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Perceptual Separability of Stimulus Dimensions: A Fechnerian Analysis

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Consider a situation in which, say, elliptically shaped visual stimuli continuously vary in the lengths of their radii, a and b, all other parameters being held fixed. This is a simple example of a two-dimensional continuous stimulus space: Each stimulus can be described by two coordinates, (x^1, x^2), taking their values within a region of Re $\times$ Re.[1] The dimensions $\langle x^1, x^2 \rangle$ can be chosen in an infinity of ways. One can put $x^1 = a$, $x^2 = b$, or $x^1 = a{:}b$, $x^2 = ab$ (aspect ratio and size), or one can even choose dimensions for which one has no conventional geometric terms, say, $x^1 = \exp(2a + 3b)$, $x^2 = \log(3a + 2b)$. The number of dimensions, in this case two, is a topological invariant (i.e., it is constant under all-continuous one-to-one transformations of the space), but the choice of the dimensions is arbitrary: With any given choice of $\langle x^1, x^2 \rangle$, one obtains other representations by arbitrarily transforming these dimensions into $\bar{x}^1 = \bar{x}^1(x^1, x^2)$, $\bar{x}^2 = \bar{x}^2(x^1, x^2)$, provided the transformations are one-to-one and smooth. If one imposes a certain "subjective"

[1]Following the traditional differential-geometric notation adopted in Dzhafarov and Colonius (1999, 2001), I use superscripts rather than subscripts to refer to point coordinates and (later) coordinates of direction vectors. The notation $\langle x^1, x^2 \rangle$, $\langle x^1 \rangle$, $\langle x^2 \rangle$ refers to frames of reference, or axes, whereas (x^1, x^2), (y^1, y^2), (x^1), (y^2), etc., denote coordinates of different stimuli with respect to specified frames of reference, or axes.

(computed from perceptual judgments) metric on the stimulus space, this metric must be invariant with respect to the choice of stimulus dimensions. In multidimensional Fechnerian scaling (MDFS), on which this work is based, this invariance is achieved automatically by the procedure of computing Fechnerian distances.

The choice of the dimensions describing elliptically shaped visual stimuli, however, may interest one from another point of view, pertaining to the focal issue of this work. One might hypothesize that with some choice of $\langle x^1, x^2 \rangle$, say, $x^1 = a{:}b$ (aspect ratio) and $x^2 = ab$ (size), the two dimensions are processed separately, so that perceptual distinctions between two ellipses can be, in some sense, computed from the perceptual distinctions between their aspect ratios (irrespective of size) and their sizes (irrespective of aspect ratio), whereas, one might hypothesize, such a reduction to individual dimensions cannot be achieved with other choices, say, $x^1 = a$ and $x^2 = b$, in which case the dimensions have to be viewed as processed integrally.

Ashby and Townsend (1986) analyze several theoretical concepts (separability, orthogonality, independence, performance parity) proposed in the literature in an attempt to capture this intuitive distinction. They propose to interpret these concepts within the framework of the General Recognition Theory (Ashby & Perrin, 1988), as different aspects of the mapping of stimuli into hypothetical random variables that take their values in some perceptual space. If one can define in this space two coordinate axes, $\langle p^1, p^2 \rangle$ (or two subspaces spanning two distinct sets of axes), such that the p^1 component and p^2 component of the random variable representing a stimulus (x^1, x^2) depend on only x^1 and x^2, respectively, then one can say that the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are perceptually separable. (The separability, perhaps by an abuse of language, is sometimes attributed to the perceptual dimensions $\langle p^1 \rangle$ and $\langle p^2 \rangle$ rather than the stimulus dimensions.) With this definition, the stochastic relationship between the p^1 and the p^2 components of the random variable representing (x^1, x^2) may depend on the (x^1, x^2) in an arbitrary fashion, provided the selective correspondence $(x^1 \leftrightarrow p^1, x^2 \leftrightarrow p^2)$ is satisfied on the level of marginal distributions. Thomas (1996) adapts this approach to the situation in which pairs of stimuli are judged on the same–different scale, which is especially relevant to the Fechnerian analysis presented in this chapter.

A different attempt to rigorously define perceptual separability is made by Shepard (1987) within the framework of multidimensional scaling. Shepard posits that stimuli are represented in a perceptual space by points separated by distances negative-exponentially related to some "stimulus generalization" measure, that, for our purposes, can be thought of as a probability of confusing one stimulus with another. It is traditionally postulated in multidimensional scaling, or derived from equivalent premises (Beals, Krantz, & Tversky, 1968; Tversky & Krantz, 1970), that one can define in this perceptual-space coordinate axes $\langle p^1, \ldots, p^k \rangle$ with respect to which the interstimulus distances D in the space form a Minkowski power metric:

$$D\left[(p^1, \ldots, p^k), (q^1, \ldots, q^k)\right]^r = \left|p^1 - q^1\right|^r + \cdots + \left|p^k - q^k\right|^r.$$

From the multidimensional scaling of several stimulus spaces, Shepard (1987) suggested that the exponent r of this power metric equals 1 (city-block metric) if the stimuli are analyzable into separately processed dimensions, and it equals 2 (Euclidean metric) if they are not. Although the relationship between subjective distances and stimulus confusion probabilities is central to Shepard's theory, he did not define the perceptual separability in terms of these confusion probabilities, relying instead on operational criteria external to his theory (such as those described in Garner, 1974). He also did not seem to consider the possibility that, just as stimuli with perceptually separable dimensions (by some criteria) can always be presented in a frame of reference whose axes are not perceptually separable (by the same criteria), so the "perceptual integrality" of stimuli corresponding to $r = 2$ could generally be a function of a specific choice of stimulus dimensions, rather than a property of the stimuli per se.

In this chapter I present a new approach to the issue of perceptual separability of stimulus dimensions, based on the theory of MDFS (Dzhafarov, 2001, 2002a, 2002b, 2002c, 2002d; Dzhafarov & Colonius, 1999, 2001). This chapter closely follows Dzhafarov (2002c).

Historical precursors of MDFS can be traced back to Helmholtz's (1891) and Schrödinger's (1920) reconstructions of color metrics from color-discrimination data. In MDFS, subjective (Fechnerian) distances among stimuli are computed from the probabilities with which stimuli are discriminated from their close neighbors in a continuous stimulus space. Accordingly, the concepts explicating the intuitive idea of perceptual separability are formulated in this chapter solely in terms of discrimination probabilities. Specifically, I propose to treat dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ as perceptually separable if the following two conditions are met:

1. The probability with which a stimulus $\mathbf{x} = (x^1, x^2)$ is discriminated from nearby stimuli $\mathbf{y} = (y^1, y^2)$ can be computed from the probabilities with which $\mathbf{x}$ is discriminated from $\mathbf{y}_1 = (y^1, x^2)$ (differing from $\mathbf{x}$ along the first dimension only) and from $\mathbf{y}_2 = (x^1, y^2)$ (differing from $\mathbf{x}$ along the second dimension only);
2. The difference between the probabilities with which $\mathbf{x} = (x^1, x^2)$ is discriminated from nearby $\mathbf{y}_1 = (y^1, x^2)$ and with which $\mathbf{x}$ is discriminated from itself does not depend on x^2; and analogously for $\mathbf{x} = (x^1, x^2)$ and nearby $\mathbf{y}_2 = (x^1, y^2)$.

