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Multidimensional Fechnerian Scaling: Probability-Distance Hypothesis

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The probability-distance hypothesis states that the probability with which one stimulus is discriminated from another is a function of some subjective distance between these stimuli. The analysis of this hypothesis within the framework of multidimensional Fechnerian scaling yields the following results. If the hypothetical subjective metric is internal (which means, roughly, that the distance between two stimuli equals the infimum of the lengths of all paths connecting them), then the underlying assumptions of Fechnerian scaling are satisfied and the metric in question coincides with the Fechnerian metric. Under the probability-distance hypothesis, the Fechnerian metric exists (i.e., the underlying assumptions of Fechnerian scaling are satisfied) if and only if the hypothetical subjective metric is internalizable, which means, roughly, that by a certain transformation it can be made to coincide in the small with an internal metric; and then this internal metric is the Fechnerian metric. The specialization of these results to unidimensional stimulus continua is closely related to the so-called Fechner problem proposed in 1960's as a substitute for Fechner's original theory.

INTRODUCTION

The historical origins of the probability-distance hypothesis lie in what Luce and Edwards (1958, p. 232) refer to as "the old, famous psychological rule of thumb: equally often noticed differences are equal, unless always or never noticed." The source of this formulation remains obscure. Several authors (Bock & Jones, 1968,

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p. 24; Edwards, 1957, p. 41; Guilford, 1954, p. 39) attribute it to Fullerton and Cattell (1892), but in that paper the issue is not even mentioned.

Being applied to an n-dimensional continuous space $\mathfrak{M}^{(n)} \subseteq \mathrm{Re}^n$ of stimuli $\mathbf{x} = (x^1, ..., x^n)$ endowed with psychometric functions

$$\psi_{\mathbf{x}}(\mathbf{y}) = \Pr[\mathbf{y} \text{ is discriminated from } \mathbf{x}],$$

the assertion, stated less aphoristically, is as follows: *There is a subjective metric* $D(x, y)$ *imposed on the stimulus space, such that*

$$\psi_{\mathbf{x}}(\mathbf{y}) = \begin{cases} 0 & \text{if} \quad D(\mathbf{x}, \mathbf{y}) \leqslant D_*, \\ f[D(\mathbf{x}, \mathbf{y}] & \text{if} \quad D_* < D(\mathbf{x}, \mathbf{y}) < D^*, \\ 1 & \text{if} \quad D(\mathbf{x}, \mathbf{y}) \geqslant D^*, \end{cases} \tag{1}$$

where f *is a continuously increasing function, while* $D_* \geqslant 0$, $D^* \leqslant \infty$ *are two constants* (on the extended set of reals).

It should be emphasized that, unlike in Luce and Edwards (1958) and most of the related literature, the discrimination probabilities $\psi_{\mathbf{x}}(\mathbf{y})$ considered in the present paper do not involve a semantically unidimensional subjective property (such as "intensity," "size," or "attractiveness") with respect to which $\mathbf{x}$ and $\mathbf{y}$ are compared in terms of "greater than." Rather, whichever of the variety of suitable experimental procedures or computations is used to obtain the judgment "$\mathbf{y}$ is discriminated from $\mathbf{x}$ in a given trial," the latter is interpreted as indicating that $\mathbf{x}$ and $\mathbf{y}$ do not appear to the observer as one and the same stimulus, whatever and however many subjective attributes their perception may involve.

If $D_* \geqslant 0$, (1) implies that each stimulus $\mathbf{x}$ is surrounded by a neighborhood of stimuli, $\{\mathbf{y}: D(\mathbf{x}, \mathbf{y}) \leqslant D_*\}$, that can never be discriminated from $\mathbf{x}$, while outside this neighborhood every stimulus is discriminated from $\mathbf{x}$ with some nonzero probability (see Fig. 1). This naive version of a "just-noticeable difference" has long since been abandoned in psychophysics, and I rule it out in this paper by putting $D_* = 0$.

Looking at the other end of the equation, the possibility that D^* is finite and that, consequently, all stimuli outside the spherical neighborhood $\{\mathbf{y}: D(\mathbf{x}, \mathbf{y}) < D^*\}$ are discriminated from $\mathbf{x}$ with probability 1 is debatable. Fortunately, the value of D^* makes no difference for the analysis to be presented, as this analysis only makes use of the relationship between $D(\mathbf{x}, \mathbf{y})$ and $\psi_{\mathbf{x}}(\mathbf{y})$ in the region of arbitrarily small values of $D(\mathbf{x}, \mathbf{y})$.

For our purposes, therefore, one can reformulate (1) as follows: *There is a metric* $D(\mathbf{x}, \mathbf{y})$ *imposed on the stimulus space, such that (at least) on some interval* $0 < D(\mathbf{x}, \mathbf{y}) < D^* \leqslant \infty$,

$$\psi_{\mathbf{x}}(\mathbf{y}) = f[D(\mathbf{x}, \mathbf{y})], \tag{2}$$

where f *is a continuously increasing function*. This statement does not say anything about the value of $\psi_{\mathbf{x}}(\mathbf{y})$ outside the interval $0 < D(x, y) < D^*$.

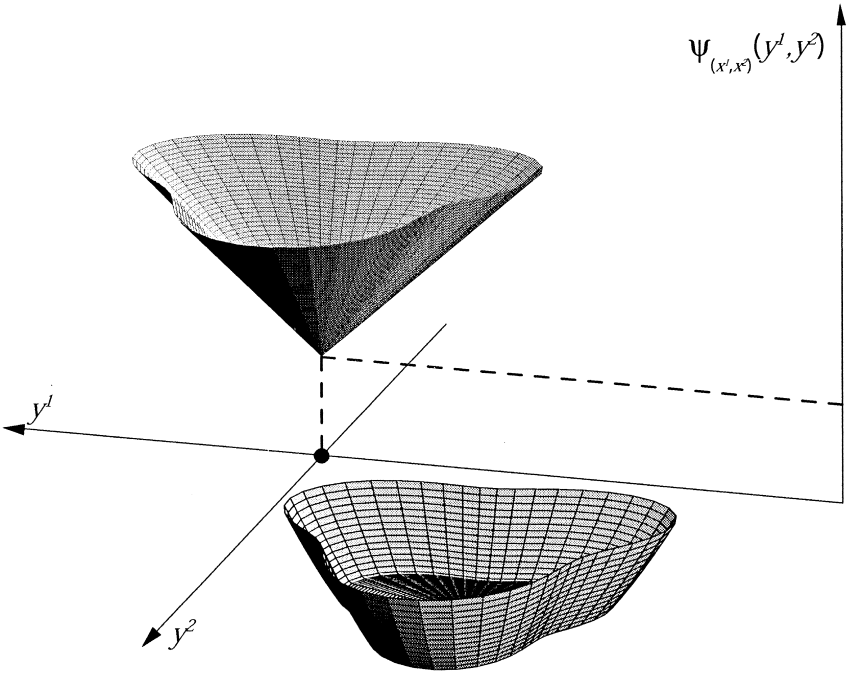

FIG. 1. Examples of psychometric functions satisfying Eq. (1), with $D^* > 0$ (right front) and $D^* = 0$ (left back). The functions are shown in small vicinities of their minima in a two-dimensional stimulus space.

Following Dzhafarov and Colonius (2001), the term "metric" in this paper is understood in the generalized sense of an *oriented metric*, which means that $D(\mathbf{x}, \mathbf{y})$ is assumed to satisfy the properties

$$\begin{array}{ll} \text{(positivity)} & \mathbf{x} \neq \mathbf{y} \Rightarrow D(\mathbf{x}, \mathbf{y}) > 0 \\ \text{(zero-value)} & D(\mathbf{x}, \mathbf{x}) = 0 \\ \text{(triangle inequality)} & D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}) \geqslant D(\mathbf{x}, \mathbf{z}), \end{array} \tag{3}$$

whereas the property

$$\text{(symmetry)} \qquad D(\mathbf{x}, \mathbf{y}) = D(\mathbf{y}, \mathbf{x}) \tag{4}$$

is treated as optional: its use is always stated separately, as an addendum to a statement involving oriented metrics. The reason for this is that the symmetry property is not an integral part of the theory of metrics we are concerned with in this paper (the internal and internalizable metrics, as explained below). By dispensing with this requirement in (2) one loses the unnecessarily restrictive prediction $\psi_{\mathbf{x}}(\mathbf{y}) = \psi_{\mathbf{y}}(\mathbf{x})$, but hardly anything else. By the same token, if $\psi_{\mathbf{x}}(\mathbf{y}) = \psi_{\mathbf{y}}(\mathbf{x})$ is found to hold empirically for a certain discrimination task, the symmetry can be added to the list of the defining properties of $D(\mathbf{x}, \mathbf{y})$ with virtually no other modifications. (See Comment 1 in the Appendix.)

