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# Multidimensional Fechnerian Scaling: Pairwise Comparisons, Regular Minimality, and Nonconstant Self-Similarity

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Stimuli presented pairwise for same–different judgments belong to two distinct observation areas (different time intervals and/or locations). To reflect this fact the underlying assumptions of multidimensional Fechnerian scaling (MDFS) have to be modified, the most important modification being the inclusion of the requirement that the discrimination probability functions possess regular minima. This means that the probability with which a fixed stimulus in one observation area (a reference) is discriminated from stimuli belonging to another observation area reaches its minimum when the two stimuli are identical (following, if necessary, an appropriate transformation of the stimulus measurements in one of the two observation areas). The remaining modifications of the underlying assumptions are rather straightforward, their main outcome being that each of the two observation areas has its own Fechnerian metric induced by a metric function obtained in accordance with the regular variation version of MDFS. It turns out that the regular minimality requirement, when combined with the empirical fact of nonconstant self-similarity (which means that the minimum level of the discrimination probability function for a fixed reference stimulus is generally different for different reference stimuli), imposes rigid constraints on the interdependence between discrimination probabilities and metric functions within each of the observation areas and on the interdependence between Fechnerian metrics and metric functions belonging to different observation areas. In particular, it turns out that the psychometric order of the stimulus space cannot exceed 1. © 2002 Elsevier Science (USA)

## 1. INTRODUCTION

This paper adapts the theory of multidimensional Fechnerian scaling (MDFS) to the empirical paradigm in which stimuli are presented pairwise and the observer is

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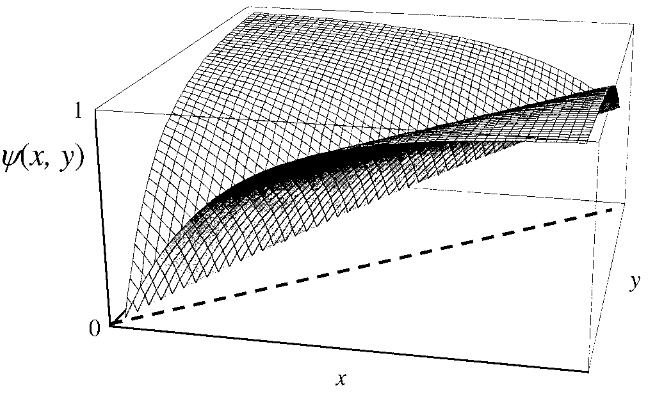


**FIG. 1.** A possible appearance of a discrimination probability function for unidimensional stimuli.

asked to determine whether they are the same or different. Although the discrimination probabilities

$$\psi(\mathbf{x}, \mathbf{y}) = \Pr[\text{stimulus } \mathbf{y} \text{ is discriminated from stimulus } \mathbf{x}]$$

can be computed from other experimental paradigms as well, the same–different one with pairwise presented stimuli seems to be most naturally suited for this purpose. Throughout this paper discrimination probabilities $\psi(\mathbf{x}, \mathbf{y})$ are always assumed to be obtained from same–different judgments. A possible appearance of such a discrimination probability function is shown in Fig. 1.

In the previous papers on MDFS (Dzhafarov, 2002, in press a, b; Dzhafarov & Colonius, 1999a, 1999b, 2001) the two stimuli, $\mathbf{x}$ and $\mathbf{y}$, have been treated as belonging to one and the same $n$-dimensional stimulus space, the $n$ dimensions of the space representing the stimulus characteristics that vary in a given experiment. This is, however, an idealization, less appropriate for the same–different paradigm than it is for one in which stimuli are presented and categorized by an observer one at a time, the pairwise discrimination probabilities being computed from the distributions of the categories assigned to individual stimuli.

The key fact about pairwise presentations is that $\mathbf{x}$ and $\mathbf{y}$ belong to two distinct *observation areas*, in essence two different stimulus spaces: thus, $\mathbf{x}$ (say, a tone) may be presented first and followed by $\mathbf{y}$ (another tone), or $\mathbf{x}$ and $\mathbf{y}$ (visual objects) may be presented to the left and to the right of a fixation mark. Fechner (1887/1987, p. 217) calls this key fact the "non-removable spatiotemporal non-coincidence" of the stimuli being compared. It gives an empirical meaning to treating $\mathbf{x}$ and $\mathbf{y}$ as an *ordered pair*, with the implication that $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{y}, \mathbf{x})$ are distinct pairs, and $(\mathbf{x}, \mathbf{x})$ is a pair rather than a single stimulus.

To reflect the existence of two distinct observation areas, the underlying assumptions of MDFS have to be modified. The modifications in question are simple, but they have surprisingly far-reaching implications. The most important modification occurs in the First Assumption of Fechnerian scaling whose original formulation has to be complemented by a fundamental qualitative constraint, termed *regular minimality*, both intuitively plausible and corroborated by empirical evidence. This and other, more trivial, emendations of the underlying assumptions, when taken in conjunction with the empirical fact that another qualitative constraint, *constant self-similarity*, does *not* hold for discrimination probabilities, impose rigid limitations on the shape and smoothness of the discrimination probability functions in the vicinity of their minima, inducing thereby rigid constraints on the ensuing Fechnerian computations.

A brief account of the application of the theory presented in this paper to unidimensional stimulus continua can be found in Dzhafarov (2001).

## 2. REGULAR MINIMALITY AND NONCONSTANT SELF-SIMILARITY

To emphasize the symmetrical treatment of the two stimuli $\mathbf{x}$ and $\mathbf{y}$ and the arbitrariness of assigning to one of them the status of a reference stimulus, the discrimination probabilities are denoted by $\psi(\mathbf{x}, \mathbf{y})$, rather than $\psi_{\mathbf{x}}(\mathbf{y})$ used in the previous publications. Formally, a complete characterization of a stimulus $\mathbf{x}$ in a same–different experiment can be presented as

$$\mathbf{x} = (x^1, ..., x^n, \mathbf{F}, \mathfrak{I}),$$

where $(x^1, ..., x^n)$ are stimulus characteristics that vary in the experiment, $\mathbf{F}$ denotes all stimulus characteristics that could vary independent of $(x^1, ..., x^n)$ but are kept at fixed values, while $\mathfrak{I}$ stands for the stimulus characteristics that determine the observation area to which the stimulus belongs (e.g., first-second, left-right). The values of $(x^1, ..., x^n)$ are assumed to belong to an open connected region of $\mathrm{Re}^n$, referred to as an $n$-dimensional continuous stimulus space, $\mathfrak{M}$. In accordance with common practice, it is convenient to identify $\mathbf{x}$ with its varying part,

$$\mathbf{x} = (x^1, ..., x^n),$$

leaving $\mathbf{F}$ unmentioned altogether and $\mathfrak{I}$ implied by the position of $\mathbf{x}$ within the ordered pair: $(\mathbf{x}, \cdot)$ or $(\cdot, \mathbf{x})$. In the present context, however, the observation area often has to be mentioned explicitly, in which case I use a modified "belongs to" sign and write $\mathbf{x} \epsilon \mathfrak{I}_1$ or $\mathbf{x} \epsilon \mathfrak{I}_2$ to indicate that $\mathbf{x}$ belongs to the observation area $\mathfrak{I}_1$ (or $\mathfrak{I}_2$). Thus, for any $\mathbf{x}, \mathbf{y} \in \mathfrak{M}$, the expression $\psi(\mathbf{x}, \mathbf{y})$ implies $\mathbf{x} \epsilon \mathfrak{I}_1$ and $\mathbf{y} \epsilon \mathfrak{I}_2$.

The notion of regular minimality is somewhat easier to introduce in the context of the *sensory-physical matching* paradigm rather than the same–different comparisons. Refer to Fig. 2. Let, for a fixed $\mathbf{x} \epsilon \mathfrak{I}_1$, the subject be asked to find the value of $\mathbf{y} \epsilon \mathfrak{I}_2$ that appears as close to $\mathbf{x}$ as possible (closer than any other $\mathbf{y} \epsilon \mathfrak{I}_2$); vice versa, for a fixed $\mathbf{y} \epsilon \mathfrak{I}_2$ the subject seeks the value of $\mathbf{x} \epsilon \mathfrak{I}_1$ that appears as close as possible to $\mathbf{y}$. Let both these tasks have unique solutions: for a fixed $\mathbf{x} \epsilon \mathfrak{I}_1$ the closest match in $\mathfrak{I}_2$ occurs at $\mathbf{y} = \mathbf{h}(\mathbf{x})$, while for a fixed $\mathbf{y} \epsilon \mathfrak{I}_2$ the closest match in $\mathfrak{I}_1$ occurs at $\mathbf{x} = \mathbf{g}(\mathbf{y})$. The regular minimality constraint is the requirement that

$$\mathbf{g} \equiv \mathbf{h}^{-1}.$$

In other words, if $\mathbf{y}_0$ is the closest match to $\mathbf{x}_0 \epsilon \mathfrak{I}_1$ among all $\mathbf{y} \epsilon \mathfrak{I}_2$, then $\mathbf{x}_0$ is the closest match to $\mathbf{y}_0 \epsilon \mathfrak{I}_2$ among all $\mathbf{x} \epsilon \mathfrak{I}_1$. The simplest form of regular minimality

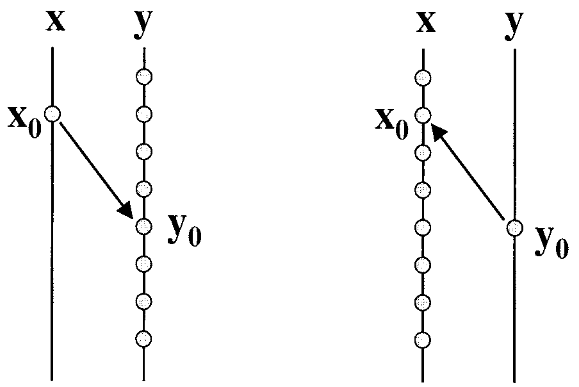


**FIG. 2.** Regular minimality for sensory-physical matching: $\mathbf{y}_0$ is closer to $\mathbf{x}_0$ than any other $\mathbf{y}$ if and only if $\mathbf{x}_0$ is closer to $\mathbf{y}_0$ than any other $\mathbf{x}$ (where $\mathbf{x} \in \mathfrak{I}_1$, $\mathbf{y} \in \mathfrak{I}_2$).

holds when the matched stimuli are identical (in their varying part), that is, when the functions $\mathbf{h}$ and $\mathbf{g}$ are identities.