If the probabilities with which each stimulus is discriminated from nearby stimuli are known, then MDFS allows one to uniquely compute the Fechnerian distances among all stimuli comprising the stimulus space. The following question therefore is a natural one to ask in relation to the definition of perceptual separability just outlined: Given that $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are separable, how is the Fechnerian distance between stimuli $\mathbf{a} = (a^1, a^2)$ and $\mathbf{b} = (b^1, b^2)$, not necessarily close, related to the corresponding coordinatewise Fechnerian distances, between $\mathbf{a}$ and $\mathbf{b}_1 = (b^1, a^2)$ and between $\mathbf{a}$ and $\mathbf{b}_2 = (a^1, b^2)$? The answer to this question is the main result of this work. It turns out, with the Fechnerian distances denoted by G, that

$$G(\mathbf{a}, \mathbf{b})^r = G(\mathbf{a}, \mathbf{b}_1)^r + G(\mathbf{a}, \mathbf{b}_2)^r, r \geq 1.$$

This means that the Fechnerian metric in a stimulus space with perceptually separable dimensions is a Minkowski power metric with respect to these dimensions.

This result may appear similar to Shepard's (1987) suggestion. The resemblance, however, is rather superficial. First, in MDFS the metric is imposed directly on the stimulus space rather than on a hypothetical perceptual space (which may even have a different dimensionality). Second, it is the power function form per se of the Fechnerian metric that is indicative of perceptual separability, rather than a specific value of the exponent r. I show below that the value of r is determined by the value of the fundamental characteristic of MDFS, μ, the *psychometric order of stimulus space*. Specifically, $r = \mu$ if $\mu \geq 1$, and $r = 1$ otherwise. Roughly, the psychometric order μ determines the degree of flatness/cuspidality of discrimination probability functions at their minima, and this characteristic has nothing to do with perceptual separability.[2]

The theory to be presented is formulated for two-dimensional stimulus spaces, but it can be readily generalized to arbitrary dimensionality, or even to an arbitrary number of subspaces spanning several dimensions each. This generalizability is the main reason why I keep in this chapter the notation adopted in Dzhafarov and Colonius (1999, 2001) for n-dimensional stimulus spaces.

PERCEPTUAL SEPARABILITY: DEFINITION AND PROPERTIES

Consider a two-dimensional stimulus space $\mathfrak{M}$, an open connected region of Re^2, and let $\langle x^1, x^2 \rangle$ be a coordinate system imposed on this space. The stimulus space is assumed to be endowed with *psychometric* (discrimination probability) functions

$$\psi_{\mathbf{x}}(\mathbf{y}) = \text{Prob}\,(\mathbf{y} \text{ is discriminated from } \mathbf{x}),$$

[2]A new theoretical development described in Dzhafarov (2002d), based on two fundamental properties of perceptual discrimination (called regular minimality and nonconstant self-similarity), shows that μ generally cannot exceed unity. This means that in the case of perceptual separability $r = 1$, precisely as Shepard suggested but with a very different justification. This and other relevant results from Dzhafarov (2002d) are not reflected in the present chapter, as they were obtained long after the chapter had been accepted for publication.

where $\mathbf{x} = (x^1, x^2) \in \mathfrak{M}$, $\mathbf{y} = (y^1, y^2) \in \mathfrak{M}$. I refer to the stimulus space together with the psychometric functions defined on it as the *discrimination system* $\langle \mathfrak{M}, \psi \rangle$.

Given a stimulus $\mathbf{x} = (x^1, x^2)$, the stimulus that lies $s \geq 0$ units away from $\mathbf{x}$ in the direction $\mathbf{u} = (u^1, u^2)$ can be denoted by $\mathbf{x} + \mathbf{u}s = (x^1 + u^1 s, x^2 + u^2 s)$; u^1 and u^2 may be any real numbers, except that they cannot vanish simultaneously. The difference

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) = \psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}), \quad s \geq 0$$

is referred to as the *psychometric differential* (at $\mathbf{x}$, $\mathbf{u}$), and it plays a central role in Fechnerian computations. The underlying assumptions of MDFS ensure that following, if necessary, a certain "elimination of constant error" procedure whose description can be found in, e.g., Dzhafarov & Colonius, 1999, 2001, the psychometric differentials are positive (for $s > 0$) and continuously decrease to zero as $s \to 0+$. The definition of perceptual separability is formulated below in terms of the psychometric differentials Ψ rather than the discrimination probabilities ψ per se.

In the subsequent presentation I also use the following convention. Given a direction vector $\mathbf{u} = (u^1, u^2)$, I denote its coordinate projections $(u^1, 0)$ and $(0, u^2)$ by $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively.

Definition A (Refer to Fig. 1.1). The coordinate system $\langle x^1, x^2 \rangle$ forms a *dimensional basis* for the discrimination system $\langle \mathfrak{M}, \psi \rangle$ if for any stimulus $\mathbf{x}$ one can find an open neighborhood $\mathfrak{N}_{\mathbf{x}} \subseteq \mathfrak{M}$ of $\mathbf{x}$ such that, whenever $\mathbf{x} + \mathbf{u}s \in \mathfrak{N}_{\mathbf{x}}$,

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) = H_{\mathbf{x}} \left[\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_1 s), \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_2 s) \right], \tag{1}$$

where H_x is some function differentiable on $\mathfrak{N}_{\mathbf{x}}$.

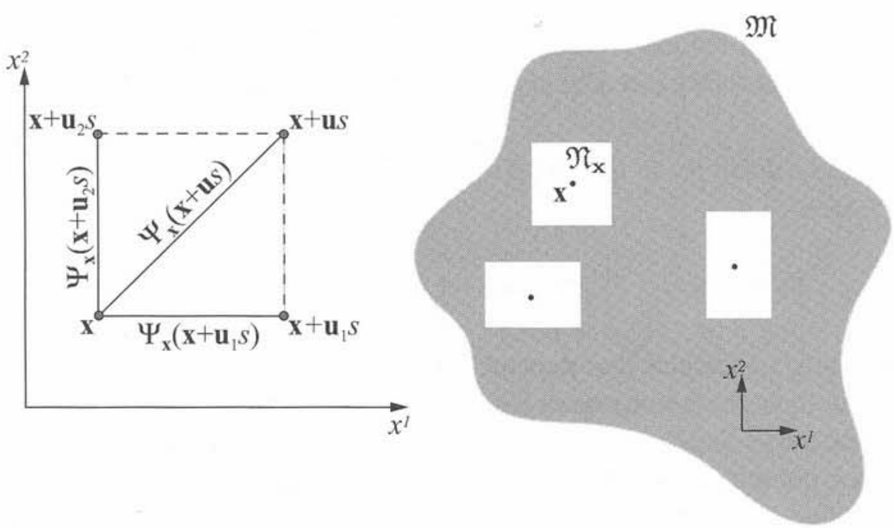

FIG. 1.1. A diagram for Definition A (dimensional basis).

