., amplitude values of tones presented chronologically first and those of tones presented second). We impose very mild restrictions on these sets: we assume them to be *Hausdorff arc-connected* topological spaces.[1] We assume that ψ is continuous with respect to the product topology on $X \times Y$.

We say that ψ satisfies the law of *Regular Minimality* if:

($\mathcal{P}1$) for some function $h : X \to Y$ and all $x \in X, y \in Y$,

$$y \neq h(x) \Longrightarrow \psi(x, h(x)) < \psi(x, y), \tag{1}$$

($\mathcal{P}2$) for some function $g : Y \to X$ and all $x \in X, y \in Y$,

$$x \neq g(y) \Longrightarrow \psi(g(y), y) < \psi(x, y), \tag{2}$$

($\mathcal{P}3$) the two functions are each other's inverses,

$$g \equiv h^{-1}. \tag{3}$$

The stimulus $h(x) \in Y$ is called the *point of subjective equality* (PSE) for x, and for any $y \in Y$, the stimulus $g(y) \in X$ is called the PSE for y. The functions h and g are referred to as the *PSE functions* ($X \to Y$ and $Y \to X$, respectively).

The function

$$\omega(x) = \psi(x, h(x)) \tag{4}$$

is called the *minimum level function* (along the PSE function h).[2] It is easy to see that if ψ is sufficiently "well-behaved" (e.g., if X and Y are regions of $\mathbb{R}^n$ and ψ is continuously differentiable) then Regular Minimality implies $\omega(x) \equiv const$.[3] For stimuli representable by points in $\mathbb{R}^n$ the notion of well-behavedness (which generalizes the continuous differentiability of ψ) was introduced in Dzhafarov (2003). In Kujala and Dzhafarov (2008) we generalized it further to the class of continuous functions on Hausdorff arc-connected stimulus spaces.[4]

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1 Recall that a space is Hausdorff if any two distinct points in it belong to disjoint open sets. The arc-connectedness of a Hausdorff space means that any two points in it belong to an image of a continuous mapping of an interval of reals into the space.

2 The minimum level function can be equivalently, in the sense of describing the same graph, defined as $\psi(g(y), y)$. The choice makes no difference for our analysis.

3 This is of interest because in empirical data, to the extent they support Regular Minimality, the minimum level function is not constant, implying that a model aimed at accounting for both Regular Minimality and the nonconstancy of $\omega(x)$ should generate functions which are not well-behaved (see Dzhafarov, 2003).

4 Kujala and Dzhafarov (2008) also posited that stimulus spaces were first countable. This was not, strictly speaking, necessary. In the present paper we will only use the first countability to simplify the description of a certain transfinite-recursive construction (see footnote 12).

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It turns out, however, that the definition given in Kujala and Dzhafarov (2008) is unnecessarily restrictive (and its formulation unnecessarily complicated). As it serves little purpose to analyze a construction we now consider unsatisfactory, we will instead simply go ahead and present a significantly more general and better constructed definition (but all relevant facts and results from Kujala & Dzhafarov, 2008, will be recapitulated).

1. Preliminaries

We focus on continuous discrimination functions ψ with *homeomorphic PSE functions h*. The continuity of ψ, if the latter satisfies Regular Minimality, does not imply the continuity of h or h^{-1} (see Kujala & Dzhafarov, 2008, Section 2.2).[5]

Let Z denote either X or Y, and $p, q \in Z$ be two distinct points. It is convenient to present an arc[6] $z : [0,1] \to Z$ with $z(0) = p$ and $z(1) = q$ as z_p^q, to indicate its endpoints and to distinguish it from points in Z. So we will speak of arcs z_p^q with points $z(t)$, $t \in [0,1]$ (of course, the notation z_p^q itself does not determine the arc, only its endpoints). To distinguish an arc as a mapping, $z_p^q : [0,1] \to Z$, from its image $z_p^q([0,1])$ in Z, we will denote the image $\left[z_p^q\right]$. Clearly, different arcs z_p^q and $\widetilde{z}_p^q$ may have the same image, $\left[z_p^q\right] = \left[\widetilde{z}_p^q\right]$.

Given any two arcs $x_u^{u'} : [0,1] \to X$ and $y_v^{v'} : [0,1] \to Y$ (with endpoints u, u' and v, v', respectively), the function

$$\varphi(s,t) = \psi(x(s), y(t)) \tag{5}$$

is called an *arc-parametrized facet* (*AP-facet*, for short) of ψ. Since ψ is continuous, $\varphi(s,t)$ is continuous (hence uniformly continuous) on $[0,1] \times [0,1]$.

Given an AP-facet φ, we use the following notation for finite differences of the first and second order. For any $s, s', t, t' \in [0,1]$,

$$\begin{aligned}\Delta^1_{s'}\varphi(s,t) &= \varphi(s',t) - \varphi(s,t),\\ \Delta^2_{t'}\varphi(s,t) &= \varphi(s,t') - \varphi(s,t),\end{aligned} \tag{6}$$

with the superscripts referring to the position of the arguments changed. Analogously,

$$\begin{aligned}\Delta^{12}_{(s',t')}\varphi(s,t) &= \Delta^1_{s'}\Delta^2_{t'}\varphi(s,t) = \Delta^2_{t'}\Delta^1_{s'}\varphi(s,t)\\ &= \varphi(s',t') - \varphi(s',t) - \varphi(s,t') + \varphi(s,t).\end{aligned} \tag{7}$$

Another notation convention: we use double arrows $(s',t') \rightrightarrows (s,t)$ to indicate that s' and t' approach, respectively, s and t from the same side. Specifically:

$$\begin{aligned}&(s',t') \rightrightarrows (s,t)\pm \quad \text{means one of the two: } \begin{matrix} s' \to s+ \text{ and } t' \to t+,\\ s' \to s- \text{ and } t' \to t-,\end{matrix}\\ &(s',t') \rightrightarrows (s,t) \quad \text{means } s' \to s \text{ and } t' \to t \text{ and } (s'-s)(t'-t) \geq 0.\end{aligned} \tag{8}$$

The definition below is unchanged from Kujala and Dzhafarov (2008, the motivation for this definition can be found on p. 125). This is an intermediate concept, needed to formulate the notion (generalized from Kujala & Dzhafarov, 2008) of a continuous discrimination function well-behaved with respect to the PSE function h.