As an example, let $x, y$ be unidimensional stimuli, say, weights placed on two palms of a subject (left, $\mathfrak{I}_1$, and right, $\mathfrak{I}_2$, respectively), and let, for whatever reason, among all $y$ placed on the right palm the closest match to $x$ on the left palm be achieved at $y = cx$, where $c$ is a constant. The regular minimality then simply means that among all $x$ placed on the left palm the one to appear the closest to $y$ on the right palm will be $x = y/c$. The regular minimality constraint, therefore, allows one to speak of matched stimuli ($\mathbf{x} \in \mathfrak{I}_1$, $\mathbf{y} \in \mathfrak{I}_2$) without specifying which of the two stimuli was matched to which.

Returning to the same–different comparisons, the value of $\mathbf{y} \in \mathfrak{I}_2$ that appears as close as possible to a fixed $\mathbf{x} \in \mathfrak{I}_1$ is naturally defined as

$$\arg\min_{\mathbf{y}} \psi(\mathbf{x}, \mathbf{y}),$$

the value of $\mathbf{y}$ at which the mapping $\mathbf{y} \to \psi(\mathbf{x}, \mathbf{y})$ reaches its minimum. Following a tradition, this value can be called the *point of subjective equality* (in $\mathfrak{I}_2$) for $\mathbf{x} \in \mathfrak{I}_1$. The point of subjective equality in $\mathfrak{I}_1$ for $\mathbf{y} \in \mathfrak{I}_2$ is defined analogously,

$$\arg\min_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{y}).$$

According to the First Assumption of MDFS (Dzhafarov & Colonius, 2001), for a fixed $\mathbf{x}$, the function $\mathbf{y} \to \psi(\mathbf{x}, \mathbf{y})$ achieves its global minimum at some value $\mathbf{y} = \mathbf{h}(\mathbf{x})$ (see Fig. 3, left), $\mathbf{h}$ being continuously differentiable. By symmetry, the First Assumption also states (see Fig. 3, right) that for a fixed $\mathbf{y}$, the function $\mathbf{x} \to \psi(\mathbf{x}, \mathbf{y})$ achieves its global minimum at some value $\mathbf{x} = \mathbf{g}(\mathbf{y})$, $\mathbf{g}$ being continuously differentiable. The regular minimality property holds, or, equivalently, $\psi(\mathbf{x}, \mathbf{y})$ has *regular minima*, if $\mathbf{g} \equiv \mathbf{h}^{-1}$ (which implies that both $\mathbf{g}$ and $\mathbf{h}$ are diffeomorphisms $\mathfrak{M} \to \mathfrak{M}$). Rather than treating it as a separate assumption, it is convenient to consider this requirement as part of the (amended) First Assumption of MDFS.

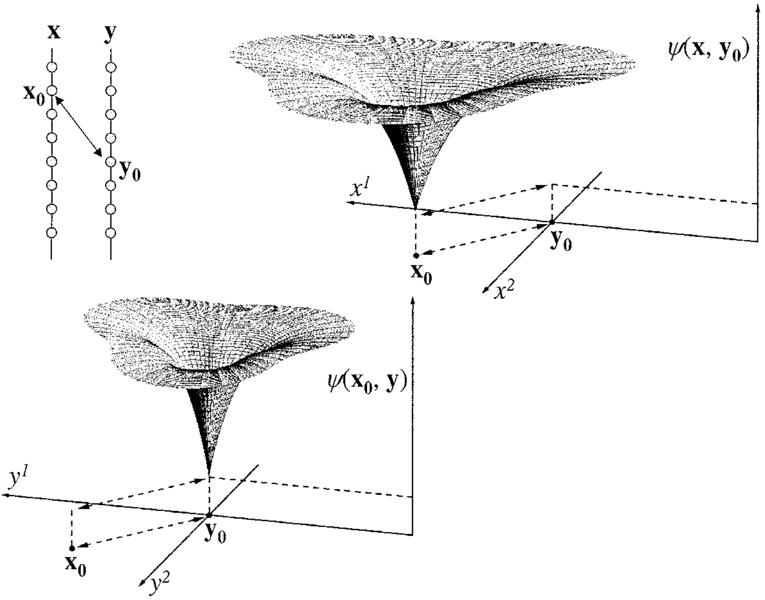


**FIG. 3.** Regular minimality for discrimination probabilities (two-dimensional stimuli): $\mathbf{y}_0 = \arg\min_{\mathbf{y}} \psi(\mathbf{x}_0, \mathbf{y})$ if and only if $\mathbf{x}_0 = \arg\min_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{y}_0)$.

Once the regular minimality constraint is in place, the discrimination probability functions can be brought to a *canonical form* by either of the following transformations (Dzhafarov, in press a; Dzhafarov & Colonius, 2001):

$$\psi(\mathbf{x}, \mathbf{y}) = \psi_1[\mathbf{h}(\mathbf{x}), \mathbf{y}] = \psi_2[\mathbf{x}, \mathbf{g}(\mathbf{y})]. \tag{1}$$

Clearly, both $\psi_1(\mathbf{x}, \mathbf{y})$ and $\psi_2(\mathbf{x}, \mathbf{y})$ achieve their minima at $\mathbf{x} = \mathbf{y}$, that is, the subjective equality functions $\mathbf{g}$ and $\mathbf{h}$ for these canonical forms are identities (see Fig. 4). As explained in Section 4, since

$$\psi_2(\mathbf{x}, \mathbf{y}) = \psi_1[\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{y})] \tag{2}$$

and $\mathbf{h}$ is a diffeomorphism, it is immaterial for the Fechnerian theory which of these two (or other possible) canonical forms one uses. Assuming that $\psi(\mathbf{x}, \mathbf{y})$ is already in a canonical form, the regular minimality constraint can be presented as

**(A1a)** (*regular minimality*) for any $\mathbf{x}$ and any $\mathbf{y} \neq \mathbf{x}$,

$$\psi(\mathbf{x}, \mathbf{x}) < \begin{cases} \psi(\mathbf{x}, \mathbf{y}) \\ \psi(\mathbf{y}, \mathbf{x}). \end{cases} \tag{3}$$

In this form regular minimality has been introduced in Dzhafarov (in press a). The First Assumption of MDFS includes, in addition, the following two statements:

**(A1b)** (*continuity*) $\psi(\mathbf{x}, \mathbf{y})$ is continuous in $(\mathbf{x}, \mathbf{y})$;

**(A1c)** (*monotonicity*) for any $\mathbf{x}$ and any direction vector $\mathbf{u} \neq \mathbf{0}$, $\psi(\mathbf{x}, \mathbf{x} + \mathbf{u}s)$ and $\psi(\mathbf{x} + \mathbf{u}s, \mathbf{x})$ increase with $s > 0$ in a sufficiently small vicinity of $s = 0$.

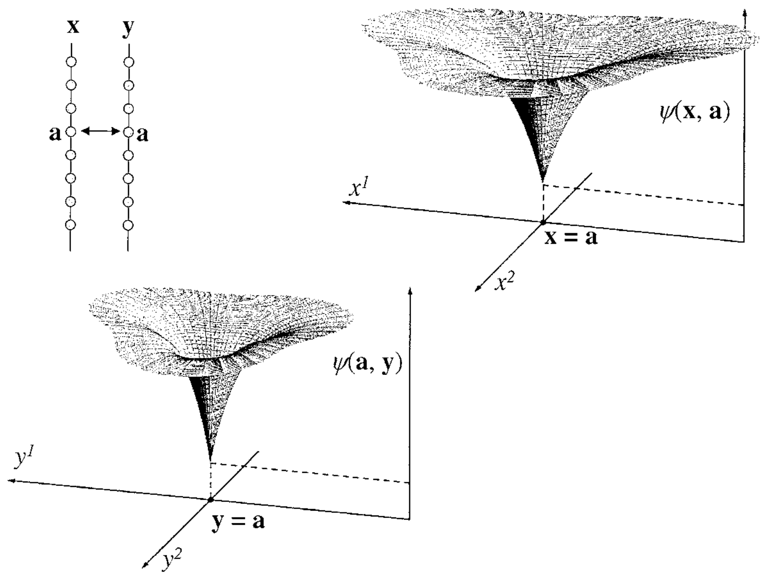


**FIG. 4.** Regular minimality in a canonical form (two-dimensional stimuli): $\arg\min_{\mathbf{y}} \psi(\mathbf{a}, \mathbf{y}) = \arg\min_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{a}) = \mathbf{a}$.