Observe that the functions $H_{\mathbf{x}}$ are allowed to be different for different $\mathbf{x}$. One should note the following two important properties of $H_{\mathbf{x}}$. First,

$$H_{\mathbf{x}}\left[\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s), 0\right] = \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s),\ H_{\mathbf{x}}\left[0, \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_2 s)\right] = \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_2 s),$$
$$H_{\mathbf{x}}(0,0) = 0 \tag{2}$$

The second property is

$$\left.\frac{\partial H_{\mathbf{x}}(a,b)}{\partial a}\right|_{a=b=0} = \left.\frac{\partial H_{\mathbf{x}}(a,b)}{\partial b}\right|_{a=b=0} = 1. \tag{3}$$

See Dzhafarov (2002c) for the proof.

As a simple example of $H_{\mathbf{x}}$, consider the discrimination system in which

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = 1 - \left[1 - \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s)\right]\left[1 - \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_2 s)\right]. \tag{4}$$

$H_{\mathbf{x}}$ here does not depends on $\mathbf{x}$. If, in addition, $\psi_{\mathbf{x}}(\mathbf{x}) \equiv 0$, this equation can be rewritten by substituting ψ for Ψ, and it can be interpreted as saying that the discriminations along $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are stochastically independent and that two stimuli are discriminated whenever they are discriminated along either of these dimensions.

One can obtain a wealth of special cases for $H_{\mathbf{x}}$ by choosing an arbitrary strictly monotone (differentiable) function $T_{\mathbf{x}}(a)$, $0 \le a \le 1$, vanishing at $a = 0$, and putting

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = T_{\mathbf{x}}^{-1}\{T_{\mathbf{x}}[\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s)] + T_{\mathbf{x}}[\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_2 s)]\}. \tag{5}$$

In particular, this equation reduces to Equation 4 if $T_{\mathbf{x}}(a) = T(a) = \log(1-a)$.

The following lemma is part of the foundation of the Fechnerian analysis of perceptual separability (see Dzhafarov, 2002c for the proof).

Lemma A (Additivity in the small). If $\langle x^1, x^2 \rangle$ forms a dimensional basis for $\langle \mathfrak{M}, \psi \rangle$, then

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) \sim \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s) + \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_2 s)\ (\text{as } s \to 0+). \tag{6}$$

(The symbol $\sim$ indicates that the two expressions it connects are asymptotically equal, i.e., their ratio tends to 1. The term in the small means "at the limit" or "asymptotically".) As $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, one recognizes in Equation 6 an asymptotic version of the conventional factorial additivity (of the main effects of changes along the two dimensions on Ψ).

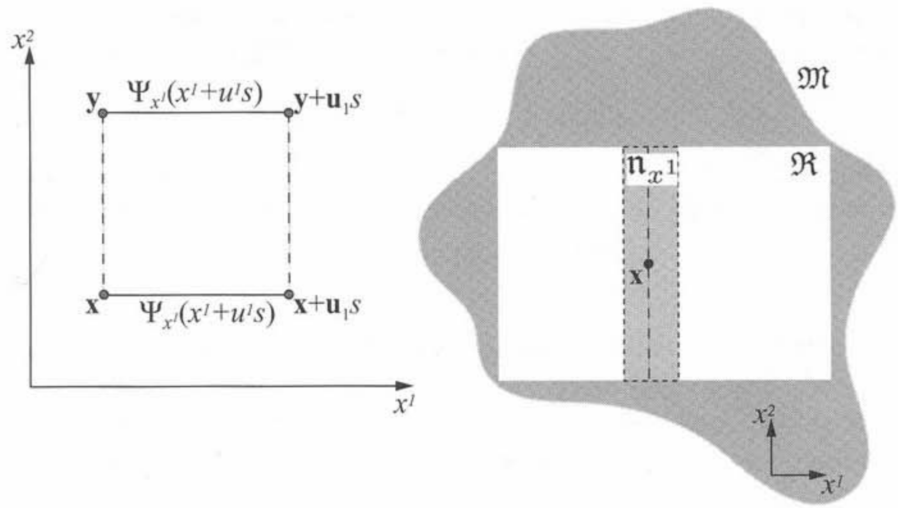

FIG. 1.2. A diagram for Definition B (detachability, shown for the horizontal axis only).

The next definition requires that we confine the consideration to some open rectangular area $\mathcal{R} = (x^1_{\inf}, x^1_{\sup}) \times (x^2_{\inf}, x^2_{\sup})$ of $\mathcal{M}$. Accordingly, the discrimination system $\langle \mathcal{M}, \psi \rangle$ is restricted to $\langle \mathcal{R}, \psi \rangle$. The area $\mathcal{R}$ may be infinite in either or both dimensions (i.e., some or all of the symbols $x^1_{\inf}, x^1_{\sup}, x^2_{\inf}, x^2_{\sup}$ may stand for $\pm\infty$), and it may coincide with the entire $\mathcal{M}$.

Definition B (Refer to Fig. 1.2). The dimension $\langle x^1 \rangle$ is *detachable* from the discrimination system $\langle \mathcal{R}, \psi \rangle$ if for any value of x^1 one can find an open vicinity $\mathfrak{n}_{x^1} \subseteq (x^1_{\inf}, x^1_{\sup})$ of x^1, such that whenever $x^1 + u^1 s \in \mathfrak{n}_{x^1}$, $\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_1 s)$ does not depend on $x^2 \in (x^2_{\inf}, x^2_{\sup})$. In other words, whenever $\mathbf{x} \in \mathcal{R}$ and $x^1 + u^1 s \in \mathfrak{n}_{x^1}$,

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_1 s) = \Psi_{x^1}(x^1 + u^1 s). \tag{7}$$

Analogously for the detachability of $\langle x^2 \rangle$ from $\langle \mathcal{R}, \psi \rangle$: Whenever $\mathbf{x} \in \mathcal{R}$ and $x^2 + u^2 s \in \mathfrak{n}_{x^2}$,

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_2 s) = \Psi_{x^2}(x^2 + u^2 s). \tag{8}$$

The reason for confining the definition of detachability to a rectangular area $\mathcal{R}$ is simple: if $\mathcal{M}$ is shaped differently, one can find $x^1, x^1 + u^1 s, x^2_1$, and x^2_2 such that $(x^1, x^2_1) \in \mathcal{M}$, $(x^1 + u^1 s, x^2_1) \in \mathcal{M}$, whereas $(x^1 + u^1 s, x^2_2) \notin \mathcal{M}$. As a result, $\Psi_x(\mathbf{x} + \mathbf{u}_1 s)$ would be defined for some and not defined for other values of x^2, which would mean that it does depend on x^2, contrary to the definition.