Definition 1. Given a continuous function ψ and a pair of arc images, $\left[x_u^{u'}\right]$ and $\left[y_v^{v'}\right]$, we say that the restriction $\psi|\left[x_u^{u'}\right] \times \left[y_v^{v'}\right]$ of ψ is *well-behaved* on $\left[x_u^{u'}\right]$ if, for some parametrizations[7] $x_u^{u'} : [0,1] \to \left[x_u^{u'}\right]$ and $y_v^{v'} : [0,1] \to \left[y_v^{v'}\right]$, the resulting AP-facet φ of ψ has the following properties:

($\mathcal{R}1$) for all $(s,t) \in [0,1] \times [0,1]$ except for an at most denumerable set,

$$\limsup_{(s',t') \rightrightarrows (s,t)} \left| \frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s} \right| < \infty; \tag{9}$$

($\mathcal{R}2$) for almost all $s \in [0,1]$ and almost all $t \in [0,1]$,[8]

$$\lim_{(s',t') \rightrightarrows (s,t)\pm} \frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{s'-s} = 0, \tag{10}$$

where the choice of $+$ or $-$ may be different for different (s,t).

The definition of a restriction $\psi|\left[x_u^{u'}\right] \times \left[y_v^{v'}\right]$ well-behaved on $\left[y_v^{v'}\right]$ is obtained by replacing the quotient in (9) and (10) with

$$\frac{\Delta^{12}_{(s',t')}\varphi(s,t)}{t'-t}.$$

In Kujala and Dzhafarov (2008, Lemma 5) it was shown that for the well-behavedness on $\left[x_u^{u'}\right]$ the parametrization $y_v^{v'} : [0,1] \to \left[y_v^{v'}\right]$ is irrelevant and can be chosen arbitrarily (and, symmetrically, for the well-behavedness on $\left[y_v^{v'}\right]$ the parametrization $x_u^{u'}$ of $\left[x_u^{u'}\right]$ is irrelevant). This is apparent from inspecting (9) and (10).

2. Well-behavedness of discrimination functions: A new definition

Definition 2. Given a continuous discrimination function ψ with a homeomorphic PSE function h, let us define $E(\psi, h)$ as the set of all pairs (u, u') in $X \times X$ such that for some arc $x_u^{u'}$, the restriction $\psi|\left[x_u^{u'}\right] \times h\left(\left[x_u^{u'}\right]\right)$ is well-behaved on at least one of the two arc images, $\left[x_u^{u'}\right]$ or $h\left(\left[x_u^{u'}\right]\right)$.[9]

In Kujala and Dzhafarov (2008, Theorem 5) we showed that if $(u, u') \in E(\psi, h)$, then

$$\omega(u) = \psi(u, h(u)) = \psi(u', h(u')) = \omega(u'),$$

with the obvious implication: if $E(\psi, h) = X \times X$, then the minimum level function $\omega(x) = \psi(x, h(x))$ is constant. We defined

5 The continuity of h and h^{-1} had been part of the original formulation of Regular Minimality (in Dzhafarov, 2002, 2003) but the formulation was made more general (referring to any bijective h) in subsequent publications. In most of these later publications h is transformed into an identity function by means of a so-called *canonical transformation* of ψ. We adopt a compromise approach in which Regular Minimality is formulated in complete generality and the homeomorphic nature of the PSE function h is stipulated additionally.

6 We define an arc z as a function which is either a homeomorphism from $[0,1]$ to a subset of Z or a constant mapping from $[0,1]$ to a singleton in Z (the reason for including the latter case is simply to make each point of Z arc-connected to itself).

7 A parametrization of an arc image $[x]$ is simply a choice of an arc $x : [0,1] \to X$ whose image is $[x]$.

8 "Almost all" here refers to the Lebesgue measure on $[0,1]$. Note that the condition is more restrictive than "for almost all (s,t) in $[0,1] \times [0,1]$".

9 Throughout this paper, the reference to ψ with a homeomorphic PSE function implies that ψ satisfies Regular Minimality. Consequently, the notation $E(\psi, h)$ always implies that ψ is such a function. Note that the choice of $X \times X$ over $Y \times Y$ in the definition of $E(\psi, h)$ is arbitrary: we could very well consider instead $E(\psi, h^{-1})$ as the set of $(v, v') \in Y \times Y$ such that for some arc $y_v^{v'}$, the restriction $\psi|h^{-1}\left(\left[y_v^{v'}\right]\right) \times \left[y_v^{v'}\right]$ is well-behaved on $\left[y_v^{v'}\right]$ or on $h^{-1}\left(\left[y_v^{v'}\right]\right)$.

then a well-behaved discrimination function ψ as one for which $E(\psi, h) = X \times X$. As mentioned above, this formulation is unnecessarily restrictive.[10] The theorem below shows that rather than identifying $X \times X$ with $E(\psi, h)$ itself it is sufficient to adopt the following definition.