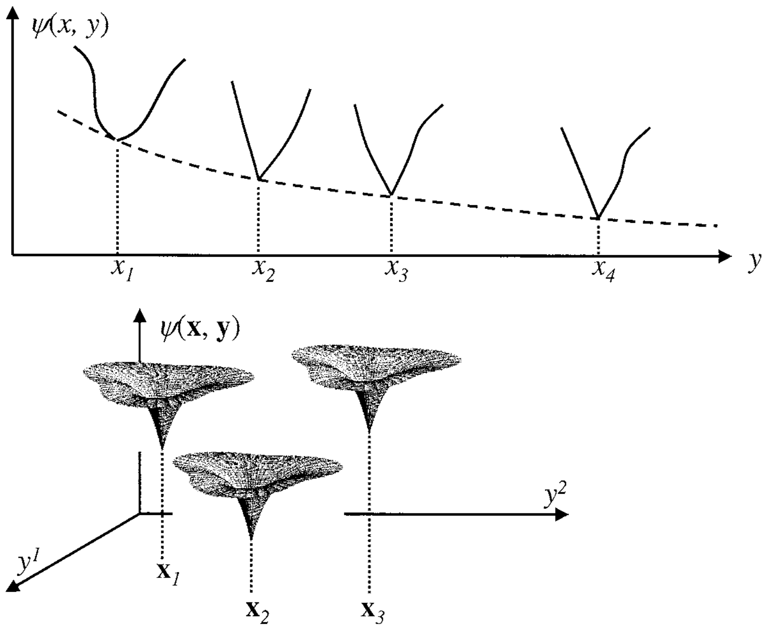


**FIG. 5.** The nonconstant self-similarity property of discrimination probabilities (one- and two-dimensional stimuli).

The discrimination probability function shown in Fig. 1 satisfies all the constraints constituting the First Assumption, including the regular minimality in a canonical form. The First Assumption does not imply, however, that the minimum level $\psi(\mathbf{x}, \mathbf{x})$ is constant across different $\mathbf{x}$. If this does happen,

$$\psi(\mathbf{x}, \mathbf{x}) \equiv const,$$

we say that $\psi(\mathbf{x}, \mathbf{y})$ possesses the *constant self-similarity* (or self-dissimilarity) property. Otherwise $\psi(\mathbf{x}, \mathbf{y})$ exhibits *nonconstant self-similarity* (see Fig. 5):

$$\psi(\mathbf{x}, \mathbf{x}) \neq const \tag{4}$$

at least in some subregion of the stimulus space $\mathfrak{M}$. This is, for instance, the case for $\psi(\mathbf{x}, \mathbf{y})$ shown in Fig. 1: the value of $\psi(\mathbf{x}, \mathbf{x})$ there increases with $\mathbf{x}$.

## 3. EMPIRICAL EVIDENCE

Empirical data on same–different discrimination probabilities in continuous stimulus spaces are scarce, but what is available seems to uphold the principle of regular minimality while ruling out constant self-similarity as a general constraint.

In an experiment by Zimmer and Colonius (2000) listeners made same–different judgments in response to successively presented pure fixed-frequency tones varying in intensity (see Fig. 6). For fixed intensity values of the first tone, $x$, the discrimination probabilities $\psi(x, y)$ in this experiment form characteristically V-shaped curves with the minimum point at $y = x$; and the analogous fact is found for

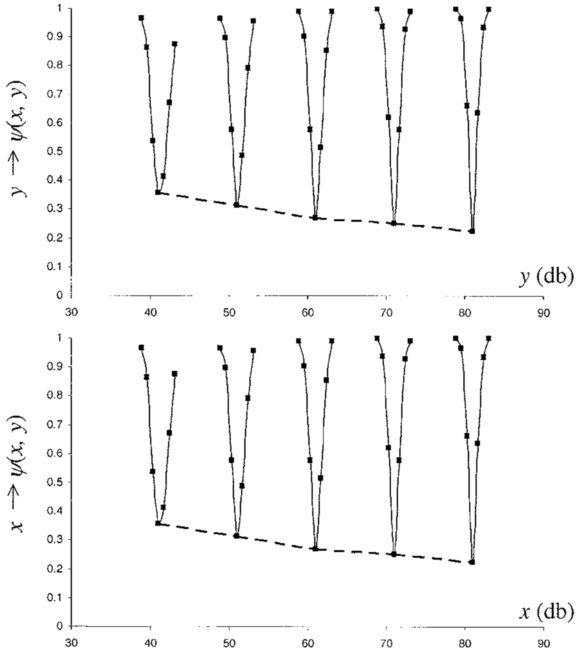


**FIG. 6.** Discrimination probabilities in the experiment by Zimmer and Colonius (2000). The data are shown for one observer in the format of Fig. 5 (upper panel). $x$ ($y$) is the intensity of the tone presented first (second).

$\psi(x, y)$ with fixed intensity values of the second tone. The regular minimality, therefore, holds here in a canonical form. At the same time, the minimum level $\psi(x, x)$ of the discrimination probability function prominently changes with intensity $x$ (nonconstant self-similarity).

The same pattern (regular minimality in a canonical form and nonconstant self-similarity) is demonstrated in Fig. 7 which presents results of one of my recent experiments. The observers made same–different judgments in response to two horizontal synchronous apparent motions (two-flash stimuli) presented collinearly at a 10 deg arc separation in a frontoparallel plane. The only possible physical difference between the two motions (the left-hand and the right-hand one) was their amplitude (the distance between the two flashes), that varied between 5 and 45 min arc. In this particular case the discrimination probabilities are to a high degree of precision *order-balanced*,

$$\psi(\mathbf{x}, \mathbf{y}) \equiv \psi(\mathbf{y}, \mathbf{x}), \tag{5}$$

because of which they can be shown in a single graph, rather than in two, as in Fig. 6. The results involving other observers and/or modified experimental designs (e.g., successive presentation of two motions) generally do not show this order-balance. In addition, they typically exhibit some constant error, a systematic left–right or first–second asymmetry. In no case, however, can one reject the regular minimality hypothesis, while the constancy of self-similarity can be rejected in most cases (examples of data with a constant error are not shown because it is very difficult to visually assess a noncanonical form of regular minimality).

Indow, Robertson, von Grunau, and Fielder (1992) and Indow (1998) report discrimination probabilities for side-by-side presented colors varying in CIE chromaticity–luminance coordinates (a three-dimensional continuous stimulus space). With the right-hand color $\mathbf{y}$ serving as a reference stimulus, the discrimination probabilities in this study reached their minimum level at $\mathbf{x} = \mathbf{y}$. The experiment was not replicated with the left-hand color $\mathbf{x}$ used as a reference, so one cannot check for the regular minimality constraint directly. It is reasonable to assume, however, that $\psi(\mathbf{x}, \mathbf{y})$ for side-by-side presented colors is order-balanced, (5), and

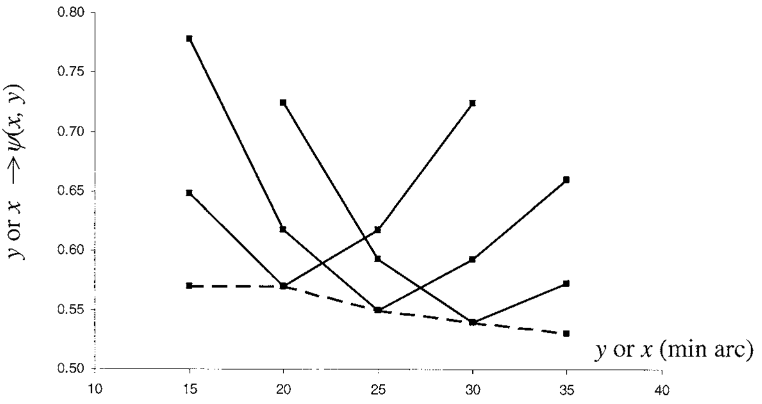


**FIG. 7.** Discrimination probabilities for apparent motions varying in amplitude. The data are shown for one observer in the format of Fig. 5 (upper panel). $x$ ($y$) is the amplitude of the left-hand (right-hand) motion.

consequently regular minimality holds here in a canonical form. Nonconstant self-similarity is prominently present here too: thus, when the colors were presented on a dark background and the reference color $\mathbf{y}$ changed from gray to red to yellow to green to blue, the probability $\psi(\mathbf{y}, \mathbf{y})$ in one observer increased from 0.07 to 0.33.

With some caution one can also add to this list the often cited data by Rothkopf (1957), whose listeners made same–different judgments in response to successively presented Morse-coded letters. This stimulus space is discrete, and our definition of regular minimality strictly speaking does not apply. It is still relevant, however, that the probability of discriminating a given Morse letter (whether presented first or second) is at its minimum when paired with the same Morse letter, but that this minimum level changes from one letter to another. Additional evidence of the same nature can be found in Krumhansl (1978) and Tversky (1977).

## 4. OTHER ASSUMPTIONS OF FECHNERIAN SCALING

MDFS is based on four assumptions, of which the First, Second, and Fourth Assumptions constitute the core of the theory while the third assumption is treated as optional. The modification of the first assumption that incorporates the distinction between two observation areas and the fundamental notion of regular minimality is described in Section 2. Here, I consider the induced changes in the remaining assumptions.

The discrimination probability function $\psi(\mathbf{x}, \mathbf{y})$ in the remainder is considered to be in a canonical form, perhaps following an appropriate transformation, (1).