Irrespective of whether the symmetry is adopted or dropped, (2) has too little empirical content to be interesting. As pointed out by Kiener (unpublished manuscript), based on a more general remark by Beals, Krantz, and Tversky (1968), if $\psi_{\mathbf{x}}(\mathbf{x}) \equiv const < \psi_{\mathbf{x}}(\mathbf{y})$ whenever $\mathbf{x} \neq \mathbf{y}$, then (2) is trivially satisfied by putting, for example,

$$D(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if} \quad \mathbf{x} = \mathbf{y}, \\ 1 + \psi_{\mathbf{x}}(\mathbf{y}) & \text{if} \quad \mathbf{x} \neq \mathbf{y}. \end{cases}$$

It is easy to verify that D thus defined is a (generally oriented) metric and that f continuously increases on the the interval $(0, 2)$, which includes the range of nonzero values of D.

There are many ways of constraining the distance D in (2) to exclude such "pathological" constructs. The weakest constraint that suits the purposes of the present development is as follows. Presenting $\mathbf{y}$ as $\mathbf{x} + \mathbf{u}s$ $(s \geqslant 0)$, where $\mathbf{u} = (u^1, ..., u^n)$ is an arbitrary nonzero direction vector, let $D(\mathbf{x}, \mathbf{y})$ have the following properties:

$$\begin{array}{ll} \text{(continuity)} & D(\mathbf{x}, \mathbf{y}) \text{ is continuous in } (\mathbf{x}, \mathbf{y}) \\ \text{(monotonicity)} & D(\mathbf{x}, \mathbf{x} + \mathbf{u}s) \text{ decreases to 0 with } s \to 0+. \end{array} \tag{5}$$

Following Dzhafarov (2001), I use the term "decreasing to zero with $s \to 0+$" to designate that the function vanishes at $s = 0$ (which in our case is automatic) and that it strictly increases on some right-hand vicinity of $s = 0$ (the width of which generally depends on $\mathbf{x}, \mathbf{u}$). I call the constraint (5) imposed on the distance D the *monotonic continuity* of D. In essence, it precludes the possibility that the value of $D(\mathbf{x}, \mathbf{y})$ may jump in response to small changes in $(\mathbf{x}, \mathbf{y})$ or that it may infinitely oscillate, alternately decreasing and increasing, as $\mathbf{y}$ approaches $\mathbf{x}$ along a straight line. With this constraint in place, we arrive at the final formulation of the

PROBABILITY-DISTANCE HYPOTHESIS. *There is a (generally oriented) monotonically continuous metric $D(\mathbf{x}, \mathbf{y})$ imposed on the stimulus space, such that (at least) on some interval $0 < D(\mathbf{x}, \mathbf{y}) < D^* \leqslant \infty$ Eq.* (2) *holds with a continuously increasing function f.*

The probability-distance hypothesis is of a special interest in the context of the general theory of multidimensional Fechnerian scaling, MDFS (Dzhafarov, 2001; Dzhafarov & Colonius, 1999a, 1999b, 2001). In this theory the discrimination probabilities $\psi_{\mathbf{x}}(\mathbf{y})$ are used to compute a certain (*Fechnerian*) metric $G(\mathbf{x}, \mathbf{y})$ on the stimulus space. As stated in Dzhafarov and Colonius (2001), MDFS is motivated by an expectation (in no way, however, derivable from the theory itself) that Fechnerian distances computed from discrimination probabilities should underlie a variety of behavioral measurements. It is reasonable to expect then that the psychometric functions themselves, $\psi_{\mathbf{x}}(\mathbf{y})$, should be first and foremost among the behavioral measurements to be expressible as functions of Fechnerian distances among stimuli. In the simplest case, $G(\mathbf{x}, \mathbf{y})$ may play the role of $D(\mathbf{x}, \mathbf{y})$ in the formulation of the probability-distance hypothesis, that is, $\psi_{\mathbf{x}}(\mathbf{y})$ may be a continuously increasing function of $G(\mathbf{x}, \mathbf{y})$ (that can be shown then to satisfy the monotonic continuity constraint). It is also possible, however, that the dependence of $\psi_{\mathbf{x}}(\mathbf{y})$ on Fechnerian distances is less direct. As an example, it is conceivable that $\psi_{\mathbf{x}}(\mathbf{y})$ in certain stimulus spaces could be a function of three Fechnerian distances, $G(\mathbf{x}, \mathbf{y})$, $G(\mathbf{x}, \mathbf{o})$, and $G(\mathbf{o}, \mathbf{y})$, where $\mathbf{o}$ is a fixed, "special" stimulus (say, the white color in a color space).

The computation of a Fechnerian metric only depends on the shapes of the psychometric functions in arbitrarily small vicinities of their minima, which means that one can deform $\psi_{\mathbf{x}}(\mathbf{y})$ outside such vicinities without affecting the ensuing Fechnerian metric. As a result, the general theory of MDFS neither predicts nor rules out any of the possible relationships between G and ψ. A study of such relationships, therefore, can be viewed as one of the main lines in the development of the theory of MDFS toward a network of competing empirically testable models. This paper takes a first step in this direction by posing the following question: *Assuming that the probability-distance hypothesis is true, so that* $\psi_{\mathbf{x}}(\mathbf{y})$ *is a continuously increasing function of a monotonically continuous metric* $D(\mathbf{x}, \mathbf{y})$*, what is the relationship between this* $D(\mathbf{x}, \mathbf{y})$ *and the Fechnerian metric* $G(\mathbf{x}, \mathbf{y})$? In particular, does $G(\mathbf{x}, \mathbf{y})$ have to exist (i.e., all the underlying assumptions of MDFS be satisfied) if the probability-distance hypothesis holds true, and if $G(\mathbf{x}, \mathbf{y})$ exists does it have to coincide with $D(\mathbf{x}, \mathbf{y})$?

The first of the main results obtained in this paper (by a fairly straightforward argument) is that if $D(\mathbf{x}, \mathbf{y})$ is an *internal metric*, then $G(\mathbf{x}, \mathbf{y})$ exists and coincides with $D(\mathbf{x}, \mathbf{y})$. In other words, $\psi_{\mathbf{x}}(\mathbf{y})$ cannot be determined by an internal metric other than the Fechnerian one. Roughly, $D(\mathbf{x}, \mathbf{y})$ is internal if its value equals the infimum of the lengths of all sufficiently smooth paths connecting $\mathbf{x}$ with $\mathbf{y}$. As discussed in Section 5, the specialization of this result to unidimensional stimulus spaces is closely related to the so-called *Fechner problem* (Creelman, 1967; Falmagne, 1971; Krantz, 1971; Luce & Edwards, 1958; Luce & Galanter, 1963; Pfanzagl, 1962).

In Section 4 I introduce the concept of an *internalizable metric*: roughly, this is a metric that can be monotonically transformed to coincide in the small with an internal metric. The latter is determined uniquely, and I call it the *internal conjugate* of the original, internalizable metric. The second main result obtained in this paper is that if $D(\mathbf{x}, \mathbf{y})$ in the probability-distance hypothesis is an internalizable metric, then the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ exists and coincides with the internal conjugate of $D(\mathbf{x}, \mathbf{y})$.

The third main result is that if both the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ exists and the probability-distance hypothesis holds true, then $D(\mathbf{x}, \mathbf{y})$ in this hypothesis must be an internalizable metric, with $G(\mathbf{x}, \mathbf{y})$ being its internal conjugate.

In the presentation below I follow the notation, terminology, and rely on the results established in Dzhafarov and Colonius (2001). A familiarity with this work is desirable, but it is not necessary for the reader who is willing to overlook technical details.

1. IMMEDIATE CONSEQUENCES OF THE PROBABILITY-DISTANCE HYPOTHESIS

In accordance with Dzhafarov and Colonius (1999a, 2001), the stimulus space on which the distance function $D(\mathbf{x}, \mathbf{y})$ and the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ are defined is an open connected region $\mathfrak{M}^{(n)}$ of Re^n endowed with the conventional topology (see Comment 2).[1] The computation of Fechnerian distances in MDFS is contingent upon three assumptions, referred to simply as the First, Second, and the Third Assumptions of MDFS (see Comment 3). Of these, the Third Assumption is treated as optional, as it only serves to ensure that the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ is symmetric. Rather than listing these assumptions here all at once, I begin by presenting certain elementary consequences of the probability-distance hypothesis and identifying these consequences as satisfying the First Assumption and (if the metric D is symmetric) the Third Assumption of MDFS. The Second Assumption is discussed subsequently.