The following definition of perceptual separability is simply the conjunction of the previous two, except that the dimensional basis is now restricted to $\langle \Re, \psi \rangle$.

Definition AB. The dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are *perceptually separable* with respect to the discrimination system $\langle \Re, \psi \rangle$ if $\langle x^1, x^2 \rangle$ forms a dimensional basis for $\langle \Re, \psi \rangle$ and if both $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are detachable from $\langle \Re, \psi \rangle$.

Two aspects of this definition are significant for the subsequent development. First, one can readily verify that if $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are perceptually separable, then so are all the smooth monotonic transformations ("recalibrations") thereof, $\langle \bar{x}^1 \rangle$ and $\langle \bar{x}^2 \rangle$. Second, the application of this definition to Lemma A immediately yields the following result.

LEMMA AB (Detachable additivity in the small). If $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are perceptually separable with respect to $\langle \Re, \psi \rangle$, then, within $\Re$,

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) \sim \Psi_{x^1}(x^1 + u^1 s) + \Psi_{x^2}(x^2 + u^2 s) \text{ (as } s \to 0+). \qquad (9)$$

I show in the next section that these two properties directly lead one to the Minkowski power-metric structure of the Fechnerian metric.

PERCEPTUAL SEPARABILITY: FECHNERIAN ANALYSIS

The theory of MDFS is based on four assumptions about the shapes of the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ (Dzhafarov, 2002a; Dzhafarov & Colonius, 2001). Rather than describing them here, I confine the discussion to those consequences of these assumptions that are relevant for the present analysis.

First, the assumptions of MDFS guarantee that the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ look more or less as shown in Fig. 1.3 (ignore for now the values of μ): For any $\mathbf{x}$, $\psi_{\mathbf{x}}(\mathbf{y})$ is continuous, attains its global minimum at some point, and increases as one moves a small distance away from this point in any direction. Note that $\psi_{\mathbf{x}}(\mathbf{y})$ is generally allowed to be different from $\psi_{\mathbf{y}}(\mathbf{x})$, and $\psi_{\mathbf{x}}(\mathbf{x})$ is allowed to vary with $\mathbf{x}$. By a certain procedure that eliminates constant error by "renaming" reference stimuli (Dzhafarov & Colonius, 1999, 2001) one can always ensure that the minimum of $\psi_{\mathbf{x}}(\mathbf{y})$ is attained at $\mathbf{y} = \mathbf{x}$, which in this chapter is assumed to be the case. This makes all psychometric differentials $\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s)$ positive at $s > 0$ and continuously decreasing to zero as $s \to 0+$.

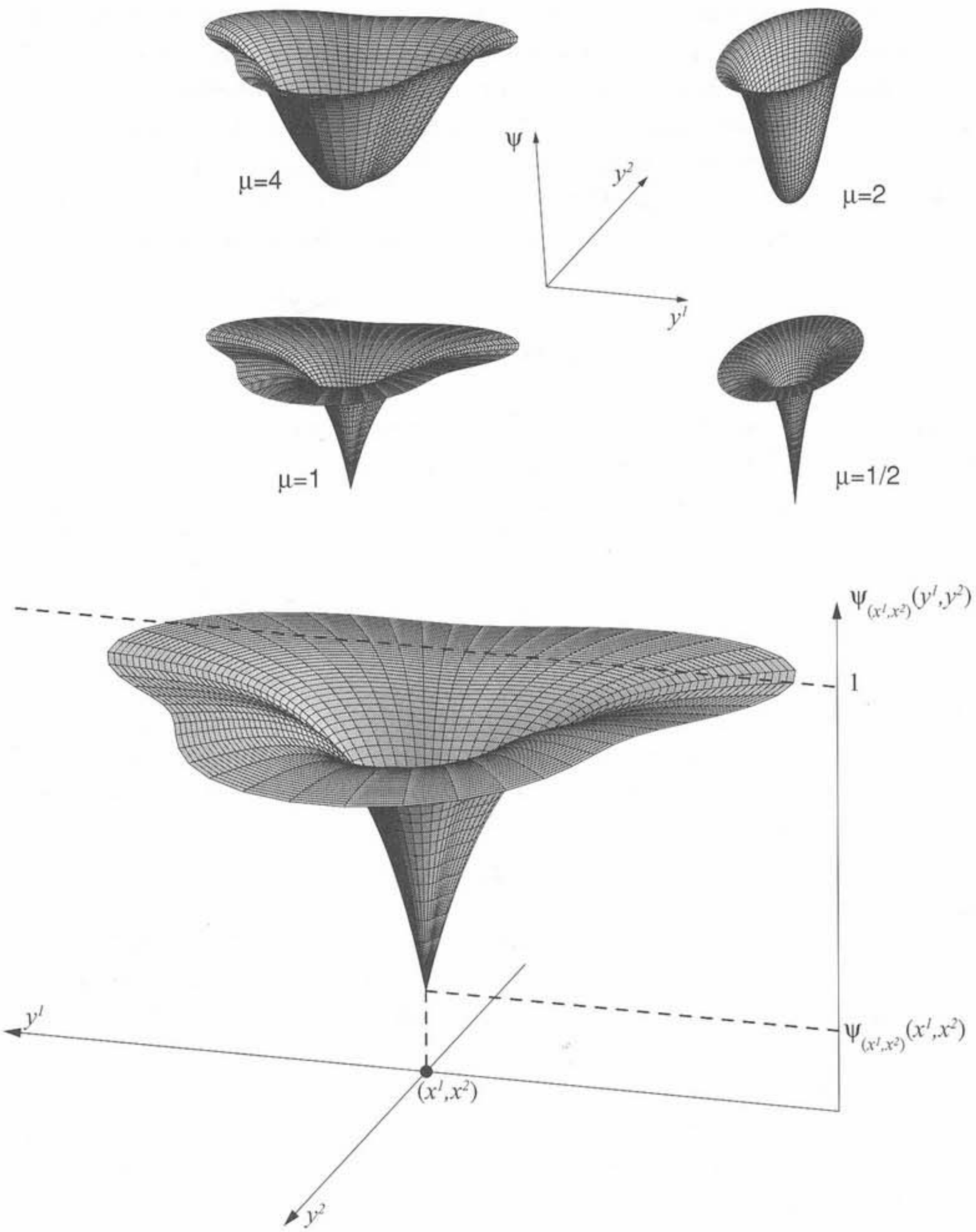

FIG. 1.3. Possible appearances of psychometric functions.