For any stimulus $\mathbf{x}$ and any direction vector $\mathbf{u} \neq \mathbf{0}$, it follows from the First Assumption (with the regular minimality in a canonical form) that the *psychometric differentials of the first and second kind,*

$$\begin{array}{l} \Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s) = \psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x}) \\ \Psi^{(2)}_{\mathbf{x},\mathbf{u}}(s) = \psi(\mathbf{x}+\mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x}) \end{array}, \qquad s \geqslant 0,$$

continuously decrease to zero with $s \to 0+$. The solutions for $s$ of the equations

$$\begin{array}{c} \Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s) = h \\ \Psi^{(2)}_{\mathbf{x},\mathbf{u}}(s) = h \end{array}$$

in an interval of sufficiently small $h > 0$ are called *stimulus differentials,*

$$\begin{array}{c} s_1 = \Phi^{(1)}_{\mathbf{x},\mathbf{u}}(h) \\ s_2 = \Phi^{(2)}_{\mathbf{x},\mathbf{u}}(h). \end{array}$$

The Second Assumption of MDFS is formulated in Dzhafarov and Colonius (2001) and in Dzhafarov (2002) as a statement of the comeasurability in the small of stimulus differentials of the first kind (or, by symmetry, second kind). It now has to be amended to ensure, in addition, that stimulus differentials of the first kind are comeasurable in the small with those of the second kind. (The comeasurability in the small means that the ratio of two quantities tends to a positive finite limit as both of them tend to zero.)

**(A2)** For some fixed line element (i.e., a stimulus–direction pair) $(\mathbf{x}_0, \mathbf{u}_0)$ and $i = 1$ or 2, the limit ratios

$$\lim_{h \to 0+} \frac{\Phi^{(i)}_{\mathbf{x}_0, \mathbf{u}_0}(h)}{\Phi^{(1)}_{\mathbf{x}, \mathbf{u}}(h)} = F_1(\mathbf{x}, \mathbf{u})$$

$$\lim_{h \to 0+} \frac{\Phi^{(i)}_{\mathbf{x}_0, \mathbf{u}_0}(h)}{\Phi^{(2)}_{\mathbf{x}, \mathbf{u}}(h)} = F_2(\mathbf{x}, \mathbf{u})$$

are finite, positive, and continuous in $(\mathbf{x}, \mathbf{u})$, for any $(\mathbf{x}, \mathbf{u})$.

Thus, instead of a single *Fechner–Finsler metric function* $F(\mathbf{x}, \mathbf{u})$ considered in the previous publications we now have two, $F_1$ and $F_2$, for the stimuli belonging to the first and the second observation area, respectively. (The designation of the metric functions includes the name of Finsler because the ensuing Fechnerian metrics are referred to in geometry as generalized Finsler metrics, with the Finsler metrics proper being their prominent special case; for details see Dzhafarov & Colonius, 1999a, 2001.)

The properties of $F_1$ and $F_2$ are precisely the same as those of $F$ in the previously published theory. Thus, the positive Euler homogeneity,

$$\begin{aligned} F_1(\mathbf{x}, k\mathbf{u}) &= kF_1(\mathbf{x}, \mathbf{u}) \\ F_2(\mathbf{x}, k\mathbf{u}) &= kF_2(\mathbf{x}, \mathbf{u}) \end{aligned}, \qquad k > 0,$$

is proved by the same argument as in Dzhafarov and Colonius (2001). The so-called *Fundamental Theorem* of MDFS is also proved as in Dzhafarov and Colonius (2001), although its formulation changes to reflect the existence of the two kinds of psychometric differentials.

(The symbol $\sim$ connecting two expressions means their asymptotic equality, i.e., that their ratio tends to 1.)

THEOREM 4.1 (Fundamental Theorem of MDFS). *There exists a global psychometric transformation $\Phi(h)$, continuously decreasing to zero with $h \to 0+$ and one and the same for all psychometric differentials of both kinds, such that*

$$\begin{aligned} \Phi[\psi(\mathbf{x}, \mathbf{x} + \mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x})] &\sim F_1(\mathbf{x}, \mathbf{u})\, s \\ \Phi[\psi(\mathbf{x} + \mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x})] &\sim F_2(\mathbf{x}, \mathbf{u})\, s \end{aligned} \qquad \textit{as} \quad s \to 0+. \tag{6}$$

*The metric functions $F_1$ and $F_2$ are determined from $\psi(\mathbf{x}, \mathbf{y})$ uniquely and $\Phi$ asymptotically uniquely, up to multiplication by one and the same positive constant,*

$$F_1^*(\mathbf{x}, \mathbf{u}) = kF_1(\mathbf{x}, \mathbf{u}),$$

$$F_2^*(\mathbf{x}, \mathbf{u}) = kF_2(\mathbf{x}, \mathbf{u}),$$

$$\Phi^*(h) \sim k\Phi(h), \qquad \textit{as} \quad h \to 0+.$$

COROLLARY 4.1. *If $\psi(\mathbf{x}, \mathbf{y})$ is order-balanced,*

$$\psi(\mathbf{x}, \mathbf{y}) \equiv \psi(\mathbf{y}, \mathbf{x}),$$

*then*

$$F_1(\mathbf{x}, \mathbf{u}) \equiv F_2(\mathbf{x}, \mathbf{u}) \equiv F(\mathbf{x}, \mathbf{u}). \tag{7}$$

This is, of course, what one should expect intuitively: if the two observation intervals are interchangeable, one does not have to distinguish $F_1$ and $F_2$.

Deferring until later a discussion of the optional Third Assumption,

$$\textbf{(A3)} \qquad \begin{cases} \Phi^{(1)}_{\mathbf{x},\mathbf{u}}(h) \sim \Phi^{(1)}_{\mathbf{x},-\mathbf{u}}(h) \\ \Phi^{(2)}_{\mathbf{x},\mathbf{u}}(h) \sim \Phi^{(2)}_{\mathbf{x},-\mathbf{u}}(h) \end{cases} \qquad \text{as} \quad h \to 0+,$$

the Fourth Assumption of MDFS states that

**(A4)** for at least one line element $(\mathbf{x}_0, \mathbf{u}_0)$ and $i = 1$ or 2, $\Psi^{(i)}_{\mathbf{x}_0,\mathbf{u}_0}(s)$ regularly varies at $s = 0+$.

The notion of *regular variation* (at the origin) is used as in Dzhafarov (2002), meaning that

$$\lim_{s \to 0+} \frac{\Psi^{(i)}_{\mathbf{x}_0,\mathbf{u}_0}(ks)}{\Psi^{(i)}_{\mathbf{x}_0,\mathbf{u}_0}(s)} = \gamma(k)$$

is finite, positive, and nonconstant. The nonconstancy requirement excludes *slowly varying functions*, those with $\gamma(k) \equiv 1$, from the class of regularly varying ones, which is a deviation from the standard mathematical usage (Bingham, Goldie, & Teugels, 1987).

To prevent any confusion, the line elements $(\mathbf{x}_0, \mathbf{u}_0)$ mentioned in the formulations of the Second and the Fourth Assumptions of MDFS have nothing to do with each other. Moreover, as shown in Dzhafarov (2002) and Dzhafarov and Colonius (2001), the choice of $(\mathbf{x}_0, \mathbf{u}_0)$ in both these assumptions is completely arbitrary.

Using the same reasoning as in Dzhafarov (2002) one arrives at the following theorem, in which the unit-regularly varying function $R(s)$ has the structure

$$R(s) = s\ell(s), \; s \geqslant 0, \tag{8}$$

with $\ell(s) \geqslant 0$ being the slowly varying component of $R(s)$,

$$\lim_{s \to 0+} \frac{\ell(ks)}{\ell(s)} = 1. \tag{9}$$

Examples of $R(s)$ continuously decreasing to zero with $s \to 0+$ are $s$, $se^s$, $s \log \frac{1}{s}$, $s/\log \frac{1}{s}$, etc.

THEOREM 4.2 (Main asymptotic representation). *There is a positive constant $\mu$, uniquely determined and called the psychometric order of the stimulus space, and a unit-regularly varying function $R(s)$, determined asymptotically uniquely and continuously decreasing to zero with $s \to 0+$, such that for any $(\mathbf{x}, \mathbf{u})$,*

$$\begin{matrix} \Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s) = \psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x}) \sim F_1^{\mu}(\mathbf{x}, \mathbf{u})\, R(s)^{\mu} \\ \Psi^{(2)}_{\mathbf{x},\mathbf{u}}(s) = \psi(\mathbf{x}+\mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x}) \sim F_2^{\mu}(\mathbf{x}, \mathbf{u})\, R(s)^{\mu} \end{matrix}, \qquad \textit{as} \quad s \to 0+. \tag{10}$$

In accordance with Dzhafarov and Colonius (2001), the contour formed by the endpoints of vectors $\mathbf{u}$ satisfying

$$F_1(\mathbf{x}, \mathbf{u}) = 1$$

for a fixed $\mathbf{x} \in \mathfrak{I}_1$ (the *Fechnerian indicatrix* associated with $F_1$) is roughly similar to the shape of the horizontal (parallel to stimulus space) cross-section of $\mathbf{y} \to \psi(\mathbf{x}, \mathbf{y})$ made at a very small elevation above the minimum level $\psi(\mathbf{x}, \mathbf{x})$; the smaller the elevation the better the similarity. The interpretation of $F_2(\mathbf{x}, \mathbf{u})$ in terms of horizontal cross-sections of $\mathbf{y} \to \psi(\mathbf{y}, \mathbf{x})$ is analogous.