Definition 3. A continuous discrimination function ψ is well-behaved with respect to its homeomorphic PSE function h if $X \times X$ coincides with the transitive–topological closure of $E(\psi, h)$, i.e., with the smallest transitively closed and topologically closed subset of $X \times X$ containing $E(\psi, h)$.[11]

The set $E(\psi, h)$ itself is generally neither transitively nor topologically closed, because of which it may very well be a proper subset of $X \times X$. At the same time, the transitive–topological closure of $E(\psi, h)$ exists and is unique. Indeed, let $\mathbb{U}$ be a collection of all transitively and topologically closed subsets of $X \times X$ containing $E(\psi, h)$ (clearly, $X \times X \in \mathbb{U}$). Then the intersection $\bigcap \mathbb{U}$ is the the transitive–topological closure of $E(\psi, h)$, as it clearly contains $E(\psi, h)$, is transitively and topologically closed, and is contained in any member of $\mathbb{U}$.

Theorem 1. *If a continuous discrimination function ψ with a homeomorphic PSE function h is well-behaved, then the minimum level function $\omega(x)$ is constant.*

Proof. Let

$$A = \{(u, u') \in X \times X : \omega(u) = \omega(u')\}.$$

Obviously, A is transitively closed. It is also topologically closed since $X \times X \setminus A$ is open: indeed, if $(u, u') \in X \times X \setminus A$, i.e., $\omega(u) \neq \omega(u')$, then the continuity of ω implies that some open neighborhood of (u, u') should lie within $X \times X \setminus A$. Since $E(\psi, h) \subset A$, it follows that A contains the transitive–topological closure of $E(\psi, h)$. But by Definition 3, this transitive–topological closure is $X \times X$. □

Separately taken, the operations of transitive closure and topological closure of $E(\psi, h)$ have transparent "procedural" meanings. To effect the transitive closure, we simply add to $E(\psi, h)$ the pair (u, u'') every time we find (u, u') and (u', u'') in $E(\psi, h)$. To effect the topological closure, we add to $E(\psi, h)$ all its limit points in $X \times X$. Definition 3 and Theorem 1 do not elucidate, however, the "procedural" meaning of the transitive–topological closure of $E(\psi, h)$: generally, a transitive (topological) closure of a subset of $X \times X$ will not preserve its topological (respectively, transitive) closedness. We will describe therefore a construction of the transitive–topological closure of $E(\psi, h)$ by means of a transfinite recursion which employs alternating operations of topological and transitive closure. The construction is confined to first countable spaces X and Y (in addition to their being Hausdorff and arc-connected).[12]

3. Ordinal numbers: A primer

We begin by recalling the notion of ordinal numbers, or *ordinals* (see, e.g., Natanson, 1964b, Chapter 14; Wolf, 2005, Section 2.4). Natural numbers $0, 1, 2, 3, \ldots$ are ordinals. They can be thought of as nested sets: $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, etc. The set $\{0, 1, 2, 3, \ldots\}$ is the first *transfinite* ordinal, ω, and $\{0, 1, 2, 3, \ldots, \omega\}$ is the next ordinal, $\omega + 1$. One continues in this manner to form $\omega + 2, \omega + 3, \ldots, \omega + \omega = \omega \cdot 2, \ldots, \omega \cdot 3, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}$, etc.

Stated rigorously, a set γ is an ordinal if it has the following property: elements of γ are sets, and if $\beta \in \gamma$ and $\alpha \in \beta$, then $\alpha \subset \beta \subset \gamma$. To mention some basic facts about ordinals: (1) the empty set is an ordinal (denoted 0); (2) every element of an ordinal is an ordinal; (3) for any two distinct ordinals α, β, either $\alpha \in \beta$ or $\beta \in \alpha$; (4) an ordinal γ is well-ordered by $\in$, i.e., every nonempty subset Γ of γ contains an ordinal $\gamma_{\min}$ such that $\beta \notin \gamma_{\min}$ for all $\beta \in \Gamma$; (5) for every set Γ of ordinals, $\bigcup \Gamma$ is an ordinal; (6) if γ is an ordinal, then $\gamma \cup \{\gamma\}$ is an ordinal.

The ordinal just mentioned, $\gamma \cup \{\gamma\}$, is called a *successor ordinal*: it is written as $\gamma + 1$ and γ is called its *predecessor*. An ordinal is called a *limit ordinal* if it has no predecessor, i.e., cannot be presented as $\gamma + 1$ for some ordinal γ. A limit ordinal is the union of all its elements.

It is common to write $\alpha < \beta$ in place of $\alpha \in \beta$.

In this paper we will only need to deal with the ordinals that are countable sets (see footnote 12). The property of these *countable ordinals* which is critical for us is that for any sequence of countable ordinals one can find a countable ordinal which is strictly greater than all elements of the sequence. The set of all countable ordinals is an uncountable ordinal, ω_1, and in fact it has the smallest uncountable cardinality.

Theorems involving ordinals are often proved by *transfinite induction*: if some statement holds for 0, and if, for any ordinal β (or any β below some ordinal γ), whenever we assume that the statement holds for all ordinals $\alpha < \beta$ it also holds for β, then the statement holds for all ordinals (respectively, all ordinals below γ). A property can be defined by means of *transfinite recursion*: if it is defined for 0, and if, for any ordinal β (below some ordinal γ), whenever we assume that it has been defined for all ordinals $\alpha < \beta$ it can also be defined for β, then the property is defined for all ordinals (below γ). Within the induction or recursion step, it is usually convenient to handle the successor and limit cases separately.