The assumptions underlying MDFS also ensure that all psychometric differentials can be asymptotically decomposed as

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) \sim [F(\mathbf{x}, \mathbf{u})R(s)]^{\mu} \text{ (as } s \to 0+), \tag{10}$$

with the following meaning of the right-hand terms. The constant $\mu > 0$, referred to as the *psychometric order* of the stimulus space, is one and the same for all reference stimuli $\mathbf{x}$ and directions of transition $\mathbf{u}$, and it is determined by psychometric differentials uniquely. $R(s)$ is some function *regularly varying (at the origin) with a unit exponent.*[3] It, too, is one and the same for all psychometric differentials and is determined by them asymptotically uniquely. The latter means that $R(s)$ in Equation 10 can be replaced only with $R^*(s) \sim R(s)$ (as $s \to 0+$). Finally, $F(\mathbf{x}, \mathbf{u})$ in Equation 10 is the (Fechner–Finsler) *metric function*, also determined uniquely. $F(\mathbf{x}, \mathbf{u})$ is positive (for $\mathbf{u} \neq 0$), continuous, and Euler homogeneous, the latter meaning that, for any k,

$$F(\mathbf{x}, k\mathbf{u}) = |k| F(\mathbf{x}, \mathbf{u}). \tag{11}$$

This metric function is all one needs to compute Fechnerian distances. Briefly, the logic of this computation is as follows. When any two points (stimuli) $\mathbf{a}$ and $\mathbf{b}$ are connected by a smooth path $\mathbf{x}(t)$: $[a, b] \to \mathfrak{M}$, $\mathbf{x}(a) = \mathbf{a}$, $\mathbf{x}(b) = \mathbf{b}$, the *psychometric length* of this path is defined as

$$L[\mathbf{x}(t)] = \int_a^b F[\mathbf{x}(t), \dot{\mathbf{x}}(t) \mathrm{d}t.$$

The *Fechnerian distance* $G(\mathbf{a}, \mathbf{b})$ is defined as the infimum of $L[\mathbf{x}(t)]$ across all smooth paths connecting $\mathbf{a}$ and $\mathbf{b}$. The thus-defined $G(\mathbf{a}, \mathbf{b})$ is a continuous distance function, invariant with respect to all possible smooth transformations of coordinates (Dzhafarov & Colonius, 1999, 2001).

The metric function $F(\mathbf{x}, \mathbf{u})$ can be given a simple geometric interpretation in terms of the shapes of psychometric functions. This is achieved through the important concept of a *Fechnerian indicatrix*. For a given stimulus $\mathbf{x}$, the Fechnerian indicatrix centered at $\mathbf{x}$ is the contour formed by the direction vectors $\mathbf{u}$ satisfying the equality $F(\mathbf{x}, \mathbf{u}) = 1$. The set of the indicatrices centered at all possible stimuli and the metric function determine each other uniquely. It turns out (Dzhafarov & Colonius, 2001) that the Fechnerian indicatrices are asymptotically similar to the contours formed by horizontally cross-secting $\psi_{\mathbf{x}}(\mathbf{y})$ at a small elevation h from their minima; the smaller the h, the better the geometric similarity (see Fig. 1.4).

Figure 1.4 and the top panel of Fig. 1.3 illustrate the geometric meaning of the psychometric order μ. As shown in Dzhafarov and Colonius (2001), if one cross-sects different psychometric functions by vertical planes passing through

[3]The unit-regular variation of $R(s)$ means that $R(ks)/R(s) \to k$ as $s \to 0+$. For example, $R(s) \equiv s$ is such a function, and in many respects any unit-regularly varying $R(s)$ is indistinguishable from s (Dzhafarov, 2002a). The reader who is willing to overlook technical details may, with no serious consequences for understanding this work, assume that $R(s) \equiv s$, and hence Equation 10 has the form

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) \sim F(\mathbf{x}, \mathbf{u})^{\mu} s^{\mu} \quad (\text{as } s \to 0+).$$

This is the so-called *power-function version* of MDFS (Dzhafarov & Colonius, 1999). The more general theory adopted in the this chapter is called the *regular variation version* of MDFS (Dzhafarov, 2002a).

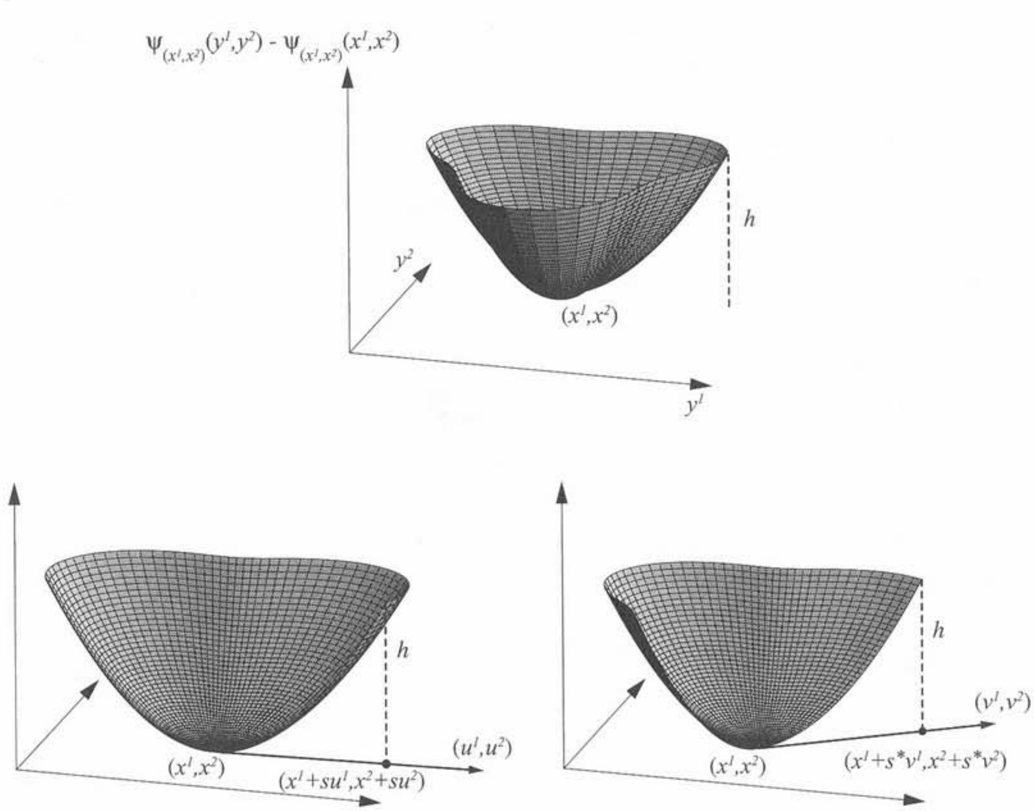

FIG. 1.4. A horizontal and two vertical cross sections of a psychometric function near its minimum.