The psychometric order $\mu$ determines another aspect of the shape of the discrimination probability function, the degree of smoothness/cuspidality of the vertical (perpendicular to stimulus space) cross-sections of $\psi(\mathbf{x}, \mathbf{y})$, for a fixed $\mathbf{x} \in \mathfrak{I}_1$ or a fixed $\mathbf{y} \in \mathfrak{I}_2$, made through its point of minimum, $(\mathbf{x}, \psi(\mathbf{x}, \mathbf{x}))$ or $(\mathbf{y}, \psi(\mathbf{y}, \mathbf{y}))$, respectively. Roughly, the lower tips of such cross-sections may range from Y-shaped (needle-sharp, $\mu < 1$) to V-shaped (pencil-sharp, $\mu = 1$) to U-shaped (rounded, $\mu > 1$). According to the theorem above, this characteristic is one and the same for all reference stimuli (fixed $\mathbf{x} \in \mathfrak{I}_1$ or $\mathbf{y} \in \mathfrak{I}_2$) and all directions $\mathbf{u}$ in which the cross-section is made.

The logic of Fechnerian computations dictates that the metric functions $F_1$ and $F_2$ induce two generally different *Fechnerian (oriented) metrics*, $G_1$ and $G_2$: one for the stimuli belonging to the observation area $\mathfrak{I}_1$, the other for those belonging to $\mathfrak{I}_2$. Put briefly (see Dzhafarov & Colonius, 2001, for an exhaustive treatment), the logic is as follows. Let $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ denote an allowable path connecting $\mathbf{a}$ to $\mathbf{b}$, that is, a piecewise continuously differentiable function $\mathbf{x}(t)$, $0 \leqslant t \leqslant 1$, taking values in the stimulus space $\mathfrak{M}$, with $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$. The psychometric lengths $L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b})$ and $L_2(\mathbf{a} \to \mathbf{x} \to \mathbf{b})$ of the path $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ in the two observation areas are defined by

$$\begin{aligned} \mathbf{a}, \mathbf{b} \in \mathfrak{I}_1 &\Rightarrow L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) = \int_0^1 F_1[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt \\ \mathbf{a}, \mathbf{b} \in \mathfrak{I}_2 &\Rightarrow L_2(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) = \int_0^1 F_2[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt. \end{aligned} \tag{11}$$

The function $\mathbf{x}(1-t)$ is the path "opposite" to $\mathbf{x}(t)$, the same trajectory but oriented from $\mathbf{b}$ to $\mathbf{a}$; let it be denoted by $\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}$. For the subsequent development it is useful to mention the easily verifiable fact that when the definition above is applied to this opposite path $\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}$, we have

$$\begin{aligned} \mathbf{a}, \mathbf{b} \in \mathfrak{I}_1 &\Rightarrow L_1(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = \int_0^1 F_1[\mathbf{x}(t), -\dot{\mathbf{x}}(t)]\, dt \\ \mathbf{a}, \mathbf{b} \in \mathfrak{I}_2 &\Rightarrow L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = \int_0^1 F_2[\mathbf{x}(t), -\dot{\mathbf{x}}(t)]\, dt. \end{aligned} \tag{12}$$

The Fechnerian metrics are the infima (whose metric properties can be easily proved, see Dzhafarov & Colonius, 1999a, 2001) taken over all allowable paths,

$$\begin{aligned} \mathbf{a}, \mathbf{b} \in \mathfrak{I}_1 &\Rightarrow G_1(\mathbf{a}, \mathbf{b}) = \inf L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) \\ \mathbf{a}, \mathbf{b} \in \mathfrak{I}_2 &\Rightarrow G_2(\mathbf{a}, \mathbf{b}) = \inf L_2(\mathbf{a} \to \mathbf{x} \to \mathbf{b}). \end{aligned} \tag{13}$$

As shown in Dzhafarov and Colonius (2001), thus defined metrics are invariant under all diffeomorphisms of the stimulus space. In other words, the Fechnerian distances $G_1(\mathbf{a}, \mathbf{b})$ and $G_2(\mathbf{a}, \mathbf{b})$ remain constant when one redefines the discrimination probability functions by

$$\psi(\mathbf{x}, \mathbf{y}) = \tilde{\psi}[\mathbf{H}(\mathbf{x}), \mathbf{H}(\mathbf{y})],$$

where $\mathbf{H}$ is a diffeomorphism $\mathfrak{M} \to \mathfrak{M}$. This is, in view of (2), the reason why either of the two canonizing transformations in (1) can be used for computing both $G_1$ and $G_2$.

Note that unless one adopts the Third Assumption of MDFS, (A3), which implies (see Dzhafarov & Colonius, 2001)

$$\begin{aligned} F_1(\mathbf{x}, \mathbf{u}) &\equiv F_1(\mathbf{x}, -\mathbf{u}) \\ F_2(\mathbf{x}, \mathbf{u}) &\equiv F_2(\mathbf{x}, -\mathbf{u}), \end{aligned} \tag{14}$$

the Fechnerian distances are generally *oriented,*

$$\begin{aligned} G_1(\mathbf{a}, \mathbf{b}) &\neq G_1(\mathbf{a}, \mathbf{b}) \\ G_2(\mathbf{a}, \mathbf{b}) &\neq G_2(\mathbf{a}, \mathbf{b}). \end{aligned}$$

The necessary and sufficient conditions for symmetry of Fechnerian metrics are given in Dzhafarov and Colonius (2001).

## 5. BASIC CONSEQUENCES

Without mentioning this every time, all formal results stated in the remainder are predicated on the core assumptions of MDFS: the First, Second, and Fourth. The use of the Third Assumption, however, is always specified explicitly. The clause "under the Third Assumption" therefore should always be taken to mean "under the Third Assumption added to the core assumptions of MDFS." No assumptions of MDFS imply the order-balance property, (5). The use of this property, therefore, is also indicated explicitly.

Another convention adopted in the remainder is that all limit statements and asymptotic equations of the type $f(s) \sim g(s)$ or $f(s) = o\{g(s)\}$ (i.e., $f(s)/g(s) \to 1$ and $f(s)/g(s) \to 0$, respectively) are tacitly predicated upon $s \to 0+$.

The function

$$\omega(\mathbf{x}) = \psi(\mathbf{x}, \mathbf{x})$$

is referred to as the *minimum level function*. The differential

$$\Omega_{\mathbf{x}, \mathbf{u}}(s) = \omega(\mathbf{x} + \mathbf{u}s) - \omega(\mathbf{x}), \qquad s > 0,$$

of this function at the line element $(\mathbf{x}, \mathbf{u})$ can be presented as

$$\begin{aligned}\Omega_{\mathbf{x},\mathbf{u}}(s) &= \omega(\mathbf{x}+\mathbf{u}s)-\omega(\mathbf{x})\\ &= [\psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s)-\psi(\mathbf{x}, \mathbf{x})]-[\psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s)-\psi(\mathbf{x}+\mathbf{u}s, \mathbf{x}+\mathbf{u}s)]\\ &= [\psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s)-\psi(\mathbf{x}, \mathbf{x})]-\{\psi[(\mathbf{x}+\mathbf{u}s)-\mathbf{u}s, (\mathbf{x}+\mathbf{u}s)]-\psi[(\mathbf{x}+\mathbf{u}s), (\mathbf{x}+\mathbf{u}s)]\}\\ &= \Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s)-\Psi^{(2)}_{\mathbf{x}+\mathbf{u}s, -\mathbf{u}}(s).\end{aligned} \tag{15}$$

Making use of (10) and the continuity of the metric functions $F_1$ and $F_2$, we have

$$\Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s) \sim F_1(\mathbf{x}, \mathbf{u})^\mu R^\mu(s)$$
$$\Psi^{(2)}_{\mathbf{x}+\mathbf{u}s, -\mathbf{u}}(s) \sim F_2(\mathbf{x}+\mathbf{u}s, -\mathbf{u})^\mu R^\mu(s) \sim F_2(\mathbf{x}, -\mathbf{u})^\mu R^\mu(s).$$

Then it follows from (15) that

$$\begin{aligned}&F_1(\mathbf{x}, \mathbf{u}) = F_2(\mathbf{x}, -\mathbf{u}) \Leftrightarrow \Omega_{\mathbf{x},\mathbf{u}}(s) = o\{R^\mu(s)\}\\ &F_1(\mathbf{x}, \mathbf{u}) \neq F_2(\mathbf{x}, -\mathbf{u}) \Leftrightarrow \Omega_{\mathbf{x},\mathbf{u}}(s) \sim [F_1(\mathbf{x}, \mathbf{u})^\mu - F_2(\mathbf{x}, -\mathbf{u})^\mu]\, R^\mu(s).\end{aligned} \tag{16}$$