4. Construction of transitive–topological closure

To provide the intuition for the construction of the transitive–topological closure of $E(\psi, h)$, denote $E^0 = E(\psi, h)$. We know that if $(u, u') \in E^0$, then $\omega(u) = \omega(u')$. This does not mean, of course, that $\omega(u) \neq \omega(u')$ if (u, u') is not in E^0. Thus, if (u, u') is the limit of a sequence (u_n, u'_n) each element of which is in E^0, then $\omega(u) = \omega(u')$ as a consequence of $\omega(u_n) = \omega(u'_n)$ for every n. Insofar as the constancy of ω is concerned therefore, we can replace E^0 with its topological closure $\overline{E^0}$. This is not, however, the best we can do. Suppose that $(u, u') \notin \overline{E^0}$ but one can find a sequence $u = u_0, u_1, \ldots, u_k = u'$ with $(u_{i-1}, u_i) \in \overline{E^0}$ for all $i = 1, \ldots, k$. Then $\omega(u) = \omega(u')$ by the transitivity of equality. We can therefore replace $\overline{E^0}$ with its transitive closure $\mathrm{tc}\overline{E^0}$ (i.e., the set of the pairs connectable by finite chains whose successive links belong to $\overline{E^0}$). Let us denote this transitive closure $\mathrm{tc}\overline{E^0}$ by E^1.

It is clear that we still have not done our best in determining all pairs with $\omega(u) = \omega(u')$: if $(u, u') \notin E^1$ but some sequence (u_n, u'_n) in E^1 converges to (u, u'), then $\omega(u) = \omega(u')$, leading us to form

[10] It turns out that the original definition is equivalent to the following formulation: either for each (u, u') there is an arc image $\left[x_u^{u'}\right] \subset X$ such that $\psi \,|\, \left[x_u^{u'}\right] \times h\left(\left[x_u^{u'}\right]\right)$ is well-behaved on $\left[x_u^{u'}\right]$; or for each (u, u') there is an arc image $\left[x_u^{u'}\right] \subset X$ such that $\psi \,|\, \left[x_u^{u'}\right] \times h\left(\left[x_u^{u'}\right]\right)$ is well-behaved on $h\left(\left[x_u^{u'}\right]\right)$.

[11] A relation $E \subset X \times X$ is called transitively closed if the conjunction of $(u, u') \in E$ and $(u', u'') \in E$ implies $(u, u'') \in E$.

[12] The first countability of a topological space Z means that, for any subset $Z' \subset Z$ and any $z \in Z$, Z' must contain a sequence $z_n \to z$ if z is a limit point of Z' (i.e., every open neighborhood of z intersects Z'). The first countability is not essential for our construction, but by adopting this restriction we ensure that the transfinite recursion to be described has an identifiable termination point in the set of ordinal numbers: namely, the recursion is guaranteed to produce the transitive–topological closure of $E(\psi, h)$ before reaching the first uncountable ordinal (see below).

the topological closure $\overline{E^1}$. But then it is possible that $(u, u') \notin \overline{E^1}$ but one can find a sequence $u = u_0, u_1, \ldots, u_k = u'$ with all links $(u_{i-1}, u_i) \in \overline{E^1}$, implying again $\omega(u) = \omega(u')$. This leads us to the transitive closure $\mathrm{tc}\overline{E^1}$, denoted E^2. We can continue this process (topological closure–transitive closure–topological closure–...) and form the successive sets

$$E^1 \subset E^2 \subset E^3 \subset \cdots \subset X \times X.$$

Let E^ω be the union of these E-sets. It can be shown (see Section 5) that E^ω may very well not be topologically closed. And if we close it, forming $\overline{E^\omega}$, the latter is not necessarily transitively closed. The transitive closure $\mathrm{tc}\overline{E^\omega}$ requires from us to go beyond natural numbers to index it: $E^{\omega+1}$. This leads to the *transfinite* continuation of the process, forming

$$E^{\omega+1} \subset E^{\omega+2} \subset E^{\omega+3} \subset \cdots \subset X \times X.$$

In this way we obtain sets indexed by all countable ordinals. Denoting the union of all these E-sets by U we finally reach the desired generality: U can be shown (Theorem 2 below) to be closed both topologically and transitively (within any first countable Hausdorff space X). The formal definition is as follows (note that it is formulated for an arbitrary relation E^0, not necessarily $E(\psi, h)$).

Definition 4 (*By Transfinite Recursion*). Let $E^0 \subset X \times X$. Let β be a countable ordinal, and let the sets $E^\alpha \subset X \times X$ have been defined for all ordinals $\alpha < \beta$. Then, if β has a predecessor, define

$$E^\beta = \mathrm{tc}\overline{E^{\beta-1}}, \tag{11}$$

and if β is a limit ordinal, define

$$E^\beta = \bigcup_{\alpha<\beta} E^\alpha, \tag{12}$$

where tc denotes transitive closure. Denoting by Ω the set of all countable ordinals, we define

$$U = \bigcup_{\beta\in\Omega} E^\beta. \tag{13}$$

In particular, if $E^0 = E(\psi, h)$ of Definition 2, then the final product of this transfinite recursion, U, can be denoted $U(\psi, h)$.