Let the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ satisfy (2), with $D(\mathbf{x}, \mathbf{y})$ satisfying (3) and (5). The positivity and zero-value properties of $D(\mathbf{x}, \mathbf{y})$ imply that, for any given $\mathbf{x}$, $\psi_{\mathbf{x}}(\mathbf{y})$ assumes its global minimum at $\mathbf{y} = \mathbf{x}$, that is,

$$\mathbf{x} \neq \mathbf{y} \Rightarrow \psi_{\mathbf{x}}(\mathbf{x}) < \psi_{\mathbf{x}}(\mathbf{y}). \tag{6}$$

Moreover, on putting

$$f(0) = \lim_{D \to 0+} f(D)$$

and observing that

$$\psi_{\mathbf{x}}(\mathbf{x}) = \lim_{\mathbf{y} \to \mathbf{x}} f[D(\mathbf{x}, \mathbf{y})] = f(0),$$

we also have the following *constant self-similarity* property:

$$\text{The value of } \psi_{\mathbf{x}}(\mathbf{x}) \text{ is the same for all } \mathbf{x}. \tag{7}$$

The continuity of $D(\mathbf{x}, \mathbf{y})$ implies that

$$\psi_{\mathbf{x}}(\mathbf{y}) \text{ is continuous in } (\mathbf{x}, \mathbf{y}), \tag{8}$$

while the monotonicity implies that, for any fixed $\mathbf{x}$ and $\mathbf{u}$,

$$\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}) \text{ decreases to zero with } s \to 0+. \tag{9}$$

The properties (6), (8), and (9) (but not the constant self-similarity property) constitute the First Assumption of MDFS (see Comment 4). This part of the discussion therefore can be summarized as

[1] In the following, all references to numbered comments refer the reader to the Appendix.

THEOREM 1.1. *Under the probability-distance hypothesis, the First Assumption of MDFS is satisfied.*

Assume now that $D(\mathbf{x}, \mathbf{y})$, in addition, has the property of symmetry, (4). Then

$$\psi_{\mathbf{x}}(\mathbf{y}) = \psi_{\mathbf{y}}(\mathbf{x}),$$

from which it follows, of course, that

$$\frac{\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})}{\psi_{\mathbf{x}+\mathbf{u}s}(\mathbf{x})-\psi_{\mathbf{x}}(\mathbf{x})} = 1, \qquad s > 0. \tag{10}$$

On presenting $\psi_{\mathbf{x}+\mathbf{u}s}(\mathbf{x})-\psi_{\mathbf{x}}(\mathbf{x})$ as $\psi_{\mathbf{x}+\mathbf{u}s}[(\mathbf{x}+\mathbf{u}s)-\mathbf{u}s]-\psi_{\mathbf{x}}(\mathbf{x})$ and observing that, by continuity,

$$\lim_{s \to 0+} \frac{\psi_{\mathbf{x}+\mathbf{u}s}[(\mathbf{x}+\mathbf{u}s)-\mathbf{u}s]-\psi_{\mathbf{x}+\mathbf{u}s}(\mathbf{x}+\mathbf{u}s)}{\psi_{\mathbf{x}}(\mathbf{x}-\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})} = 1,$$

we conclude that (10) implies

$$\lim_{s \to 0+} \frac{\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})}{\psi_{\mathbf{x}}(\mathbf{x}-\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})} = 1. \tag{11}$$

This limit statement constitutes the Third Assumption of MDFS, and we have

THEOREM 1.2. *Under the probability-distance hypothesis, if $D(\mathbf{x}, \mathbf{y})$ is symmetric then the Third Assumption of MDFS is satisfied.*

2. FUNDAMENTAL THEOREM OF FECHNERIAN SCALING

Since

$$h = \psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})$$

(the quantity referred to as the *psychometric differential*) continuously decreases to zero with $s \to 0+$, it has the inverse function

$$s = \Phi_{\mathbf{x}, \mathbf{u}}(h), \qquad h \geqslant 0,$$

continuously decreasing to zero with $h \to 0+$ and referred to as the *stimulus differential*. ($\Phi_{\mathbf{x}, \mathbf{u}}$ may be defined only below a certain h-value, depending on $\mathbf{x}, \mathbf{u}$.) The Second Assumption of MDFS is that, for some fixed $(\mathbf{x}_0, \mathbf{u}_0)$ and any $(\mathbf{x}, \mathbf{u})$, the limit ratio

$$\lim_{h \to 0+} \frac{\Phi_{\mathbf{x}_0, \mathbf{u}_0}(h)}{\Phi_{\mathbf{x}, \mathbf{u}}(h)}$$

is finite, positive, and continuous in $(\mathbf{x}, \mathbf{u})$. In other words, stimulus differentials corresponding to equal psychometric differentials are comeasurable in the small (with a continuous coefficient of asymptotic proportionality). This assumption is equivalent to the following statement, proved in Dzhafarov and Colonius (2001). (The symbol $\sim$ connecting two expressions indicates that they are asymptotically equal; i.e., their ratio tends to 1.)

THEOREM 2.1 (Fundamental Theorem of MDFS). *There exists a transformation $\Phi(h)$, continuously decreasing to zero with $h \to 0+$, such that, when applied to psychometric differentials $\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})$, it makes them all comeasurable in the small with s,*

$$\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})] \sim F(\mathbf{x}, \mathbf{u})\, s \qquad (as\ s \to 0+), \tag{12}$$

where $F(\mathbf{x}, \mathbf{u})$ is positive and continuous.

For the purposes of the present analysis this statement is more convenient to deal with than the (equivalent to it) formulation of the Second Assumption itself. $F(\mathbf{x}, \mathbf{u})$ is referred to as the (Fechner–Finsler) *metric function*. Intuitively, it determines the Fechnerian distances $G(\mathbf{x}, \mathbf{x}+\mathbf{u}s)$ between "infinitesimally close" stimuli (as $s \to 0+$). For the following it is important to note that $F(\mathbf{x}, \mathbf{u})$ is determined by (12) uniquely, and $\Phi(h)$ asymptotically uniquely (as $h \to 0+$), up to multiplication by one and the same arbitrary constant $k>0$. That is, all allowable substitutions for $F(\mathbf{x}, \mathbf{u})$ and $\Phi(h)$ in (12) are given by

$$\begin{aligned} F^*(\mathbf{x}, \mathbf{u}) &= kF(\mathbf{x}, \mathbf{u}), \\ \Phi^*(h) &\sim k\Phi(h) \qquad (\text{as } h \to 0+). \end{aligned} \tag{13}$$

(In Dzhafarov and Colonius (2001), this uniqueness statement is given as part of the Fundamental Theorem.) The transformation Φ is referred to as the *global psychometric transformation.*

The properties (3) and (5) of $D(\mathbf{x}, \mathbf{y})$, with or without (4), are too general to ensure that the statement of the Fundamental Theorem holds true under the probability-distance hypothesis. As an example, let the stimulus space be represented by the interval $(0, \infty)$ and the metric $D(x, y)$ be defined on this interval as

$$D(x, y) = \begin{cases} \sqrt{y-x} & \text{if} \quad x \leqslant 1, y \leqslant 1, x \leqslant y \\ \sqrt{1-x}+\sqrt{y-1} & \text{if} \quad x \leqslant 1, y > 1 \\ \sqrt{y-1}-\sqrt{x-1} & \text{if} \quad x > 1, y > 1, x \leqslant y \\ D(y, x) & \text{if} \quad x > y. \end{cases}$$

One can easily verify that $D(x, y)$ satisfies all properties of the (symmetric) monotonically continuous metric. Let

$$\psi_x(y) = f[D(x, y)],$$

with some continuously increasing function f differentiable at zero. Then, as $s \to 0+$,

$$\psi_x(x+us) - \psi_x(x) = f[D(x, x+us)] - f(0) \sim \begin{cases} f'(0)\sqrt{|u|}\sqrt{s} & \text{if } \; x \leqslant 1, \\ \dfrac{f'(0)}{2\sqrt{x-1}}\,|u|\,s & \text{if } \; x > 1, \end{cases}$$

and it is clear that no transformation Φ can make all psychometric differentials comeasurable in the small with s (see Comment 5).

As the metric function $F(\mathbf{x}, \mathbf{u})$ (from which a Fechnerian metric is computed) is only defined through (12), this example shows that *in its general form the probability-distance hypothesis does not imply the existence of a Fechnerian metric*. The situation changes, however, if one assumes that the monotonically continuous metric $D(\mathbf{x}, \mathbf{y})$ is internal, or internalizable, as discussed below.