The same differential $\Omega_{\mathbf{x},\mathbf{u}}(s)$ can also be decomposed as

$$\begin{aligned}\Omega_{\mathbf{x},\mathbf{u}}(s) &= \omega(\mathbf{x}+\mathbf{u}s)-\omega(\mathbf{x})\\ &= [\psi(\mathbf{x}+\mathbf{u}s, \mathbf{x})-\psi(\mathbf{x}, \mathbf{x})]-[\psi(\mathbf{x}+\mathbf{u}s, \mathbf{x})-\psi(\mathbf{x}+\mathbf{u}s, \mathbf{x}+\mathbf{u}s)]\\ &= [\psi(\mathbf{x}+\mathbf{u}s, \mathbf{x})-\psi(\mathbf{x}, \mathbf{x})]-\{\psi[(\mathbf{x}+\mathbf{u}s), (\mathbf{x}+\mathbf{u}s)-\mathbf{u}s]-\psi[(\mathbf{x}+\mathbf{u}s), (\mathbf{x}+\mathbf{u}s)]\}\\ &= \Psi^{(2)}_{\mathbf{x},\mathbf{u}}(s)-\Psi^{(1)}_{\mathbf{x}+\mathbf{u}s, -\mathbf{u}}(s),\end{aligned} \tag{17}$$

and by the same reasoning as above we get

$$\begin{aligned}&F_1(\mathbf{x}, -\mathbf{u}) = F_2(\mathbf{x}, \mathbf{u}) \Leftrightarrow \Omega_{\mathbf{x},\mathbf{u}}(s) = o\{R^\mu(s)\}\\ &F_1(\mathbf{x}, -\mathbf{u}) \neq F_2(\mathbf{x}, \mathbf{u}) \Leftrightarrow \Omega_{\mathbf{x},\mathbf{u}}(s) \sim [F_2(\mathbf{x}, \mathbf{u})^\mu - F_1(\mathbf{x}, -\mathbf{u})^\mu]\, R^\mu(s).\end{aligned} \tag{18}$$

As $\Omega_{\mathbf{x},\mathbf{u}}(s)$ cannot be simultaneously $o\{R^\mu(s)\}$ and asymptotically proportional to $R^\mu(s)$, from (16) and (18) we obtain

THEOREM 5.1. *For any line element* $(\mathbf{x}, \mathbf{u})$*, there are two possibilities: either*

$$\begin{aligned}&F_1(\mathbf{x}, \mathbf{u}) = F_2(\mathbf{x}, -\mathbf{u})\\ &F_1(\mathbf{x}, -\mathbf{u}) = F_2(\mathbf{x}, \mathbf{u})\end{aligned} \quad \textit{and} \quad \Omega_{\mathbf{x},\mathbf{u}}(s) = o\{R^\mu(s)\} \tag{19}$$

*or*

$$\begin{aligned}&F_1(\mathbf{x}, \mathbf{u}) \neq F_2(\mathbf{x}, -\mathbf{u})\\ &F_1(\mathbf{x}, -\mathbf{u}) \neq F_2(\mathbf{x}, \mathbf{u})\end{aligned} \quad \textit{and} \quad \Omega_{\mathbf{x},\mathbf{u}}(s) \sim \begin{cases}[F_1(\mathbf{x}, \mathbf{u})^\mu - F_2(\mathbf{x}, -\mathbf{u})^\mu]\, R^\mu(s)\\ [F_2(\mathbf{x}, \mathbf{u})^\mu - F_1(\mathbf{x}, -\mathbf{u})^\mu]\, R^\mu(s).\end{cases} \tag{20}$$

I refer to the possibility (19) by saying that $F_1$ and $F_2$ are *cross-balanced at* $(\mathbf{x}, \mathbf{u})$; otherwise, if (20), the two metric functions are *cross-unbalanced at* $(\mathbf{x}, \mathbf{u})$. In either of these cases

$$F_1(\mathbf{x}, \mathbf{u})^\mu - F_2(\mathbf{x}, -\mathbf{u})^\mu = F_2(\mathbf{x}, \mathbf{u})^\mu - F_1(\mathbf{x}, -\mathbf{u})^\mu,$$

whence we have

THEOREM 5.2. *The psychometric order $\mu$ and the Fechner–Finsler metric functions $F_1$ and $F_2$ are related by*

$$F_1(\mathbf{x}, \mathbf{u})^\mu + F_1(\mathbf{x}, -\mathbf{u})^\mu \equiv F_2(\mathbf{x}, \mathbf{u})^\mu + F_2(\mathbf{x}, -\mathbf{u})^\mu. \tag{21}$$

If the Third Assumption of MDFS holds, the combination of (21) and (14) implies the cross-balanced relationship (19), whence we have

THEOREM 5.3. *Under the Third Assumption of MDFS, the metric functions $F_1$ and $F_2$ are identical and symmetrical at all line elements,*

$$F_1(\mathbf{x}, \mathbf{u}) \equiv F_2(\mathbf{x}, \mathbf{u}) \equiv F_1(\mathbf{x}, -\mathbf{u}) \equiv F_2(\mathbf{x}, -\mathbf{u}), \tag{22}$$

*and the differentials $\Omega_{\mathbf{x}, \mathbf{u}}(s)$ are all $o\{R^\mu(s)\}$.*

Due to Corollary 4.1, for order-balanced $\psi(\mathbf{x}, \mathbf{y})$ the relationship (21) holds trivially. In general, however, the single metric function $F(\mathbf{x}, \mathbf{u})$ in (7) does not have to be cross-balanced at any line element $(\mathbf{x}, \mathbf{u})$.

THEOREM 5.4. *For order-balanced discrimination probability functions, the metric function $F_1$ and $F_2$ are cross-balanced at $(\mathbf{x}, \mathbf{u})$ if and only if at this line element they are symmetrical, (14).*

The truth of this statement is obvious.

## 6. CROSS-BALANCE AND SELF-SIMILARITY

The metric functions $F_1$ and $F_2$ are said to be *cross-balanced* if they are cross-balanced at every line element $(\mathbf{x}, \mathbf{u})$,

$$F_1(\mathbf{x}, \mathbf{u}) \equiv F_2(\mathbf{x}, -\mathbf{u}). \tag{23}$$

$F_1$ and $F_2$ are said to be *cross-unbalanced* if this identity does not hold, that is, if the two metric functions are cross-unbalanced at least at one line element $(\mathbf{x}, \mathbf{u})$. It should be apparent that in the case of cross-balanced $F_1$ and $F_2$ all Fechnerian computations may be confined to just one of them, the Fechnerian computations involving the other one being merely their mirror-reflection. In particular, for cross-balanced $F_1$ and $F_2$,

$$G_1(\mathbf{a}, \mathbf{b}) \equiv G_2(\mathbf{b}, \mathbf{a}).$$

Theorem 5.3 tells us that the Third Assumption of MDFS implies this case. In fact it implies

$$G_1(\mathbf{a}, \mathbf{b}) \equiv G_1(\mathbf{b}, \mathbf{a}) \equiv G_2(\mathbf{a}, \mathbf{b}) \equiv G_2(\mathbf{b}, \mathbf{a}).$$

I relate now the notion of cross-(un)balance to that of (non)constant self-similarity.

### 6.1. *Case I: Constant Self-Similarity*

If the discrimination probability function $\psi(\mathbf{x}, \mathbf{y})$ possesses the constant self-similarity property, then

$$\Omega_{\mathbf{x},\mathbf{u}}(s) = \omega(\mathbf{x}+\mathbf{u}s) - \omega(\mathbf{x}) \equiv 0,$$

whence we conclude, due to Theorem 5.1, that $F_1$ and $F_2$ are necessarily cross-balanced. The Third Assumption of MDFS may or may not hold in this case, $\mu$ can be any positive number, and $R(s)$ any unit-regularly varying function. As a result, this case allows for all possible shapes considered in Dzhafarov (2002) and Dzhafarov and Colonius (2001) for the discrimination probability functions taken in the vicinity of their minima.

This is by far the simplest theoretical possibility, but we know from the empirical data considered in Section 3 that it cannot be posited as a general rule.

### 6.2. *Case IIa: Nonconstant Self-Similarity with Cross-Unbalanced Metric Functions*

Under nonconstant self-similarity, if $F_1$ and $F_2$ are cross-unbalanced (which, as we know from Theorem 5.3, rules out the Third Assumption of MDFS), then there is a line element $(\mathbf{x}, \mathbf{u})$ at which one of the metric functions exceeds the other, say,

$$F_1(\mathbf{x}, \mathbf{u}) > F_2(\mathbf{x}, -\mathbf{u}).$$

By continuity of the two functions this inequality should be preserved in some neighborhood of $(\mathbf{x}, \mathbf{u})$. In particular, there should exist a sufficiently small interval $0 \leqslant t < T$ on which

$$F_1(\mathbf{x}+\mathbf{u}t, \mathbf{u}) > F_2(\mathbf{x}+\mathbf{u}t, -\mathbf{u}).$$

Then, by Theorem 5.1, at this $(\mathbf{x}, \mathbf{u})$ and for $0 \leqslant t < t+s < T$,

$$\begin{aligned}\omega[\mathbf{x}+\mathbf{u}(t+s)] - \omega(\mathbf{x}+\mathbf{u}t) &= \Omega_{\mathbf{x},\mathbf{u}}(t+s) - \Omega_{\mathbf{x},\mathbf{u}}(t) \\ &\sim [F_1(\mathbf{x}+\mathbf{u}t, \mathbf{u})^{\mu} - F_2(\mathbf{x}+\mathbf{u}t, -\mathbf{u})^{\mu}]\, R^{\mu}(s). \end{aligned} \tag{24}$$

As $\Omega_{\mathbf{x},\mathbf{u}}(t+s) - \Omega_{\mathbf{x},\mathbf{u}}(t)$ consequently has to be positive for sufficiently small $s > 0$ at every $t$, the function $\Omega_{\mathbf{x},\mathbf{u}}(t)$ is strictly increasing on $0 \leqslant t < T$.