Remark 1. One might wonder about the significance of starting the process by forming first the topological closure of E^0 rather than its transitive closure. Would the outcome U of the process be different if, starting with $E^0_\bullet = E^0$, the E-set for a successor ordinal β was defined as $E^\beta_\bullet = \overline{\mathrm{tc}E^{\beta-1}_\bullet}$ rather than $E^\beta = \mathrm{tc}\overline{E^{\beta-1}}$ (with the rest of Definition 4 unchanged, except for the "dotted" notation)? The answer is it would make no difference. The identity

$$U = \bigcup_{\beta\in\Omega} E^\beta = \bigcup_{\beta\in\Omega} E^\beta_\bullet$$

is an immediate consequence of the fact that

$$\begin{cases} E^\alpha_\bullet \subset \overline{E^\alpha} \subset E^{\alpha+1} \\ E^\alpha \subset \mathrm{tc}E^\alpha_\bullet \subset E^{\alpha+1}_\bullet \end{cases} \tag{14}$$

for all countable ordinals α. To prove (14), observe that it holds for $\alpha = 0$, and assume that it holds for all $\alpha < \beta$. If β has a predecessor, then (14) holds for β since it holds for $\beta - 1$: from $E^{\beta-1}_\bullet \subset \overline{E^{\beta-1}}$ we derive $E^\beta_\bullet = \overline{\mathrm{tc}E^{\beta-1}_\bullet} \subset \overline{\mathrm{tc}\overline{E^{\beta-1}}} = \overline{E^\beta} \subset E^{\beta+1}$, and from $E^{\beta-1} \subset \mathrm{tc}E^{\beta-1}_\bullet$ we derive $E^\beta = \mathrm{tc}\overline{E^{\beta-1}} \subset \mathrm{tc}\overline{\mathrm{tc}E^{\beta-1}_\bullet} = \mathrm{tc}E^\beta_\bullet \subset E^{\beta+1}_\bullet$. If β is a limit ordinal, then $E^\beta = \bigcup_{\alpha<\beta} E^\alpha = \bigcup_{\alpha<\beta} E^\alpha_\bullet = E^\beta_\bullet$, and (14) holds for β trivially. By transfinite induction, (14) holds for all countable ordinals.

The following properties of E^α are easily established and given here without proof.

Lemma 1. *If $\alpha < \beta$, then*

$$E^\alpha \subset \overline{E^\alpha} \subset \mathrm{tc}\overline{E^\alpha} \subset E^\beta \subset U.$$

Note that for $\alpha < \beta$ it is possible that $E^\alpha = E^\beta$ (in which case $E^\alpha = U$). In particular, it may be true for some functions ψ and h that E^0 itself is both topologically and transitively closed, in which case $E^0 = E(\psi, h) = U(\psi, h)$. Even then, however, E^α is well-defined (and equal to E^0) for any countable ordinal α.

Theorem 2. *U is the transitive–topological closure of E^0.*

Proof. To see that U is transitively closed, observe that if $(u, u') \in E^\alpha$ and $(u', u'') \in E^\beta$ for some countable ordinals $\alpha < \beta$, then, by Lemma 1, $(u, u'), (u', u'') \in E^\beta$. This implies $(u, u'') \in E^\beta \subset U$ since E^β is transitively closed.

To see that U is topologically closed, observe that, X being first countable, $(u, u') \in X \times X$ is a limit point of U if and only if there is a sequence $(u_n, u'_n) \to (u, u')$ as $n \to \infty$, with all (u_n, u'_n) in U. Then there is a sequence α_n of countable ordinals such that $(u_n, u'_n) \in E^{\alpha_n}$, and, as for any sequence of countable ordinals, there is a countable ordinal β exceeding all α_n. This means that $(u_n, u'_n) \in E^\beta$ for all n, whence $(u, u') \in \overline{E^\beta} \subset U$.

We show now by transfinite induction that U is contained in any topologically and transitively closed relation U' containing E^0. Let β be a countable ordinal, and let $E^\alpha \subset U'$ for all ordinals $\alpha < \beta$. Then, if β has a predecessor, $\overline{E^{\beta-1}} \subset U'$ because U' is topologically closed, and $E^\beta = \mathrm{tc}\overline{E^{\beta-1}} \subset U'$ because U' is transitively closed. If β is a limit ordinal, then $E^\beta = \bigcup_{\alpha<\beta} E^\alpha \subset U'$. By transfinite induction, $E^\alpha \subset U'$ for all countable ordinals α, whence $U \subset U'$. □

5. Two examples

In this concluding section we show, by examples, that Definition 4 is not excessively general. Our first example demonstrates that for every (countable) ordinal β there is an equivalence (hence transitively closed) relation $E^0 \subset X \times X$ such that E^β is a topologically closed equivalence relation (and is therefore equal to U), but E^α is not topologically closed for any $\alpha < \beta$. The second example shows that there is an equivalence relation E^0 such that E^α is not topologically closed for any countable α: the topological closure is only achieved when all countable ordinals have been exhausted and one has formed U.

In the following the terms "closed" and "closure" always refer to topological closedness; transitive closedness is always specified explicitly.

The relations E, E^α, and U in these examples need not be related to well-behavedness, although we make sure that they can be, by considering spaces which are Hausdorff, arc-connected, and first countable.

5.1. Segmentwise relations

We need some preliminaries. Given an interval of reals $I = |p, q|$ (we use $|$ to indicate that the endpoints may be excluded or included, no matter which) endowed with the conventional topology, a binary relation E on I is called a *segmentwise relation* if it is an equivalence relation whose quotient set I/E (of equivalence classes) contains only positive-length intervals (open, half-open, or closed) and, possibly, the endpoints $[p, p]$ and/or $[q, q]$. Clearly, if $(x, y) \in E$ then $(a, b) \in E$ for all a, b between x and y. For $I' = |a, b| \subset I$ we denote the restriction of E to I' by $E(I')$.

Obviously, $E(I')$ is a segmentwise relation on I', with the quotient set $I'/E(I') = \{A \cap I' : A \in I/E \text{ and } A \cap I' \neq \emptyset\}$. We can apply Definition 4 with $X = I$ to $E = E^0$ to obtain relations E^α for all countable ordinals; analogously, we can put $X = I'$ and $E(I') = E^0$ to obtain relations $E(I')^\alpha$.

The following properties of segmentwise relations are rather obvious on a moment's consideration, and we present them without detailed proofs.