3. INTERNAL METRICS

A systematic theory of internal metrics is presented in Dzhafarov and Colonius (2001). Here, I briefly recapitulate the aspects of this theory that are needed in the context of the probability-distance hypothesis.

An internal metric D (generally oriented) on a stimulus space $\mathfrak{M}^{(n)}$ (an open connected region of Re^n) is induced by a metric function $\lambda(\mathbf{x}, \mathbf{u})$, defined on all pairs of $\mathbf{x} \in \mathfrak{M}^{(n)}$, $\mathbf{u} \in \mathrm{Re}^n - \{\mathbf{0}\}$, and satisfying the requirements

$$\begin{array}{ll} \text{(positivity)} & \lambda(\mathbf{x}, \mathbf{u}) > 0, \\ \text{(continuity)} & \lambda(\mathbf{x}, \mathbf{u}) \text{ is continuous in } (\mathbf{x}, \mathbf{u}), \\ \text{(positive Euler homogeneity)} & \lambda(\mathbf{x}, k\mathbf{u}) = k\lambda(\mathbf{x}, \mathbf{u}) \text{ for } k > 0. \end{array} \tag{14}$$

If, in addition, one wishes to ensure that D is symmetric, one can also posit

$$\text{(symmetry)} \qquad \lambda(\mathbf{x}, \mathbf{u}) = \lambda(\mathbf{x}, -\mathbf{u}). \tag{15}$$

In this paper, however, this property is treated as optional, on a par with (4). (See Comment 6.)

Connecting any two points $\mathbf{x}$ and $\mathbf{y}$ by a piecewise differentiable path $\mathbf{z}$: $[a, b] \to \mathfrak{M}^{(n)}$, $\mathbf{z}(a) = \mathbf{x}$, $\mathbf{z}(b) = \mathbf{y}$, the (oriented) length of this path is defined as

$$L[\mathbf{z}(t)] = \int_a^b \lambda[\mathbf{z}(t), \dot{\mathbf{z}}(t)]\, dt,$$

and the function $D(\mathbf{x}, \mathbf{y})$ is defined as the infimum of $L[\mathbf{z}(t)]$ across all paths leading from $\mathbf{x}$ to $\mathbf{y}$. (See Comment 7.) Thus defined $D(\mathbf{x}, \mathbf{y})$ is automatically a continuous distance function, that is, it satisfies the positivity, zero-value, triangle inequality, and continuity constraints of (3) and (5). Therefore an internal metric is monotonically continuous if and only if it satisfies the monotonicity constraint in (5).

An important property of an internal metric D is that $D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)$ is right-differentiable at $s=0+$,

$$\left.\frac{dD(\mathbf{x}, \mathbf{x}+\mathbf{u}s)}{ds}\right|_{s=0+} = \lim_{s \to 0+} \frac{D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)}{s} = \hat{\lambda}(\mathbf{x}, \mathbf{u}), \tag{16}$$

where $\hat{\lambda}(\mathbf{x}, \mathbf{u})$ is a metric function, as it satisfies all the properties in (14). In addition, $\hat{\lambda}(\mathbf{x}, \mathbf{u})$ is (generally nonstrictly) *convex*, that is,

$$\hat{\lambda}(\mathbf{x}, \mathbf{u}_1+\mathbf{u}_2) \leqslant \hat{\lambda}(\mathbf{x}, \mathbf{u}_1)+\hat{\lambda}(\mathbf{x}, \mathbf{u}_2), \tag{17}$$

for any direction vectors $\mathbf{u}_1$, $\mathbf{u}_2$. This metric function is called the *min-metric function* associated with the internal metric $D(\mathbf{x}, \mathbf{y})$. If $\lambda(\mathbf{x}, \mathbf{u})$, the original metric function by means of which $D(\mathbf{x}, \mathbf{y})$ is obtained, is convex itself, then $\hat{\lambda}(\mathbf{x}, \mathbf{u}) = \lambda(\mathbf{x}, \mathbf{u})$. If $\lambda(\mathbf{x}, \mathbf{u})$ is not convex, so that $\hat{\lambda}(\mathbf{x}, \mathbf{u}) \neq \lambda(\mathbf{x}, \mathbf{u})$, $\hat{\lambda}(\mathbf{x}, \mathbf{u})$ can be used to construct a metric, say, $\hat{D}(\mathbf{x}, \mathbf{y})$, by the above described procedure of defining lengths of connecting paths and taking their infima. A natural question to ask is what is the relationship between this metric $\hat{D}$, induced by $\hat{\lambda}(\mathbf{x}, \mathbf{u})$, and the original metric D, induced by $\lambda(\mathbf{x}, \mathbf{u})$. A remarkable property of $\hat{\lambda}(\mathbf{x}, \mathbf{u})$ is that the two metrics coincide, $\hat{D} \equiv D$ (a proof of this fact can be found in Dzhafarov & Colonius, 2001, where it is called the *Busemann–Mayer identity*). (See Comment 8.)

For a given stimulus $\mathbf{x}$, the set of direction vectors $\mathbf{u} \in \mathrm{Re}^n - \{\mathbf{0}\}$ satisfying the equality $\lambda(\mathbf{x}, \mathbf{u}) = 1$ is called the *indicatrix centered at* $\mathbf{x}$. The closed contour of the indicatrix, formed by the endpoints of its vectors $\mathbf{u}$, is convex in the usual geometric sense (no points of a chord connecting any its two points fall outside the contour) if and only if $\lambda(\mathbf{x}, \mathbf{u})$ is convex in the sense of (17). The metric function $\lambda(\mathbf{x}, \mathbf{u})$ and the collection of the indicatrices attached to all possible locations $\mathbf{x}$ determine each other uniquely.

The Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ in the theory of MDFS is an internal metric induced by the (Fechner–Finsler) metric function

$$F(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s}. \tag{18}$$

That $F(\mathbf{x}, \mathbf{u})$ satisfies all the properties in (14) follows from the First and Second Assumptions of MDFS. In accordance with the above described procedure,

$$G(\mathbf{x}, \mathbf{y}) = \inf \int_a^b F[\mathbf{z}(t), \dot{\mathbf{z}}(t)]\, dt,$$

the infimum being taken across all piecewise differentiable paths $\mathbf{z}(t)$ connecting $\mathbf{z}(a) = \mathbf{x}$ to $\mathbf{z}(b) = \mathbf{y}$. As $F(\mathbf{x}, \mathbf{u})$ is determined (by the psychometric functions) uniquely up to the multiplication by a positive constant, the same uniqueness statement holds for the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$.

The indicatrices corresponding to the metric function $F(\mathbf{x}, \mathbf{u})$ (the *Fechnerian indicatrices*) have a simple interpretation in terms of the shapes of the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$. Refer to Fig. 2. The contours of the Fechnerian indicatrices are

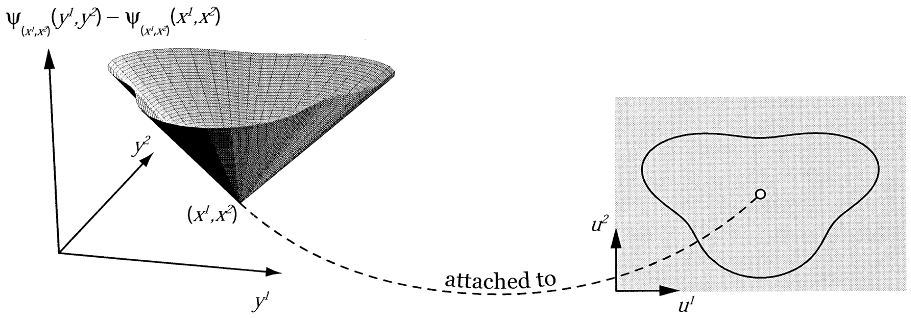

FIG. 2. A horizontal cross section of a psychometric function at a very small elevation from its minimum level is approximately geometrically similar to the Fechnerian indicatrix attached to the point at which the minimum is achieved.

asymptotically similar to the contours formed by horizontally (i.e., parallel to the stimulus space) cross-sectioning the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ at a small elevation h from their minima (as $h \to 0+$, the geometric similarity improves). In the general theory of MDFS the Fechnerian indicatrices (and hence the cross-section contours just described) need not be convex, as illustrated by Fig. 2.