their minima in various directions, then the cross sections confined between the minima and some small elevation h are horizontally scaled asymptotic replicas of each other. At the very minima of the psychometric functions these cross sections have a certain degree of flatness/cuspidality, and this degree is determined by the value of μ, from very flat (if μ is large) to pencil sharp ($\mu = 1$) to needle-sharp (if μ is close to zero). The fact that μ is one and the same for all psychometric differentials means that a specific degree of flatness/cuspidality is shared by all psychometric functions.[4]

We are now prepared to derive the main result of this work. Let the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ be perceptually separable with respect to $\langle \Re, \psi \rangle$. On applying Equation 10 to the coordinate projections $\mathbf{u}_1 = (u^1, 0)$ and $\mathbf{u}_2 = (0, u^2)$ of $\mathbf{u} = (u^1, u^2)$, one gets

$$\begin{cases} \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_1 s) \sim [F(\mathbf{x}, \mathbf{u}_1)R(s)]^{\mu} \\ \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_2 s) \sim [F(\mathbf{x}, \mathbf{u}_2)R(s)]^{\mu} \end{cases} \text{(as } s \to 0+\text{)} \qquad (12)$$

[4]As indicated in footnote 2, certain basic properties of psychometric functions eliminate the possibility of $\mu > 1$ (see Dzhafarov, 2002d).

It follows from Equations 7 and 8 that $F(\mathbf{x}, \mathbf{u}_1)$ in Equation 12 cannot depend on x^2, whereas $F(\mathbf{x}, \mathbf{u}_2)$ cannot depend on x^1. Hence one can put

$$\begin{cases} F(\mathbf{x}, \mathbf{u}_1) = F[\mathbf{x}, u^1(1,0)] = |u^1| F[\mathbf{x},(1,0)] = F_1(x^1)|u^1| \\ F(\mathbf{x}, \mathbf{u}_2) = F[\mathbf{x}, u^2(0,1)] = |u^2| F[\mathbf{x},(0,1)] = F_2(x^2)|u^2| \end{cases},$$

where one makes use of the Euler homogeneity, Equation 11. Equation 12 now can be rewritten as

$$\begin{cases} \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_1 s) = \Psi_{x^1}(x^1 + u^1 s) \sim F_1(x^1)^\mu |u^1|^\mu R(s)^\mu \\ \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_2 s) = \Psi_{x^2}(x^2 + u^2 s) \sim F_2(x^2)^\mu |u^2|^\mu R(s)^\mu \end{cases} \text{(as } s \to 0+\text{)}.$$

Applying this to the right-hand side of Equation 9 in Lemma AB, and using Equation 10 for its left-hand side, one gets

$$F(\mathbf{x}, \mathbf{u})^\mu R(s)^\mu \sim F_1(x^1)^\mu |u^1|^\mu R(s)^\mu + F_2(x^2)^\mu |u^2|^\mu R(s)^\mu \ (as\ s \to 0+),$$

which can only be true if

$$F(\mathbf{x}, \mathbf{u})^\mu = F_1(x^1)^\mu |u^1|^\mu + F_2(x^2)^\mu |u^2|^\mu. \tag{13}$$

To see that this structure of the metric function induces the Fechnerian metric with a Minkowski power-metric structure, choose an arbitrary point $\mathbf{o} = (o^1, o^2)$ and componentwise recalibrate (x^1, x^2) into

$$\bar{x}^1(x^1) = \int_{o^1}^{x^1} F_1(x)\mathrm{d}x,\ \ \bar{x}^2(x^2) = \int_{o^2}^{x^2} F_2(x)\mathrm{d}x. \tag{14}$$

According to the remark immediately following Definition AB, the axes $\langle \bar{x}^1 \rangle$ and $\langle \bar{x}^2 \rangle$ are perceptually separable. Presenting $\mathbf{x} = (x^1, x^2)$ in Equation 13 as $\bar{\mathbf{x}} = (\bar{x}^1, \bar{x}^2)$, in new coordinates, the direction $\mathbf{u} = (u^1, u^2)$ attached to $\mathbf{x}$ also acquires new coordinates, $\bar{\mathbf{u}} = (\bar{u}^1, \bar{u}^2)$. From Equations 14, these new coordinates are

$$\bar{u}^1 = F_1(x^1)u^1,\ \ \bar{u}^2 = F_2(x^2)u^2 \tag{15}$$

It follows that $F(\mathbf{x}, \mathbf{u})$ in Equation 13, when written in new coordinates as $\bar{F}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = F(\mathbf{x}, \mathbf{u})$, has the structure

$$\bar{F}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \bar{F}(\bar{\mathbf{u}}) = \sqrt[\mu]{|\bar{u}^1|^\mu + |\bar{u}^2|^\mu}. \tag{16}$$

The Fechnerian indicatrices corresponding to this metric function,

$$|\bar{u}^1|^\mu + |\bar{u}^2|^\mu = 1, \tag{17}$$

have the same shape for all stimuli $\bar{\mathbf{x}}$ at which they are centered. This shape is completely determined by the psychometric order μ, as shown in the lower panel of Fig. 1.5 (filled contours). Recall from the preceding discussion of indicatrices that these shapes describe the horizontal cross sections of the psychometric functions (Fig. 1.5, upper panel) at a small elevation above their minima. Recall also that μ has another geometric interpretation: It determines the shape (flatness/cuspidality) of the vertical cross sections of the psychometric functions in the vicinity of their minima (Fig. 1.5, middle panel). We see therefore that in the case of perceptually separable dimensions the shapes of the horizontal and vertical cross sections (generally completely independent) are interrelated, being controlled by one and the same parameter, μ.

Figure 1.5 demonstrates that the indicatrices for $\mu \geq 1$ are *convex* in all directions (*nonstrictly convex* if $\mu = 1$). A general theory of Fechnerian indicatrices is presented in Dzhafarov and Colonius (2001). Without recapitulating it here, I simply state the fact that if a Minkowskian indicatrix corresponding to $\bar{F}(\bar{\mathbf{u}})$ is convex,[5] then the Fechnerian metric it induces is computed as $G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \bar{F}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|)$. Applying this to Equation 16, with $\mu \geq 1$, one gets

$$G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \bar{F}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|) = \sqrt[\mu]{|\bar{x}^1 - \bar{y}^1|^\mu + |\bar{x}^2 - \bar{y}^2|^\mu}. \tag{18}$$

That is, the Fechnerian metric induced by Equations 16 and 17 is a *Minkowski power metric*, with the exponent equal to μ, provided the latter is not less than 1.