Lemma 2. *If E is a segmentwise relation on I, and I' is an interval in I, then*

(i) *the closure of E is obtained segmentwise,*[13]

$$\overline{E} = \bigcup_{A \in I/E} \overline{A} \times \overline{A}; \tag{15}$$

(ii) *E is closed if and only if I/E consists of $\{I\}$ alone;*
(iii) *for any countable ordinal α, E^α is a segmentwise relation on I;*
(iv) *for any countable ordinal α,*

$$E(I')^\alpha = E^\alpha(I'). \tag{16}$$

Proof. (Outlined.) Let $(x, y) \in I \times I$ $(x \leq y)$ and let some sequence $(x_n, y_n) \in E$ converge to (x, y). To prove (i) we have to show that I/E contains an interval whose closure contains both x and y. Let $x \in |a_1, a_2| \in I/E$, $y \in |b_1, b_2| \in I/E$, $a_1 < b_2$. We exclude the possibilities $(x < a_2, y > a_2)$ and $(x < b_1, y > b_1)$ as contradicting $(x_n, y_n) \in E$. Hence either $y = b_1 = a_2$ (implying $x, y \in [a_1, a_2]$), or $x = a_2 = b_1$ (implying $x, y \in [b_1, b_2]$), or else $x = a_2 < b_1 = y$. In the latter case, $(x_n, y_n) \in E$ only if $|a_2, b_1| \in I/E$, and then $x, y \in [a_2, b_1]$.

It follows from (i) that E is closed if and only if all intervals in I/E are closed. Deny (ii) and assume that I/E contains more than one equivalence interval. Since these intervals do not overlap, the set obtained by removing from I their interiors is perfect, and has therefore the power of the continuum (see, e.g., Natanson, 1964a, Chapter 2). But the set of the intervals' endpoints is at most denumerable, so the set of the closed intervals comprising I/E cannot cover I. This contradiction proves (ii).

Statement (iii) is proved by transfinite induction. It is true for $\alpha = 0$. For $E^1 = \mathrm{tc}\overline{E}$, observe that the transition $E \to \mathrm{tc}\overline{E}$ amounts to replacing with $|a, b|$ every set of intervals $|t_i, t_{i+1}| \in I/E$ with the following property: their endpoints form two at most denumerable sequences of points $t_0 < t_1 < t_2 < \cdots$ and $t_0 > t_{-1} > t_{-2} > \cdots$ in $|a, b|$ with $\inf t_i = a$, $\sup t_i = b$, and $|a, b|$ does not share an endpoint with any other member of I/E. Clearly, such replacements create intervals partitioning I, and E^1 is therefore a segmentwise relation. Moreover, $E = E^0 \subset E^1$. Assume now that for all pairs of ordinals $\alpha < \alpha'$ below β, E^α is a segmentwise relation and $E^\alpha \subset E^{\alpha'}$. If β is a successor ordinal, then we apply to the transition $E^{\beta-1} \to E^\beta = \mathrm{tc}\overline{E^{\beta-1}}$ the above argument for $E \to \mathrm{tc}\overline{E}$. If β is a limit ordinal, it is easy to see that $E^\beta = \bigcup_{\alpha<\beta} E^\alpha$ is an equivalence relation, that every equivalence class in I/E^β is an interval, and that every member of I/E^α for every $\alpha < \beta$ is contained within such an interval.

To prove (iv) we use transfinite induction again. Observe that $E(I')^0 = E^0(I') = E(I')$, and assume that (iv) holds for all ordinals $\alpha < \beta$. If β is a limit ordinal, the induction step is obvious. If β is a successor ordinal, $E^\beta = \mathrm{tc}\overline{E^{\beta-1}}$, and we observe that $|a, b| \subset I'$ belongs to $I'/E^\beta(I')$ if and only if there are intervals $|t_i, t_{i+1}| \in I'/E^{\beta-1}(I')$ with the properties stated in the proof of (iii). But then $|t_i, t_{i+1}| \in I'/E(I')^{\beta-1}$, and the properties in question hold if and only if $|a, b|$ belongs to $I'/E(I')^\beta$. □

[13] Note that the closure $\overline{E}$ is generally *not* a segmentwise relation, as it need not be an equivalence relation.

5.2. Example: Construction of U may be completed at any countable ordinal

Definition 5. Let $X =]0, 1]$. Define E_0 as the equivalence relation $X \times X$. Assuming that relations E_α have been defined for all ordinals below a countable ordinal β, define E_β as follows. If β has a predecessor, E_β is the equivalence with the quotient set

$$X/E_\beta = \bigcup_{k=1}^{\infty} 2^{-k}(1 + X/E_{\beta-1}), \tag{17}$$

where the addition of 1 and multiplication by 2^{-k} applies to all intervals in $X/E_{\beta-1}$ pointwise. If β is a limit ordinal, we choose some increasing sequence $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots < \beta$ of ordinals converging to β and define E_β as the equivalence with the quotient set

$$X/E_\beta = \bigcup_{k=1}^{\infty} 2^{-k}(1 + X/E_{\alpha_k}), \tag{18}$$

using the same notation as in (17).

E_α is clearly a segmentwise relation on $X =]0, 1]$. Endowing X with the conventional topology (which makes X a Hausdorff, first countable, arc-connected space) we can apply Definition 4 to E_α to obtain E_α^λ and U_α for all countable ordinals λ. The theorem below states that $E_\alpha^\lambda = U_\alpha$ for all $\lambda \geq \alpha$ but that E_α^λ is not closed (and hence not equal to U_α) for any $\lambda < \alpha$. We need two auxiliary results first.