We are prepared now to discuss the probability-distance hypothesis under the assumption that $D(\mathbf{x}, \mathbf{y})$ is a (monotonically continuous) internal metric, induced by some min-metric function $\hat{\lambda}(\mathbf{x}, \mathbf{u})$. Observe first that if $\psi_{\mathbf{x}}(\mathbf{y})$ is a continuously increasing function of $D(\mathbf{x}, \mathbf{y})$, it is also a continuously increasing function of $kD(\mathbf{x}, \mathbf{y})$, for any $k > 0$. The Fechnerian metric $G(\mathbf{x}, \mathbf{y})$, as we know, also can be multiplied by an arbitrary positive constant. To simplify the comparison of the two metrics, and with no loss of generality, it is convenient to agree that $D(\mathbf{x}, \mathbf{y})$ in the probability-distance hypothesis and $G(\mathbf{x}, \mathbf{y})$, if it exists, are chosen so that

$$D(\mathbf{a}, \mathbf{b}) = G(\mathbf{a}, \mathbf{b}) = 1$$

for some fixed arbitrary stimuli $\mathbf{a}, \mathbf{b}$. This allows one to speak of *the* Fechnerian metric computed from a given set of psychometric functions. The metric function $F(\mathbf{x}, \mathbf{u})$ then is also determined uniquely, while the global psychometric transformation $\Phi(h)$ is determined asymptotically uniquely (compare with (13)).

Since the probability-distance hypothesis implies the constant self-similarity property, (7), the assumption (2) can be written as

$$D(\mathbf{x}, \mathbf{y}) = \phi[\psi_{\mathbf{x}}(\mathbf{y}) - \psi_{\mathbf{x}}(\mathbf{x})],$$

where

$$\phi(h) = f^{-1}(h + c), \qquad c \equiv \psi_{\mathbf{x}}(\mathbf{x}),$$

is a function continuously decreasing to zero with $h \to 0+$. Then, by (16),

$$\lim_{s \to 0+} \frac{\phi[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s} = \lim_{s \to 0+} \frac{D(\mathbf{x}, \mathbf{x} + \mathbf{u}s)}{s} = \hat{\lambda}(\mathbf{x}, \mathbf{u}),$$

where $\hat{\lambda}(\mathbf{x}, \mathbf{u})$ is continuous and positive. On eliminating the middle expression,

$$\lim_{s \to 0+} \frac{\phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]}{s} = \hat{\lambda}(\mathbf{x}, \mathbf{u})$$

is the statement of the Fundamental Theorem of MDFS, with $\phi \equiv \Phi$ and $\hat{\lambda} \equiv F$. Recalling Theorem 1.1 and (in the symmetric case) Theorem 1.2, we conclude that all underlying assumptions of MDFS are satisfied and the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ induced by $F(\mathbf{x}, \mathbf{u}) = \hat{\lambda}(\mathbf{x}, \mathbf{u})$ exists and coincides with $D(\mathbf{x}, \mathbf{y})$. We also conclude that under the probability-distance hypothesis the Fechner–Finsler metric function $F(\mathbf{x}, \mathbf{u}) = \hat{\lambda}(\mathbf{x}, \mathbf{u})$ must be convex (for $\hat{\lambda}$ a min-metric function). This completes the proof of

THEOREM 3.1. *Under the probability-distance hypothesis, if $D(\mathbf{x}, \mathbf{y})$ is an internal metric, then*

(i) *the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ exists and coincides with $D(\mathbf{x}, \mathbf{y})$;*

(ii) *the metric function $F(\mathbf{x}, \mathbf{u})$ that induces $G(\mathbf{x}, \mathbf{y})$ is convex;*

(iii) *the relationship (2) has the form*

$$\psi_{\mathbf{x}}(\mathbf{y}) = \Phi^{-1}[G(\mathbf{x}, \mathbf{y})] + c, \qquad 0 \leqslant G(\mathbf{x}, \mathbf{y}) \leqslant G^* \leqslant \infty,$$

where $c \equiv \psi_{\mathbf{x}}(\mathbf{x})$ and Φ is a variant of the global psychometric transformation.

(See Comment 9.)

To put this result differently, psychometric functions cannot be determined by any internal metric other than the Fechnerian metric; and if they are determined by the Fechnerian metric, then their horizontal cross sections made just above their minima have convex contours.

4. INTERNALIZABLE METRICS

I introduce now a broad generalization (to the best of my knowledge not considered in mathematics before) of the notion of an internal metric. A metric $D(\mathbf{x}, \mathbf{y})$ on a stimulus space $\mathfrak{M}^{(n)}$ is called *internalizable* if there exists a transformation $g(h)$ decreasing to zero with $h \to 0+$, such that

$$\lim_{s \to 0+} \frac{g[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]}{s} = \delta(\mathbf{x}, \mathbf{u}) \tag{19}$$

is positive and continuous for all $\mathbf{x} \in \mathfrak{M}^{(n)}$, $\mathbf{u} \in \mathrm{Re}^n - \{\mathbf{0}\}$. It is easy to see that, for any $k > 0$,

$$\delta(\mathbf{x}, k\mathbf{u}) = \lim_{s \to 0+} \frac{g[D(\mathbf{x}, \mathbf{x}+(k\mathbf{u})\, s)]}{s} = k \lim_{ks \to 0+} \frac{g[D(\mathbf{x}, \mathbf{x}+\mathbf{u}(ks))]}{ks} = k\, \delta(\mathbf{x}, \mathbf{u}),$$

that is, $\delta(\mathbf{x}, \mathbf{u})$ is positive Euler homogeneous (see (14)).

If $D(\mathbf{x}, \mathbf{y})$ is symmetric, then the continuity of both $D(\mathbf{x}, \mathbf{y})$ and g implies

$$\begin{aligned}\delta(\mathbf{x}, -\mathbf{u}) &= \lim_{s\to 0+} \frac{g[D(\mathbf{x}, \mathbf{x}-\mathbf{u}s)]}{s}\\ &= \lim_{s\to 0+} \frac{g[D(\mathbf{x}-\mathbf{u}s, (\mathbf{x}-\mathbf{u}s)+\mathbf{u}s)]}{g[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]} \lim_{s\to 0+} \frac{g[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]}{s} = \delta(\mathbf{x}, \mathbf{u}).\end{aligned}$$

That is, the symmetry of $D(\mathbf{x}, \mathbf{y})$, (4), implies the symmetry of $\delta(\mathbf{x}, \mathbf{u})$ in the sense of (15).

The function $\delta(\mathbf{x}, \mathbf{u})$ (that may but need not be convex) thus can be viewed as a metric function, and it induces on the stimulus space a certain internal metric, $\bar{D}(\mathbf{x}, \mathbf{y})$, that I call the *internal conjugate* of $D(\mathbf{x}, \mathbf{y})$. Note that $\bar{D}(\mathbf{x}, \mathbf{y})$ may but generally does not coincide with $g[D(\mathbf{x}, \mathbf{y})]$. The latter generally is not even a metric.

It is easy to see that $\delta(\mathbf{x}, \mathbf{u})$ is determined by (19) uniquely, and g asymptotically uniquely, up to the multiplication by one and the same arbitrary constant $k > 0$. That is, all allowable substitutions for $\delta(\mathbf{x}, \mathbf{u})$ and g in (19) are given by

$$\begin{aligned}\delta^*(\mathbf{x}, \mathbf{u}) &= k\,\delta(\mathbf{x}, \mathbf{u}),\\ g^*(D) &\sim kg(D) \qquad (\text{as } D \to 0+).\end{aligned}$$

Indeed, for any pair of functions g^* and δ^* that satisfy (19), one should have

$$\frac{\delta^*(\mathbf{x}, \mathbf{u})}{\delta(\mathbf{x}, \mathbf{u})} = \lim_{s\to 0+} \frac{g^*[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]}{g[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]} = \lim_{D\to 0+} \frac{g^*(D)}{g(D)} = k,$$

for some $k > 0$. This uniqueness result implies, of course, that the internal conjugate $\bar{D}(\mathbf{x}, \mathbf{y})$ of an internalizable metric $D(\mathbf{x}, \mathbf{y})$ is determined uniquely up to the multiplication by a positive constant.

As an example of an internalizable metric and its internal conjugate, consider a unidimensional stimulus space represented by $(0, \infty)$ and the function $D(x, y) = \sqrt{|x-y|}$. This function can easily be checked to be a monotonically continuous metric. It is not internal, because, as $s \to 0+$, $\frac{D(x, x+us)}{s} \to \infty$, contrary to (16). At the same time,

$$\frac{D^2(x, x+us)}{s} \to |u| \qquad (\text{as } s \to 0+),$$

which satisfies (19). Hence $D(x, y)$ is internalizable, and its internal conjugate, induced by $\delta(x, u) = |u|$, is $\bar{D}(x, y) = |x-y|$. Another, perhaps more familiar, example is the relationship between the Euclidean chord metric (an internalizable metric) and the Euclidean arc metric (its internal conjugate) on any curvilinear contour, say, a circle. In this case, the transformation g is simply the identity, and the two metrics coincide in the small.