We are in the position now to derive the first truly remarkable result of the present development: that $\mu$ in (24) cannot be anything but 1, and that $R(s)$ can always be replaced with $s$.

In view of (8) and (9), it follows from (24) and the theory of regular variation (Bingham *et al.*, 1987, pp. 44–45) that

$$\frac{\Omega_{\mathbf{x},\mathbf{u}}(t+s) - \Omega_{\mathbf{x},\mathbf{u}}(t)}{s} \sim k \cdot s^{\mu-1} \ell^{\mu}(s), \qquad k > 0.$$

The theory of regular variation also tells us (e.g., Bingham *et al.*, 1987, pp. 16, 22) that

$$s^{\mu-1}\ell^{\mu}(s) \to \begin{cases} 0 & \text{if} \quad \mu > 1 \\ \infty & \text{if} \quad \mu < 1, \end{cases} \tag{25}$$

whence

$$\lim_{s\to 0+} \frac{\Omega_{\mathbf{x},\mathbf{u}}(t+s) - \Omega_{\mathbf{x},\mathbf{u}}(t)}{s} \equiv \begin{cases} 0 & \text{if} \quad \mu > 1 \\ \infty & \text{if} \quad \mu < 1. \end{cases}$$

Both these limit values, however, are impossible: zero would contradict the fact that $\Omega_{\mathbf{x},\mathbf{u}}(t)$ is strictly increasing, while infinity would contradict the famous theorem by Lebesgue that an increasing function should have a finite derivative almost everywhere (see, e.g., Hewitt & Stromberg, 1965, pp. 264–266).

Putting, as a result, $\mu = 1$, we have

$$\frac{\Omega_{\mathbf{x},\mathbf{u}}(t+s) - \Omega_{\mathbf{x},\mathbf{u}}(t)}{s} \sim k \cdot \ell(s), \qquad k > 0,$$

and by the same reasoning as before we exclude the possibilities $\ell(s) \to 0+$ or $\ell(s) \to \infty$. The possibility that $\ell(s)$ does not tend to any limit (as $s \to 0+$) would also contradict the Lebesgue theorem just mentioned. It must be, consequently, that $\ell(s)$ tends to a positive finite quantity, because of which

$$R(s) = s\ell(s) \sim cs, \qquad c > 0.$$

Since $R(s)$ in Theorem 4.2 is determined only asymptotically uniquely, one can with no loss of generality put

$$R(s) = cs, \qquad c > 0.$$

The asymptotic representations of psychometric differentials, (10), then become

$$\Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s) = \psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x}) \sim cF_1(\mathbf{x}, \mathbf{u})\, s$$
$$\Psi^{(2)}_{\mathbf{x},\mathbf{u}}(s) = \psi(\mathbf{x}+\mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x}) \sim cF_2(\mathbf{x}, \mathbf{u})\, s.$$

As a final step, due to the uniqueness part of the fundamental theorem of MDFS (Section 4), one can always multiply $F_1$ and $F_2$ by an arbitrary positive constant, and we can choose this constant to make $c$ in the previous equations equal to 1. Thus we arrive at

THEOREM 6.1 (Linear version of MDFS). *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-unbalanced, then*

$$\begin{aligned} \Psi^{(1)}_{\mathbf{x},\mathbf{u}}(s) &= \psi(\mathbf{x}, \mathbf{x}+\mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x}) \sim F_1(\mathbf{x}, \mathbf{u})\, s \\ \Psi^{(2)}_{\mathbf{x},\mathbf{u}}(s) &= \psi(\mathbf{x}+\mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x}) \sim F_2(\mathbf{x}, \mathbf{u})\, s, \end{aligned} \tag{26}$$

*and*

$$\Omega_{\mathbf{x},\mathbf{u}}(s) = \omega(\mathbf{x}+\mathbf{u}s) - \omega(\mathbf{x}) \sim [F_1(\mathbf{x}, \mathbf{u}) - F_2(\mathbf{x}, -\mathbf{u})]\, s. \tag{27}$$

The last equation can also be written as

$$\Omega_{\mathbf{x},\mathbf{u}}(s) = \omega(\mathbf{x}+\mathbf{u}s) - \omega(\mathbf{x}) \sim [F_2(\mathbf{x}, \mathbf{u}) - F_1(\mathbf{x}, -\mathbf{u})]\, s, \tag{28}$$

because Theorem 5.2 now specializes as

THEOREM 6.2. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-unbalanced, then*

$$F_1(\mathbf{x}, \mathbf{u}) + F_1(\mathbf{x}, -\mathbf{u}) \equiv F_2(\mathbf{x}, \mathbf{u}) + F_2(\mathbf{x}, -\mathbf{u}). \tag{29}$$

Observe that due to the equality of (27) and (28),

$$\Omega_{\mathbf{x},-\mathbf{u}}(s) \sim -\Omega_{\mathbf{x},\mathbf{u}}(s),$$

and

$$\begin{aligned} \left.\frac{d\Omega_{\mathbf{x},\mathbf{u}}(s)}{ds}\right|_{s=0+} &= \lim_{s\to 0+} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s)}{s} = \lim_{s\to 0+} \frac{\Omega_{\mathbf{x},-\mathbf{u}}(s)}{-s} \\ &= \lim_{s\to 0-} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s)}{s} = \left.\frac{d\Omega_{\mathbf{x},\mathbf{u}}(s)}{ds}\right|_{s=0-} = \left.\frac{d\Omega_{\mathbf{x},\mathbf{u}}(s)}{ds}\right|_{s=0}. \end{aligned}$$

This deserves to be emphasized.

THEOREM 6.3. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-unbalanced, then the minimum level function $\omega(\mathbf{x})$ is continuously differentiable at any point in any direction, and*

$$\left.\frac{d\omega(\mathbf{x}+\mathbf{u}s)}{ds}\right|_{s=0} = \left.\frac{d\Omega_{\mathbf{x},\mathbf{u}}(s)}{ds}\right|_{s=0} = \begin{cases} F_1(\mathbf{x}, \mathbf{u}) - F_2(\mathbf{x}, -\mathbf{u}) \\ F_2(\mathbf{x}, \mathbf{u}) - F_1(\mathbf{x}, -\mathbf{u}). \end{cases} \tag{30}$$

The continuity of the directional derivatives in the statement of the theorem follows, of course, from the continuity of the metric functions.

Equation (30) acquires an especially simple form if one assumes that $\psi(\mathbf{x}, \mathbf{y})$ is order-balanced, (5), in which case, due to Corollary 4.1, we have

$$\left.\frac{d\omega(\mathbf{x}+\mathbf{u}s)}{ds}\right|_{s=0} = F(\mathbf{x}, \mathbf{u}) - F(\mathbf{x}, -\mathbf{u}). \tag{31}$$

To better appreciate the implications of the linear version of MDFS, consider the case of unidimensional stimuli (see also Dzhafarov, 2001). Equation (26) in this case acquires the form

$$\begin{aligned} \psi(x, x\pm s) - \psi(x, x) &\sim F_1(x, \pm 1)\, s = F_1^{\pm}(x)\, s \\ \psi(x\pm s, x) - \psi(x, x) &\sim F_2(x, \pm 1)\, s = F_2^{\pm}(x)\, s \end{aligned}$$

(as $s \to 0+$), with

$$F_1^+(x)+F_1^-(x)=F_2^+(x)+F_2^-(x)$$

and

$$F_1^+(x)-F_2^-(x)=F_2^+(x)-F_1^-(x)=\frac{d\omega(x)}{dx}. \tag{32}$$

Figure 8 illustrates the predicted appearance of the discrimination probability functions $\psi(x, y)$ in a very small vicinity of $x=y=a$ and its relation to the minimum level functions $\omega(x)=\psi(x, x)$ in the same vicinity. In the order-balanced case the slope of the minimum level function is the same as the difference of the slopes of the right-hand and left-hand branches of $\psi(x, y)$,

$$F^+(x)-F^-(x)=\frac{d\omega(x)}{dx}, \tag{33}$$

which follows from (32) on dropping the indices at the metric functions in accordance with Corollary 4.1.

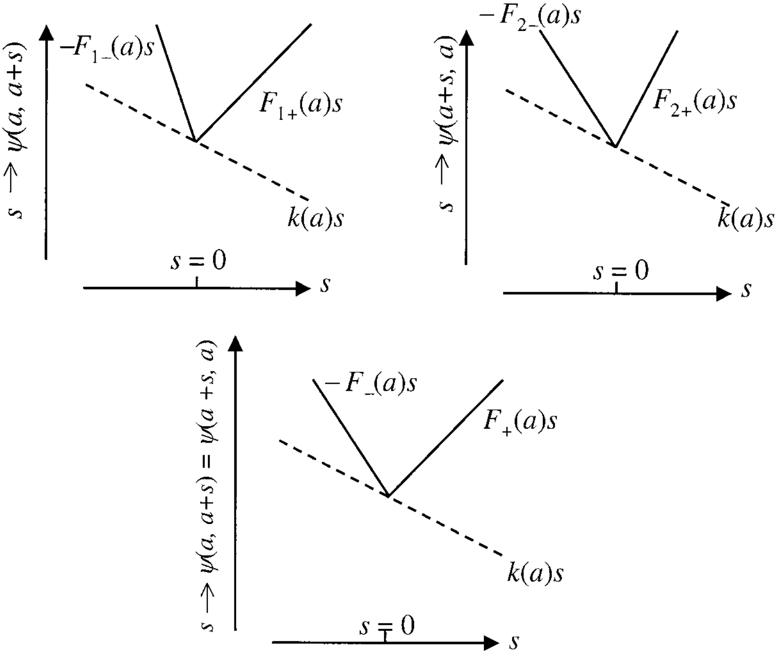


FIG. 8. Case IIa: the appearance of $\psi(x, y)$ (solid lines) and $\omega(x)$ (interrupted lines) in a very small vicinity of $x=y=a$. Left panel shows $\psi(x, y)$ at $x=a$, right panel at $y=a$; the bottom panel shows the order-balanced case. $k(a)$ denotes $\frac{d\omega(a)}{da}$ and equals $F_1^+(a)-F_2^-(a)=F_2^+(a)-F_1^-(a)$. In the order-balanced case, $k(a)=F^+(x)-F^-(x)$.