Lemma 3. *Let E be a segmentwise relation on an interval I, and let $|a, b] \in I/E$ for some $a < b$. If, for some countable ordinal α, I/E^α does not contain an interval with b as its right endpoint, then there is the smallest ordinal with this property, α'. It is a successor ordinal, such that, for some $c > b$,*

$$]b, c| \in I/E^{\alpha'-1}. \tag{19}$$

Proof. Since $|a, b]$ is contained in some element of I/E^β for every β, the interval of I/E^β containing b can only be of the forms $|b-s, b]$ or $|b - s, b + t|$, $s, t > 0$. I/E^α contains an interval of the latter type, and I/E^0 does not. So there must be the smallest ordinal, $\alpha' > 0$, such that $I/E^{\alpha'}$ contains such an interval. α' is not a limit ordinal, because then it would follow from Definition 4 that this property also holds for some $\alpha'' < \alpha'$. Hence $\alpha' - 1$ exists, and then $I/E^{\alpha'-1}$ contains an interval $|b - s, b]$. The property (19) follows from the fact that $I/E^{\alpha'}$ contains an interval $|b - s, b + t|$ which, by Definition 4 and Lemma 2(i), should include the union of $[b - s, b]$ and some $[b, b + t']$, $t' \leq t$. □

Lemma 4. *Let $X =]0, 1]$ and E_α be as in Definition 5, and let $X_k =]2^{-k}, 2^{-k+1}]$, $k = 1, 2, \ldots$. For any two countable ordinals μ, β,*

$$X_k/E_\beta^\mu(X_k) = 2^{-k}(1 + X/E_{\beta-1}^\mu) \tag{20}$$

if β has a predecessor, and

$$X_k/E_\beta^\mu(X_k) = 2^{-k}(1 + X/E_{\alpha_k}^\mu) \tag{21}$$

if β is a limit ordinal and α_k is as in Definition 5.

Proof. For a successor β, $X_k/E_\beta(X_k) = 2^{-k}(1 + X/E_{\beta-1})$ by Definition 5, whence $X_k/E_\beta(X_k)^\mu = 2^{-k}(1 + X/E_{\beta-1}^\mu)$. By Lemma 2(iv) $X_k/E_\beta(X_k)^\mu = X_k/E_\beta^\mu(X_k)$, and the statement follows. The statement for a limit β analogously follows from $X_k/E_\beta(X_k) = 2^{-k}(1 + X/E_{\alpha_k})$. □

We are ready now to prove the main result for our example.

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Theorem 3. *Let $X =]0, 1]$ and E_α be as in Definition 5. For any two countable ordinals α, λ,*

(i) *if $\lambda < \alpha$, then X/E_α^λ is not closed;*
(ii) *if $\lambda \geq \alpha$, then X/E_α^λ is closed.*

Proof. By Lemma 2(ii), X/E_α^λ is closed if and only if it consists of $\{X\}$ alone. We prove (i) by showing that with $\lambda < \alpha$, $]0, x| \notin X/E_\alpha^\lambda$ for any $x > 0$. For $\alpha = 0$ the statement holds trivially. Let it hold for all α below a countable ordinal $\beta > 0$. To prove it for β, assume the contrary: for some $\lambda < \beta$, $]0, x| \in X/E_\beta^\lambda$ for some x. As $2^{-k} \in \ ]0, x|$ for all sufficiently large k, we can apply Lemma 3 to $b = 2^{-k}$ for any such k and find a successor ordinal $\lambda_k \leq \lambda$ such that $X/E_\beta^{\lambda_k - 1}$ contains an interval $]2^{-k}, 2^{-k} + \delta_k|$. Denoting $X_k =]2^{-k}, 2^{-k+1}]$, it then follows that $X_k/E_\beta^{\lambda_k-1}(X_k)$ contains an interval $]2^{-k}, 2^{-k} + \delta_k'|$. We use Lemma 4 with $\lambda_k - 1$ replacing μ to obtain

$$X_k/E_\beta^{\lambda_k-1}(X_k) = 2^{-k}(1 + X/E_{\beta-1}^{\lambda_k-1})$$

or

$$X_k/E_\beta^{\lambda_k-1}(X_k) = 2^{-k}(1 + X/E_{\alpha_k}^{\lambda_k-1})$$

according as β is a successor or limit ordinal. We see that $X/E_{\beta-1}^{\lambda_k-1}$ or $X/E_{\alpha_k}^{\lambda_k-1}$, respectively, contains $]0, 2^k\delta_k'|$, for all sufficiently large k. But this contradicts the induction hypothesis because, if β has a predecessor, $\lambda_k - 1 \leq \lambda - 1 < \beta - 1$, and if β is a limit ordinal, then for all sufficiently large k, $\alpha_k > \lambda > \lambda_k - 1$ (where $\alpha_k < \beta$, i.e., we are within the domain of the induction hypothesis).

It is sufficient to prove (ii) for $\lambda = \alpha$ (if $E_\alpha^\alpha = X^2$ then $E_\alpha^\lambda = X^2$ for any $\lambda > \alpha$ by Lemma 1). We observe that $E_\alpha^\alpha = X^2$ for $\alpha = 0$ and then, assuming this for all α below a countable ordinal β, we show that $E_\beta^\beta = X^2$. For a successor β, by (20) with $\mu = \beta - 1$,

$$X_k/E_\beta^{\beta-1}(X_k) = 2^{-k}(1 + X/E_{\beta-1}^{\beta-1}),$$

and we see that $X_k/E_\beta^{\beta-1}(X_k) = \{X_k\}$, since $X/E_{\beta-1}^{\beta-1} = \{X\}$ by the induction hypothesis. It follows that $E_\beta^{\beta-1} \supset X_k^2$, whence $E_\beta^\beta \supset \overline{X_k^2} = \overline{X_k}^2$, for all k. For a limit β, (21) with $\mu = \alpha_k$ yields

$$X_k/E_\beta^{\alpha_k}(X_k) = 2^{-k}(1 + X/E_{\alpha_k}^{\alpha_k}),$$

and we see that $X_k/E_\beta^{\alpha_k}(X_k) = \{X_k\}$ (since $X/E_{\alpha_k}^{\alpha_k} = \{X\}$ by the induction hypothesis) and $E_\beta^{\alpha_k} \supset X_k^2$. Again, it follows that $E_\beta^\beta \supset \overline{X_k^2} = \overline{X_k}^2$, for all k. In either case we have

$$E_\beta^\beta \supset \mathrm{tc} \bigcup_{k=1}^{\infty} \overline{X_k}^2 = X^2,$$

since E_β^β is transitively closed. □

This concludes our first example.