Consider now the probability-distance hypothesis with $D(\mathbf{x}, \mathbf{y})$ being a (monotonically continuous) internalizable metric. By the same logic as in the previous section, it is convenient to agree that the internal conjugate $\bar{D}(\mathbf{x}, \mathbf{y})$ of $D(\mathbf{x}, \mathbf{y})$ and the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$, if it exists, are chosen so that

$$\bar{D}(\mathbf{a}, \mathbf{b}) = G(\mathbf{a}, \mathbf{b}) = 1,$$

for some arbitrarily chosen stimuli $\mathbf{a}$, $\mathbf{b}$.

Now, using the same argument as in the proof of Theorem 3.1, one can rewrite (2) as

$$g[D(\mathbf{x}, \mathbf{y})] = \phi[\psi_{\mathbf{x}}(\mathbf{y}) - \psi_{\mathbf{x}}(\mathbf{x})],$$

where

$$\phi(h) = g[f^{-1}(h+c)], \qquad c \equiv \psi_{\mathbf{x}}(\mathbf{x}),$$

is a function continuously decreasing to zero with $h \to 0+$. Then, by (19),

$$\lim_{s\to 0+} \frac{\phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x})]}{s} = \lim_{s\to 0+} \frac{g[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]}{s} = \delta(\mathbf{x}, \mathbf{u}),$$

with $\delta(\mathbf{x}, \mathbf{u})$ positive and continuous. Eliminating the middle expression, one obtains the statement of the Fundamental Theorem of MDFS, with $\phi \equiv \Phi$ and $\delta \equiv F$. It follows, on recalling Theorems 1.1 and (in the case of a symmetric D) 1.2, that the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ induced by $F(\mathbf{x}, \mathbf{u}) = \delta(\mathbf{x}, \mathbf{u})$ exists and, by definition, coincides with the internal conjugate $\bar{D}(\mathbf{x}, \mathbf{y})$ of $D(\mathbf{x}, \mathbf{y})$. We have therefore

THEOREM 4.1. *Under the probability-distance hypothesis, if* $D(\mathbf{x}, \mathbf{y})$ *is an internalizable metric, then*

(i) *the Fechnerian metric* $G(\mathbf{x}, \mathbf{y})$ *exists and coincides with the internal conjugate* $\bar{D}(\mathbf{x}, \mathbf{y})$ *of* $D(\mathbf{x}, \mathbf{y})$*;*

(ii) *the relationship* (2) *has the form*

$$\psi_{\mathbf{x}}(\mathbf{y}) = \Phi^{-1}\{g[G(\mathbf{x}, \mathbf{y})]\} + c, \qquad 0 \leqslant G(\mathbf{x}, \mathbf{y}) \leqslant G^* \leqslant \infty,$$

where $c \equiv \psi_{\mathbf{x}}(\mathbf{x})$*,* Φ *is a variant of the global psychometric transformation, and* g *is defined by* (19).

Clearly, this theorem includes Theorem 3.1 as its special case.

An example of a monotonically continuous metric that is not internalizable has already been given in Section 2,

$$D(x, y) = \begin{cases} \sqrt{y-x} & \text{if} \quad x \leqslant 1, y \leqslant 1, x \leqslant y, \\ \sqrt{1-x} + \sqrt{y-1} & \text{if} \quad x \leqslant 1 . y > 1, \\ \sqrt{y-1} - \sqrt{x-1} & \text{if} \quad x > 1, y > 1, x \leqslant y, \\ D(y, x) & \text{if} \quad x > y. \end{cases}$$

This metric is not internalizable because the transformation $g[D(x, y)]$ in (19) would have to be different for $x \leqslant 1$ and $x > 1$ (by the same argument as in Section 2 and Comment 5, on substituting g for f). In Section 2 this example is used to demonstrate that the probability-distance hypothesis may hold when no Fechnerian metric exists. This suggests that the relationship between the Fechnerian metrics and the internalizability of D in the probability-distance hypothesis may be deeper than is indicated by Theorem 4.1 alone. This is indeed the case, for we have the following

THEOREM 4.2. *Under the probability-distance hypothesis, if the Fechnerian metric* $G(\mathbf{x}, \mathbf{y})$ *exists, then* $D(\mathbf{x}, \mathbf{y})$ *is an internalizable metric whose internal conjugate* $\bar{D}(\mathbf{x}, \mathbf{y})$ *coincides with* $G(\mathbf{x}, \mathbf{y})$.

Proof. Being a continuously increasing function of $\psi_{\mathbf{x}}(\mathbf{y})-\psi_{\mathbf{x}}(\mathbf{x})$ (due to the constant self-similarity property), $D(\mathbf{x}, \mathbf{y})$ is also a continuously increasing function of $\Phi[\psi_{\mathbf{x}}(\mathbf{y})-\psi_{\mathbf{x}}(\mathbf{x})]$, where Φ is the (continuously increasing) global psychometric transformation (see (12) or (18)) that exists by the premise of the theorem. Presenting this fact as

$$\gamma[D(\mathbf{x}, \mathbf{y})] = \Phi[\psi_{\mathbf{x}}(\mathbf{y})-\psi_{\mathbf{x}}(\mathbf{x})],$$

we have, by (18),

$$\lim_{s \to 0+} \frac{\gamma[D(\mathbf{x}, \mathbf{x}+\mathbf{u}s)]}{s} = \lim_{s \to 0+} \frac{\Phi[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)-\psi_{\mathbf{x}}(\mathbf{x})]}{s} = F(\mathbf{x}, \mathbf{u}).$$

Since $F(\mathbf{x}, \mathbf{u})$ is a metric function, $D(\mathbf{x}, \mathbf{y})$ satisfies the definition of an internalizable metric. ∎

5. UNIDIMENSIONAL CONTINUA AND THE FECHNER PROBLEM

I consider now the specialization of the results established in Section 3 to a unidimensional stimulus space $\mathfrak{M}^{(1)}$, an open interval of reals endowed with psychometric functions

$$\psi_x(y) = \Pr\,[y \text{ is discriminated from } x].$$

Note that here, too, as in the general case, the discrimination is not assumed to be based on a semantically unidimensional subjective property along which the stimuli can be compared in terms of "greater than." There is, of course, no logical reason why if the stimuli vary along a single physical dimension their discrimination should be subjectively unidimensional, too.

Let the probability-distance hypothesis hold for $\psi_x(y)$ and an internal metric on $\mathfrak{M}^{(1)}$. Then, by Theorem 3.1, the assumptions of MDFS are satisfied and

$$\psi_x(y) = \Phi^{-1}[G(x, y)] + \psi_x(x), \qquad 0 \leqslant G(x, y) \leqslant G^*, \tag{20}$$

where G is the Fechnerian metric, Φ is (a variant of) the global psychometric transformation, and $\psi_x(x) \equiv const$. Choosing an arbitrary stimulus o, one can put

$$U_+(x) = \begin{cases} G(o, x) & \text{if } x \geqslant o \\ -G(x, o) & \text{if } x < o \end{cases}$$

and

$$U_-(x) = \begin{cases} G(x, o) & \text{if } x \geqslant o \\ -G(o, x) & \text{if } x < o. \end{cases}$$

As follows from the properties of an internal metric (Section 3), $U_+(x)$ is a continuously right-differentiable and $U_-(x)$ a continuously left-differentiable increasing function, with

$$\frac{dU_+(x)}{dx+} = \left.\frac{dG(x, x+s)}{ds}\right|_{s=0+} = F(x, 1)$$

$$\frac{dU_-(x)}{dx-} = \left.\frac{dG(x, x-s)}{ds}\right|_{s=0+} = F(x, -1),$$

where $F(x, u)$ is the (Fechner–Finsler) metric function that induces G (see Comment 10). The two functions, $U_+(x)$ and $U_-(x)$, coincide if and only if $G(x, y)$ is symmetric. Clearly,

$$G(x, y) = \begin{cases} U_+(y) - U_+(x) & \text{if } x \leqslant y \\ U_-(x) - U_-(y) & \text{if } x > y, \end{cases}$$

and one can rewrite (20) as

$$\psi_x(y) = \begin{cases} \Phi^{-1}[U_+(y) - U_+(x)] + \psi_x(x) & \text{if } x \leqslant y \\ \Phi^{-1}[U_-(x) - U_-(y)] + \psi_x(x) & \text{if } x > y. \end{cases} \tag{21}$$