One can also see in Fig. 1.5 that the Fechnerian indicatrix is not convex when $\mu < 1$ (in fact, it is then *concave* in all directions, except for the coordinate ones). The general theory (Dzhafarov & Colonius, 2001) stipulates that the metric induced by a nonconvex indicatrix is the same as the one induced by its *convex closure*, which is the minimal convex contour containing it. In our case it is obvious (see the enclosing contour in Fig. 1.5 for $\mu = 1/2$) that the convex closure of an indicatrix corresponding to any value of $\mu < 1$ is the "diamond" described by $|\bar{u}^1| + |\bar{u}^2| = 1$. As a result, when $\mu < 1$, the Fechnerian metric induced by Equations 16 and 17 is

$$G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \bar{F}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|) = |\bar{x}^1 - \bar{y}^1| + |\bar{x}^2 - \bar{y}^2|, \tag{19}$$

the city-block metric, which is familiar to psychophysicists.[6]

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[5]Indicatrices and the corresponding metric function are called Minkowskian whenever $\bar{F}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \bar{F}(\bar{\mathbf{u}})$. The power-metric structure arrived at in Equation 16 is just a special case.

[6]See footnote 2.

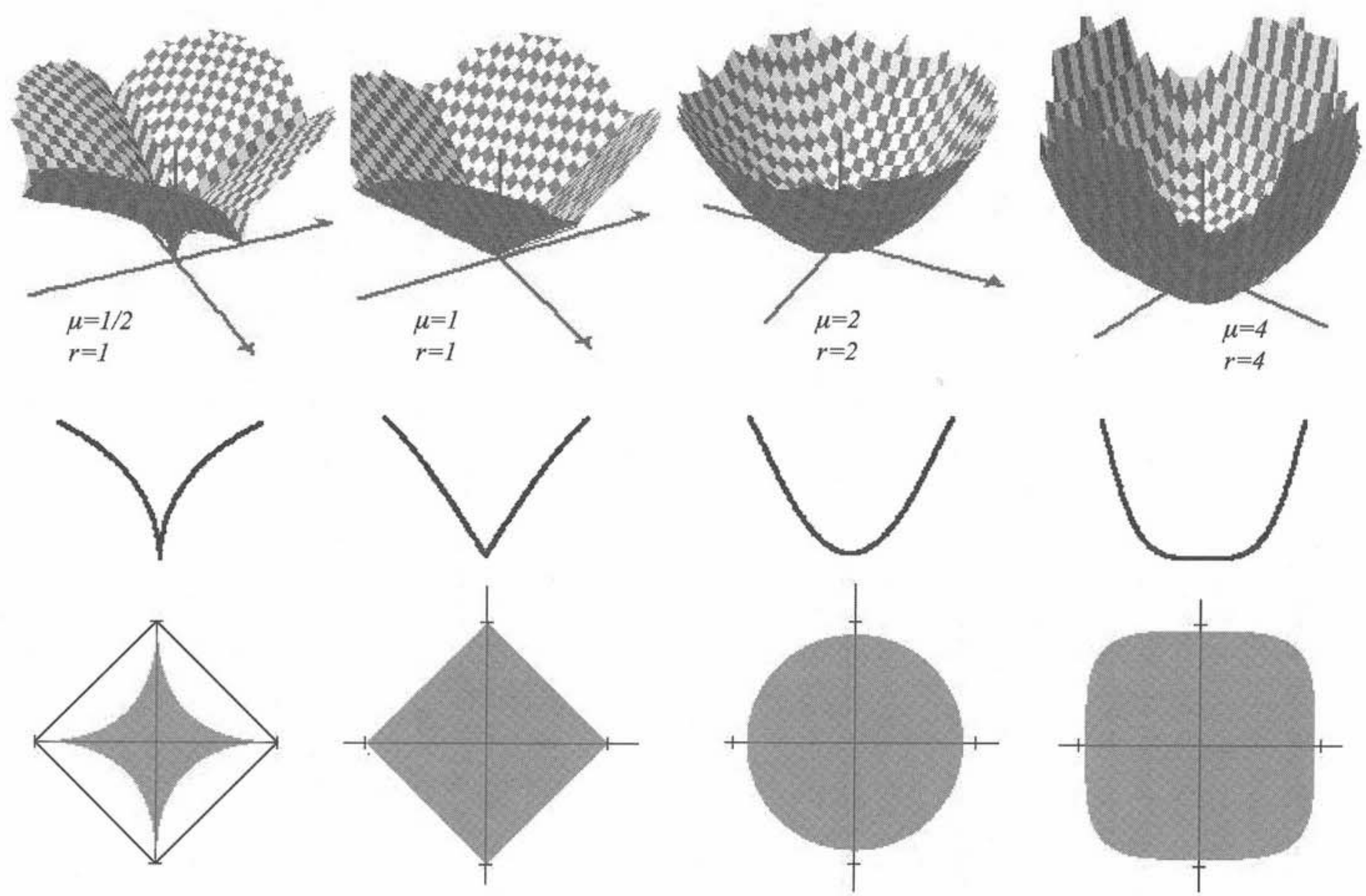

FIG. 1.5. Psychometric functions in the vicinity of their minima for different values of μ and corresponding r (upper row), their vertical cross sections (middle row), and horizontal cross sections (bottom row). The coordinate axes are calibrated in Fechnerian distances along these axes.

This equation can be combined with Equation 18 in the following statement. If $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are perceptually separable, then they can be recalibrated (smoothly transformed) into $\langle \bar{x}^1 \rangle$ and $\langle \bar{x}^2 \rangle$ in such a way that

$$G(\bar{\mathbf{x}}, \bar{\mathbf{y}})^r = |\bar{x}^1 - \bar{y}^1|^r + |\bar{x}^2 - \bar{y}^2|^r, \tag{20}$$

with the exponent $r = \max\{\mu, 1\}$.

In essence, this statement fulfills the goal of the present analysis, except that it seems more satisfying to formulate the main result of this work without mentioning the recalibration procedure (or any specific calibration at all) for the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$. This can be readily achieved. Recall that the definition of perceptual separability is formulated for some rectangular area $\mathfrak{R} \subseteq \mathfrak{M}$. One consequence of this provision is that if one chooses a point of origin in $\mathfrak{R}$, $\mathbf{o} = (o^1, o^2)$, and draws through this point the coordinate lines $\{(x^1, x^2) \in \mathfrak{R}: x^2 = o^2\}$ and $\{(x^1, x^2) \in \mathfrak{R}: x^1 = o^1\}$, then, for any $\mathbf{x} = (x^1, x^2) \in \mathfrak{R}$ and $\mathbf{y} = (y^1, y^2) \in \mathfrak{R}$, their projections $\mathbf{x}_1 = (x^1, o^2)$, $\mathbf{y}_1 = (y^1, o^2)$ on the first axis and $\mathbf{x}_2 = (o^1, x^2)$, $\mathbf{y}_2 = (o^1, y^2)$ on the second axis are stimuli belonging to $\mathfrak{R}$. Observe now, in reference to Equation 20, that

$$|\bar{x}^1 - \bar{y}^1| = G(\mathbf{x}_1, \mathbf{y}_1), |\bar{x}^2, y^2| = G(\mathbf{x}_2, \mathbf{y}_2).$$

With this, the development presented in this chapter can be summarized in the following theorem (see Fig. 1.6).