5.3. Example: Construction of U may require all countable ordinals

In the second example we choose X to be the closed (in the conventional sense) unit disk in $\mathbb{R}^2$ centered at $\mathbf{O} = (0, 0)$. We impose a topology on X as follows. For each unit vector $\mathbf{v}$ attached to $\mathbf{O}$, each point $s\mathbf{v}$, $s \in \ ]0, 1]$, has a local basis given by the sets

$$\{t\mathbf{v} : t \in \ ]s - r, s + r[\cap]0, 1]\},$$

for all $r > 0$. The origin $\mathbf{O}$ has its local basis given by all open disks centered at the origin. It is easy to check that this topology is Hausdorff, first countable, and arc-connected, that every radius $L_\mathbf{v} = \{t\mathbf{v} : t \in \ ]0, 1]\}$ is homeomorphic to $]0, 1]$ (with the usual topology), and that $\mathbf{O} = (0, 0)$ is a limit point for every $L_\mathbf{v}$. The set $\mathbb{S}^1$ of the unit vectors $\mathbf{v}$ has the power of the continuum, while the set Ω of all countable ordinals has the smallest possible uncountable power. Hence there exists a surjection $m : \mathbb{S}^1 \to \Omega$. Let

$$E_* = \{(\mathbf{O}, \mathbf{O})\} \cup \bigcup_{\mathbf{v} \in \mathbb{S}^1} \{(s\mathbf{v}, t\mathbf{v}) : (s, t) \in E_{m(\mathbf{v})}\}, \quad (22)$$

where $E_{m(\mathbf{v})}$ is as in Definition 5. Obviously E_* is an equivalence relation, and its restriction to any $L_\mathbf{v}$,

$$E(L_\mathbf{v}) = \{(s\mathbf{v}, t\mathbf{v}) : (s, t) \in E_{m(\mathbf{v})}\},$$

is (homeomorphic to) a segmentwise relation. Using Definition 4, we can form E_*^λ's and U_*. By Theorem 2, U_* is transitively and topologically closed. We will show now that this property is not attained by any E_*^λ.

Theorem 4. *E_*^λ is not closed for any countable ordinal λ.*

Proof. By an obvious analogue of Lemma 2(iv), the restriction of E_*^λ to $L_\mathbf{v}$ is

$$E_*^\lambda(L_\mathbf{v}) = E(L_\mathbf{v})^\lambda = \{(s\mathbf{v}, t\mathbf{v}) : (s, t) \in E_{m(\mathbf{v})}^\lambda\}. \quad (23)$$

It follows that E_*^λ is not closed for any λ: otherwise we would have $E_*(L_\mathbf{v})^\lambda$ closed in $L_\mathbf{v} \times L_\mathbf{v}$ for some λ and all $\mathbf{v}$, which, by Theorem 3(i), cannot be true if one chooses $\mathbf{v}$ with $m(\mathbf{v}) > \lambda$. □

6. Conclusion

We have developed a radical generalization of the notion of well-behaved discrimination functions (Definition 3). This generalization is achieved by applying a transfinite series of topological and transitive closures (Definition 4 and Theorem 2) to the well-behavedness relation (Definitions 1 and 2) as introduced in Kujala and Dzhafarov (2008). The conclusion arrived at in that paper remains unchanged in our generalized treatment: *if a discrimination function (subject to Regular Minimality) is well-behaved with respect to its homeomorphic PSE function, then its minimum level function is constant* (Theorem 1).

In Kujala and Dzhafarov (2008) it has been mentioned that this conclusion is not entirely topological, in the sense that it does not hold for all continuous discrimination functions with homeomorphic PSEs. In fact, it has been shown (Kujala & Dzhafarov, 2008, Lemma 3) that for any homeomorphism $h : X \to Y$ and any continuous (nonconstant) $\omega : X \to [0, 1]$ one can find a continuous discrimination function ψ which has h as its PSE function and ω as its minimum level function. It is important to realize, however, that the notion of well-behavedness and the conclusion in question (Theorem 1) are nevertheless *topological* in the traditional sense of the word: *they are invariant with respect to all homeomorphic transformations of the sets X and Y which define the discrimination function's domain.* This invariance is due to the fact that Definition 1 which forms the departure point for our development is formulated in terms of arc-parametrized facets (5) which do not change under homeomorphic transformations $X \to X^*$, $Y \to Y^*$. Indeed, the discrimination function $\psi(x, y)$ then transforms into $\psi^*(x^*, y^*)$ such that $\psi^*(x^*, y^*) = \psi(x, y)$ whenever $x \mapsto x^*$ and $y \mapsto y^*$. Given arcs $x(s)$, $y(t)$ in (5), the transformed arcs $x^*(s)$, $y^*(t)$ are obtained by $s \mapsto x \mapsto x^*$ and $t \mapsto y \mapsto y^*$, whence $\psi^*(x^*(s), y^*(t)) = \psi(x(s), y(t)) = \varphi(s, t)$.

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