This representation allows one to relate the unidimensional specialization of the probability-distance hypothesis to the so-called Fechner problem proposed by Luce and Edwards (1958) and Luce and Galanter (1963) as a replacement for the original Fechner theory (see Dzhafarov & Colonius, 1999a, for a discussion of possible interpretations of Fechner's theory). Stated in slightly modified terms, the Fechner problem is as follows:

Given a unidimensional stimulus space endowed with psychometric functions

$$\gamma_x(y) = \Pr\,[y \text{ is judged to be greater than } x \text{ in attribute } \mathscr{P}],$$

find continuously increasing functions U and f such that $0 < \gamma_x(y) < 1$ *implies*

$$\gamma_x(y) = f[U(y) - U(x)]. \tag{22}$$

The involvement of a semantically unidimensional subjective attribute $\mathscr{P}$ (a "subjective correlate" of the stimulus continuum) makes a direct comparison of (22) with (21) impossible. One way of dealing with this situation within the framework of MDFS is to construct a separate unidimensional Fechnerian theory based on the application of a global psychometric transformation Φ^* to the quantity $\gamma_x(y)-\gamma_x(x)$, so that $\Phi^*[\gamma_x(x+s)-\gamma_x(x)]$ becomes comeasurable in the small with s. An implementation of this approach can be found in Dzhafarov and Colonius (1999a), in the context of a "revised" interpretation of Fechner's original theory. In this paper I consider an alternative, although in the final analysis equivalent, approach:

Assuming the representation (22) *holds, find a transformation*

$$H[\gamma_x(y)]=\psi_x(y)$$

that satisfies the representation (21).

This turns out to be possible, and quite trivially so, but only if one makes an additional assumption that the function U in (22) is continuously differentiable.

Note first that unlike in (21), where $\psi_x(y)$ and $\psi_y(x)$ are not generally related to each other by any function, (22) implies that, for $x \leqslant y$, $\gamma_y(x)$ is a decreasing function of $\gamma_x(y)$, as both these quantities on this domain are monotonic functions of $U(y)-U(x)$. Consequently, and because of the fact that $\gamma_x(x) \equiv f(0)$, one can put

$$\Gamma[\gamma_x(x)-\gamma_y(x)]=\gamma_x(y)-\gamma_x(x), \qquad x \leqslant y,$$

where $\Gamma(h)$ is a positive function continuously decreasing to zero with $h \to 0+$. Define now $\psi_x(y)$ by

$$\psi_x(y)=H[\gamma_x(y)]=\begin{cases}\gamma_x(y)-\gamma_x(x) & \text{if} \quad x \leqslant y\\ \Gamma[\gamma_x(x)-\gamma_x(y)] & \text{if} \quad x>y\end{cases} \tag{23}$$

(see Comment 11), and define a function Φ by

$$f(h)-\gamma_x(x)=\Phi^{-1}(h), \qquad h \geqslant 0.$$

Then

$$\psi_x(y)=H[\gamma_x(y)]=\Phi^{-1}(|U(y)-U(x)|), \tag{24}$$

and one easily recognizes in this a special case of (21), with $\psi_x(x) \equiv 0$ and $U_+ \equiv U_- \equiv U$. The function Φ is (a variant of) the global psychometric transformation in the sense of MDFS, for (24) implies

$$\lim_{s \to 0+} \frac{\Phi[\psi_x(x+us)-\psi_x(x)]}{s}=\left.\frac{d\Phi[\psi_x(y)]}{dy}\right|_{y=x+} |u|=U'(x)\,|u|,$$

where $U'(x)\,|u|$ plays the role of the metric function $F(x, u)$ that induces the (symmetric) Fechnerian metric

$$G(x, y) = |U(y) - U(x)|.$$

To further specialize this case, assume, in addition, as is often done in the psychophysical literature, that the psychometric function $\gamma_x(y)$ is differentiable in y at $y = x$. Then (see Comment 12)

$$U'(x) = \left.\frac{d\gamma_x(y)}{dy}\right|_{y=x}, \tag{25}$$

where $U'(x)$ plays the role of $F(x, 1) = F(x, -1)$. In accordance with the general theory of MDFS, this means that the Fechnerian distance $G(a, b)$ is obtained by integrating between a and b the slopes of the psychometric functions $\gamma_x(y)$ at $y = x$. This consequence of (22) and of the smoothness assumptions complementing it essentially coincides with the result of the analysis of the Fechner problem by Pfanzagl (1962). Variants of this result can also be found in Krantz (1971), Falmagne (1971), and, with probabilities converted into d'-scores, Creelman (1967).

6. CONCLUSION

The theorems proved in this paper can be combined together to state the following.

Under the probability-distance hypothesis, the Fechnerian metric $G(\mathbf{x}, \mathbf{y})$ *exists if and only if the* (*monotonically continuous*) *metric* $D(\mathbf{x}, \mathbf{y})$ *in* (2) *is internalizable; then* $G(\mathbf{x}, \mathbf{y})$ *is the internal conjugate* $\bar{D}(\mathbf{x}, \mathbf{y})$ *of* $D(\mathbf{x}, \mathbf{y})$*. If* $D(\mathbf{x}, \mathbf{y})$ *in* (2) *is internal, then* $D(\mathbf{x}, \mathbf{y}) = \bar{D}(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y})$*, and the metric function* $F(\mathbf{x}, \mathbf{u})$ *in* (18) *is convex.*

In relation to the Fechner problem discussed in Section 5, it should be noted that the analysis of this problem in the psychophysical literature includes also sufficient conditions for the representability of $\gamma_x(y)$ in the form (22) (Falmagne, 1985). This issue is beyond the scope of the present paper, as it only deals with necessary conditions of the probability-distance hypothesis, its consequences within the framework of MDFS.

Some important work has been done to generalize the Fechner problem to psychometric functions

$$\gamma_{\mathbf{x}}(\mathbf{y}) = \Pr\,[\mathbf{y} \text{ is judged to be greater than } \mathbf{x} \text{ in attribute } \mathscr{P}],$$

involving multidimensional stimuli but retaining the unidimensionality of the subjective attribute $\mathscr{P}$ (Aczél & Falmagne, 1999; Falmagne, 1979; Falmagne & Iverson, 1979). MDFS cannot handle this paradigm directly. Rather, as shown in Dzhafarov and Colonius (1999a), MDFS requires that the multidimensional stimulus space in this situation be reparametrized into a unidimensional one, by factorizing the n-dimensional space into $(n-1)$-dimensional subspaces on which suitably defined functions

$$\varphi_{\mathbf{x}}(\mathbf{y}) = H[\gamma_{\mathbf{x}}(\mathbf{y})] = \Pr\,[\mathbf{y} \text{ differs from } \mathbf{x} \text{ in attribute } \mathscr{P}]$$

achieve their minima and by treating these subspaces as equivalence classes. Once this is done, $\varphi_{\mathbf{x}}(\mathbf{y})$ acquires a $\psi_x(y)$-form, and the treatment becomes formally equivalent to the one in Section 5. Technical details of this analysis, however, are beyond the scope of this paper.

I conclude with a commentary on two simple consequences of the probability-distance hypothesis:

(no constant error) $\psi_{\mathbf{x}}(\mathbf{y})$ achieves its minimum at $\mathbf{y} = \mathbf{x}$,

(constant self-similarity) $\psi_{\mathbf{x}}(\mathbf{x})$ does nor depend on $\mathbf{x}$.

As stated in the Appendix, Comment 4, it is part of the First Assumption of MDFS that $\psi_{\mathbf{x}}(\mathbf{y})$ attains its single minimum at some point $\mathbf{h}(\mathbf{x})$ diffeomorphically related to $\mathbf{x}$. The difference $\mathbf{h}(\mathbf{x}) - \mathbf{x}$ is traditionally referred to as the *constant error* of discrimination, whereas $\mathbf{h}(\mathbf{x})$ is considered the "*point of subjective equality*" for the reference stimulus $\mathbf{x}$. One might, therefore, interpret the first of the consequences above as a rather stringent prediction that under the probability-distance hypothesis the constant error of discrimination must always be zero. Within the framework of MDFS, however, this interpretation would be incorrect. The constant error of discrimination is a manifestation of the fact that the reference stimulus $\mathbf{x}$ and the comparison stimulus $\mathbf{y}$ in $\psi_{\mathbf{x}}(\mathbf{y})$, strictly speaking, belong to two different subspaces, in essence due to what Fechner (1887/1987, p. 217) called the "non-removable spatiotemporal non-coincidence of the [reference and comparison] stimuli" (see Comment 1). Accordingly, the "stimulus space" in the theory of MDFS is in fact the space of comparison stimuli into which the reference stimuli are projected by means of the transformation $\mathbf{h}(\mathbf{x})$. In other words (see Dzhafarov & Colonius, 1999a, 2001), before one computes Fechnerian distances from psychometric functions, one has to rename all reference stimuli $\mathbf{x}$ into comparison stimuli $\mathbf{h}(\mathbf{x})$ and redefine the psychometric functions as

$$\psi_{\mathbf{x}}(\mathbf{y}) = \hat{\psi}_{\mathbf{h}(\mathbf{x})}(\mathbf{y}).$$