Theorem AB (Minkowski power-metric structure of Fechnerian metric). Let the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ imposed on the stimulus space $\mathfrak{M}$ be perceptually

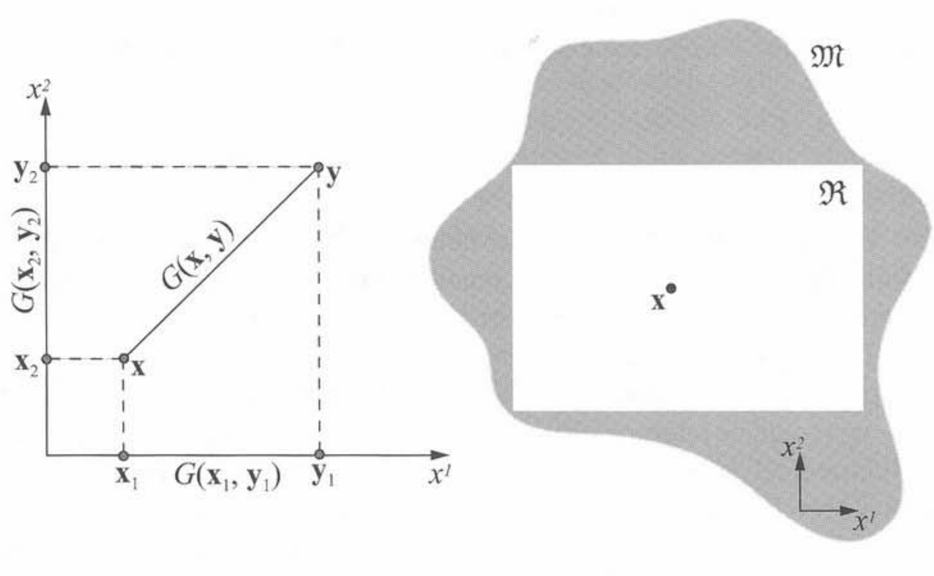

FIG. 1.6. A diagram for Theorem AB.

separable with respect to the discrimination system $\langle \Re, \psi \rangle$. Then the Fechnerian metric G on this space is a Minkowski power metric with respect to the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$:

$$G(\mathbf{x}, \mathbf{y})^r = G(\mathbf{x}_1, \mathbf{y}_1)^r + G(\mathbf{x}_2, \mathbf{y}_2)^r, \tag{21}$$

where $r = \max\{\mu, 1\}$, μ being the psychometric order of the space.

Note that the choice of the origin $\mathbf{o} = (o^1, o^2)$ used to define the coordinate projections of $\mathbf{x}$ and $\mathbf{y}$ need not be mentioned, because $G(\mathbf{x}_1, \mathbf{y}_1)$ and $G(\mathbf{x}_2, \mathbf{y}_2)$ are invariant with respect to this choice.

POSSIBLE EXTENSIONS

The definition of perceptual separability proposed in this work is mathematically unambiguous, based on observable or computable discrimination judgments, and leads to an interesting result within the framework of MDFS, the power-function Minkowski structure of the Fechnerian metric. The theory implies definite and nontrivial relationships between the horizontal and the vertical cross sections of the discrimination probability functions that can be subjected to experimental analysis. The theory is also readily extendible to an arbitrary number of perceptually separable dimensions or subspaces. None of these features, however, guarantees that the theory is empirically feasible. It is appropriate therefore to discuss in this concluding section some directions in which the theory could be generalized, if eventually found unsatisfactory in its present form.

One approach is to generalize the notion of a psychometric differential while preserving its basic properties and retaining the theory of perceptual separability as is. Specifically, one can define a generalized psychometric differential as

$$\Psi^*_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) = \Gamma^{-1}\{\Gamma[\psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s)] - \Gamma[\psi_{\mathbf{x}}(\mathbf{x})]\},$$

where Γ is some smooth monotonic function. One could interpret this as the transformation of observable discrimination probabilities into "true" probabilities, implying thereby that the latter are "contaminated" by some extraperceptual biasing factors, whose influence can be made additive by an appropriately chosen Γ transformation. Alternatively (or equivalently, depending on one's approach), $\Psi^*_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s)$ could be interpreted as a sensitivity index, on a par with the familiar d'.

A less radical approach from the standpoint of MDFS is to generalize the present theory by relaxing some of the defining properties of perceptual separability. One might argue, for example, that Definition A alone captures the essence of perceptual separability, whereas the detachability constraint can be relaxed or dropped altogether. Following this hypothetical suggestion, let us call dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ *weakly perceptually separable* if they form the dimensional basis for a discrimination system $\langle \mathfrak{M}, \psi \rangle$, in accordance with Definition A. The Fechnerian metric under weak separability does not have the Minkowski power metric structure. However, by applying Equations 10 and 12 to Lemma A, one can demonstrate the truth of the following statement.

Theorem A (Local Minkowski power-metric structure of Fechnerian metric). Let the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ imposed on the stimulus space $\mathfrak{M}$ be weakly perceptually separable with respect to the discrimination system $\langle \mathfrak{M}, \psi \rangle$. Then the (Fechner–Finsler) metric function F on this space has the structure

$$F(\mathbf{x}, \mathbf{u})^{\mu} = F(\mathbf{x}, \mathbf{u}_1)^{\mu} + F(\mathbf{x}, \mathbf{u}_2)^{\mu},$$

where μ is the psychometric order of the space. This in turn implies that the Fechnerian metric G is *locally* a Minkowski power metric with respect to the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$:

$$G(\mathbf{x}, \mathbf{x} + \mathbf{u}s)^r \sim G(\mathbf{x}, \mathbf{x} + \mathbf{u}_1 s)^r + G(\mathbf{x}, \mathbf{x} + \mathbf{u}_2 s)^r (\text{as } s \to 0+),$$

where $r = \max\{\mu, 1\}$.

This result is stronger than it might appear. The shapes of and the relationship between the vertical and the horizontal cross sections of the psychometric functions in the vicinity of their minima remain in the case of weak separability precisely the same as illustrated in Fig. 1.5, except that the calibration of the axes mentioned in the legend should now be understood in a local sense. The weak perceptual separability therefore may be sufficiently rich in consequences to be of interest.

These examples may suffice to demonstrate the generalizability of the perceptual separability theory. At this stage, however, the scientific value of the theory may to a greater extent depend on its development in the opposite direction, toward more specialized empirically falsifiable models, constructed by combining the theory's abstract and general premises with plausible constraints of a more technical and domain-specific nature.

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