For simplicity of notation, however, one may continue to write $\psi_{\mathbf{x}}(\mathbf{y})$ instead of $\hat{\psi}_{\mathbf{h}(\mathbf{x})}(\mathbf{y})$, having agreed that $\mathbf{x}$ in $\psi_{\mathbf{x}}(\mathbf{y})$ stands for the point of subjective equality for some reference stimulus rather than for this stimulus itself. This, of course, guarantees that $\psi_{\mathbf{x}}(\mathbf{y})$ achieves its minimum at $\mathbf{y} = \mathbf{x}$ (under the First Assumption). Accordingly, $D(\mathbf{x}, \mathbf{y})$ in the formulation of the probability-distance hypothesis, (2), should be understood as the distance between $\mathbf{x}$, the point of subjective equality for a reference stimulus $\mathbf{h}^{-1}(\mathbf{x})$, and $\mathbf{y}$, a comparison stimulus. Thus the "no constant error" consequence of the probability-distance hypothesis is not a falsifiable prediction; it holds true by construction.

By far more important for the empirical viability of the probability-distance hypothesis is the constant self-similarity prediction, $\psi_{\mathbf{x}}(\mathbf{x}) = const(\mathbf{x})$. Unlike the symmetry property of a metric, the zero-value property, $D(\mathbf{x}, \mathbf{x}) = 0$, from which this prediction is derived cannot be dropped without changing the very nature of the concept of a metric, and thereby the spirit and intent of the probability-distance hypothesis. Empirical evidence suggests that the constant self-similarity may be violated when discrimination probabilities $\psi_{\mathbf{x}}(\mathbf{y})$ are obtained by direct same–different judgments (Krumhansl, 1978; Rothkopf, 1957). A systematic dependence of the value of $\psi_{\mathbf{x}}(\mathbf{x})$ on $\mathbf{x}$ has been demonstrated recently in the (yet unpublished) experiments conducted in the laboratories of H. Colonius (discrimination of unidimensional tones) and J. Allik (discrimination of two-dimensional localizations of a dot).

Given the prevailing trends in the development of psychophysics subsequent to Thurstone's classical paper (1927), one might "automatically" assume that to handle violations of the constant self-similarity one would have to abandon the probability-distance hypothesis in favor of a Thurstonian approach, in which different stimuli are represented by random variables differently distributed in a hypothetical perceptual space and the judgments of similarity are determined by random representations of paired stimuli. Multidimensional variants of this approach are described in Ashby and Perrin (1988), Ennis (1992), and Ennis, Palen, and Mullen (1988). The Thurstonian mechanism of generating discrimination probabilities is perfectly compatible with the theory of MDFS, but a thorough mathematical analysis, to be presented elsewhere, shows that, given the *regular minimality* property,

$$\psi_{\mathbf{x}}(\mathbf{x}) < \begin{cases} \psi_{\mathbf{x}}(\mathbf{y}), \\ \psi_{\mathbf{y}}(\mathbf{x}), \end{cases} \qquad \mathbf{x} \neq \mathbf{y},$$

violations of the constant self-similarity cannot be accounted for by this mechanism as readily as one might expect, even in a much more general conceptual setting than is commonly adopted. In contrast, the nonconstant self-similarity can be incorporated in a radical elaboration of the probability-distance hypothesis that capitalizes on the fact that $\mathbf{x}$ and $\mathbf{y}$ in $\psi_{\mathbf{x}}(\mathbf{y})$ have their origins in different stimulus subspaces (see above). This elaboration is beyond the scope of the present paper, but its mention here illustrates my general approach to the probability-distance hypothesis: Its scientific value rests less on its ability to explain empirical data than on its usefulness in serving as a building block, or a benchmark, in constructing and characterizing more sophisticated models, with a greater claim to empirical veridicality.

APPENDIX: TECHNICAL COMMENTS

1. When considering the symmetry of the psychometric functions or such issues as the constant error of discrimination (see the Conclusion), it is critical to keep in mind that the reference stimulus $\mathbf{x}$ and the comparison stimulus $\mathbf{y}$ in $\psi_{\mathbf{x}}(\mathbf{y})$ belong to different spatial or/and temporal observation intervals (e.g., the reference is presented first, or to the left of the comparison stimulus).

2. The "conventional topology" means that the convergence of stimuli, $\mathbf{x}_i \to \mathbf{x}$ ($i = 1, 2, \ldots$), is equivalent to the simultaneous convergence of all stimulus components,

$$\max_{j=1,\ldots,n} |x_i^j \to x^j| \to 0.$$

The convergence of stimulus pairs, $(\mathbf{x}_i, \mathbf{y}_i) \to (\mathbf{x}, \mathbf{y})$, stimulus-direction pairs, $(\mathbf{x}_i, \mathbf{u}_i) \to (\mathbf{x}, \mathbf{u})$, etc., is understood in the same componentwise sense.

3. There is also a fourth assumption, added to the theory in Dzhafarov (2001), to restrict the class of possible global psychometric transformations (see below) to functions regularly varying at the origin. This assumption, however, does not bring any nontrivial modifications to the discussion of the probability-distance hypothesis, and I do not involve it here for this reason. For the reader familiar with the notion of regular variation it would suffice to note that to incorporate the Fourth Assumption of MDFS in the present treatment one has to require, in addition to (2), (3), and (5), that the composite function $f[D(\mathbf{x}_0, \mathbf{x}_0 + \mathbf{u}_0 s)]$ be regularly varying at $s = 0+$ with a positive exponent, for some fixed $(\mathbf{x}_0, \mathbf{u}_0)$.

4. In fact, the First Assumption only stipulates that $\psi_{\mathbf{x}}(\mathbf{y})$, for any given $\mathbf{x}$, attains its global minimum at some point diffeomorphically related to $\mathbf{x}$. This point need not be equal to $\mathbf{x}$. This equality, however, can be achieved by a simple "recalibration" of the reference stimuli $\mathbf{x}$ (see Dzhafarov & Colonius, 1999a, 2001), which is implicitly assumed in the text to have been done. In order to not slow the development down, I defer a more detailed explanation to the Conclusion.

5. $\Phi(h)$ must asymptotically equal kh ($k > 0$) in the region $x > 1$ (recall the uniqueness statement for Φ), and, as $\Phi(h)$ cannot depend on x, this would not work for $x \leqslant 1$.

6. The symmetry of the metric function $\lambda(\mathbf{x}, \mathbf{u})$ is sufficient but not necessary for the symmetry of the ensuing metric $D(\mathbf{x}, \mathbf{y})$. The necessary condition is formulated in terms of min-metric functions (discussed below).

7. It is readily demonstrable (see Dzhafarov & Colonius, 2001) that $D(\mathbf{x}, \mathbf{y})$ is invariant with respect to all possible diffeomorphic reparametrizations of the connecting paths and of the stimulus space as a whole.

8. In reference to Comment 6, $D(\mathbf{x}, \mathbf{y})$ is symmetric if and only if $\hat{\lambda}(\mathbf{x}, \mathbf{u})$ is symmetric.

9. In Theorem 3.1 (and later in Theorem 4.1) one has to say "a variant of Φ" because Φ in (12) and (18) is determined only asymptotically uniquely (as its argument approaches zero).

10. In the unidimensional case, $F(x, u)$ values are determined by the values of $F(x, 1)$ and $F(x, -1)$,

$$F(x, u) = F(x, \pm|u|) = F(x, \pm 1)\,|u|.$$

11. This transformation generalizes a variant of the "mirror-reflection" procedure described in Dzhafarov and Colonius (1999a),

$$\psi_x(y) = \gamma_x(y) - \tfrac{1}{2},$$

which pertains to the special case when $\gamma_x(y) + \gamma_y(x) = 1$.

12. From (23), for $s > 0$,

$$\psi_x(x+s) - \psi_x(x) = \gamma_x(x+s) - \gamma_x(x) \sim \left.\frac{d\gamma_x(y)}{dy}\right|_{y=x} s,$$

while from (24),

$$\Phi[\psi_x(x+s) - \psi_x(x)] \sim U'(x)\, s.$$

It follows that

$$\lim_{h \to 0+} \frac{\Phi(h)}{h} = \Phi'(h)|_{h=0+}$$

exists, and putting it, with no loss of generality, equal to 1, we obtain (25).

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