### *6.3. Case IIa (continued): Consequences for Psychometric Length and Fechnerian Distance*

Returning to the general statement of Theorem 6.1, consider an allowable path $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ as defined in Section 4. With $\mathbf{a}, \mathbf{b} \in \mathfrak{I}_1$, the psychometric length $L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b})$ of this path is defined by (11):

$$L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) = \int_0^1 F_1[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt.$$

With $\mathbf{a}, \mathbf{b} \in \mathfrak{I}_2$, using (12), the psychometric length $L_2$ of the opposite path $\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}$ is

$$L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = \int_0^1 F_2[\mathbf{x}(t), -\dot{\mathbf{x}}(t)]\, dt.$$

Then

$$L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) - L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = \int_0^1 \{F_1[\mathbf{x}(t), \dot{\mathbf{x}}(t)] - F_2[\mathbf{x}(t), -\dot{\mathbf{x}}(t)]\}\, dt.$$

But it follows from (30) that

$$\begin{aligned} F_1[\mathbf{x}(t), \dot{\mathbf{x}}(t)] - F_2[\mathbf{x}(t), -\dot{\mathbf{x}}(t)] &= \left.\frac{d\omega[\mathbf{x}(t) + \dot{\mathbf{x}}(t)\, s]}{ds}\right|_{s=0} \\ &= \lim_{s \to 0} \frac{\omega[\mathbf{x}(t) + \dot{\mathbf{x}}(t)\, s] - \omega[\mathbf{x}(t)]}{s} \\ &= \lim_{s \to 0} \frac{\omega[\mathbf{x}(t+s)] - \omega[\mathbf{x}(t)]}{s} = \frac{d\omega[\mathbf{x}(t)]}{dt} \end{aligned}$$

whence

$$L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) - L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = \int_0^1 \frac{d\omega[\mathbf{x}(t)]}{dt}\, dt = \int_0^1 d\omega[\mathbf{x}(t)] = \omega(\mathbf{b}) - \omega(\mathbf{a}).$$

This establishes yet another remarkable fact: the difference between the psychometric lengths in question is path-invariant, it only depends on the endpoints $\mathbf{a}$ and $\mathbf{b}$. By analogous reasoning one shows that

$$L_2(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) - L_1(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = \omega(\mathbf{b}) - \omega(\mathbf{a})$$

and proves thereby

THEOREM 6.4. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-unbalanced, then*

$$L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) - L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = L_2(\mathbf{a} \to \mathbf{y} \to \mathbf{b}) - L_1(\mathbf{a} \leftarrow \mathbf{y} \leftarrow \mathbf{b}) = \omega(\mathbf{b}) - \omega(\mathbf{a}), \tag{34}$$

*irrespective of the paths $\mathbf{x}(t)$ and $\mathbf{y}(t)$ connecting the stimuli $\mathbf{a}$ and $\mathbf{b}$.*

It follows that

$$L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) + L_1(\mathbf{a} \leftarrow \mathbf{y} \leftarrow \mathbf{b}) = L_2(\mathbf{a} \to \mathbf{y} \to \mathbf{b}) + L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}),$$

and on observing that $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ and $\mathbf{a} \leftarrow \mathbf{y} \leftarrow \mathbf{b}$ can always be redefined as an allowable closed loop $\mathbf{a} \to \mathbf{x} \to \mathbf{a}$, and vice versa, any such closed loop can be redefined as a pair of allowable paths $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$, $\mathbf{a} \leftarrow \mathbf{y} \leftarrow \mathbf{b}$, we have

THEOREM 6.5. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-unbalanced, then for any closed loop* $\mathbf{a} \to \mathbf{x} \to \mathbf{a}$,

$$L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{a}) = L_2(\mathbf{a} \to \mathbf{x} \to \mathbf{a}). \qquad (35)$$

A relationship analogous to that stated in Theorem 6.4 also exists between the Fechnerian metrics $G_1$ and $G_2$. In reference to (13), the equation

$$\inf L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) = G_1(\mathbf{a}, \mathbf{b})$$

implies, of course,

$$\inf\{L_1(\mathbf{a} \to \mathbf{x} \to \mathbf{b}) - [\omega(\mathbf{b}) - \omega(\mathbf{a})]\} = G_1(\mathbf{a}, \mathbf{b}) - [\omega(\mathbf{b}) - \omega(\mathbf{a})],$$

which, due to (34), leads to

$$G_2(\mathbf{b}, \mathbf{a}) = \inf L_2(\mathbf{a} \leftarrow \mathbf{x} \leftarrow \mathbf{b}) = G_1(\mathbf{a}, \mathbf{b}) - [\omega(\mathbf{b}) - \omega(\mathbf{a})].$$

Using the same reasoning with $G_2(\mathbf{a}, \mathbf{b})$ and $G_1(\mathbf{b}, \mathbf{a})$ we arrive at (use Fig. 9 as an illustration)

THEOREM 6.6. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-unbalanced, then*

$$G_1(\mathbf{a}, \mathbf{b}) - G_2(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) - G_1(\mathbf{b}, \mathbf{a}) = \omega(\mathbf{b}) - \omega(\mathbf{a}) \qquad (36)$$

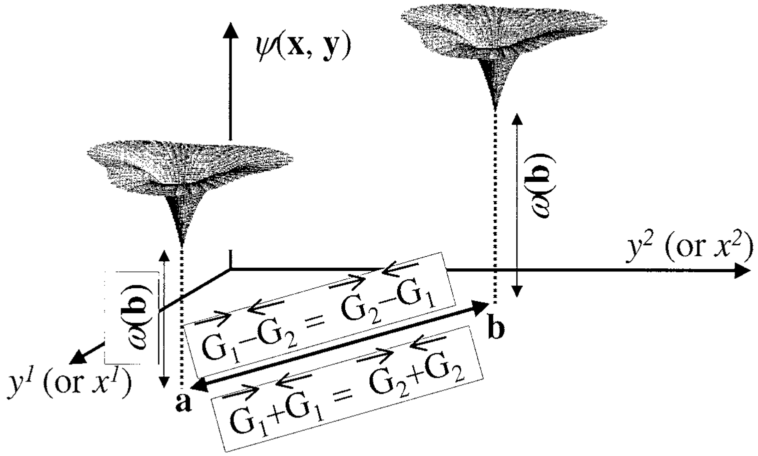


**FIG. 9.** The difference between the $G_1$ distance from $\mathbf{a}$ to $\mathbf{b}$ and the $G_2$ distance from $\mathbf{b}$ to $\mathbf{a}$ equals $\omega(\mathbf{b}) - \omega(\mathbf{a})$. Exchanging $G_1$ with $G_2$ leaves this statement true. The total $G_1$ distance from $\mathbf{a}$ to $\mathbf{b}$ and back is the same as the total $G_2$ distance from $\mathbf{a}$ to $\mathbf{b}$ and back.

*and*

$$G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a}). \tag{37}$$

If the Fechnerian metric is derived from order-balanced $\psi(\mathbf{x}, \mathbf{y})$, (5), then $G_1 \equiv G_2 \equiv G$, and the two formulas in the theorem reduce to

$$G(\mathbf{a}, \mathbf{b}) - G(\mathbf{b}, \mathbf{a}) = \omega(\mathbf{b}) - \omega(\mathbf{a}). \tag{38}$$

### 6.4. *Case IIb: Nonconstant Self-Similarity with Cross-Balanced Metric Functions*

As we know, constant self-similarity implies that the metric functions are cross-balanced. The reverse, however, is not true: it is possible that the minimum level function $\omega(\mathbf{x})$ is nonconstant while

$$F_1(\mathbf{x}, \mathbf{u}) \equiv F_2(\mathbf{x}, -\mathbf{u}).$$

If the Third Assumption of MDFS holds true, the nonconstant $\omega(\mathbf{x})$ may even coexist with metric functions $F_1$ and $F_2$ that are both identical and symmetrical and hence also cross-balanced (see Theorem 5.3). From Theorem 5.1 we know, however, that with cross-balanced metric functions all variations in the value of $\omega(\mathbf{x})$ should have a higher order of infinitesimality than $R^\mu(s)$,

$$\Omega_{\mathbf{x}, \mathbf{u}}(s) = \omega(\mathbf{x} + \mathbf{u}s) - \omega(\mathbf{x}) = o\{R^\mu(s)\}, \tag{39}$$

for all $(\mathbf{x}, \mathbf{u})$.

The restrictions imposed on $\mu$ and $R(s)$ by (39) are not as rigid as in the previously considered Case IIa. Nevertheless one can show by essentially the same argument as above that $\mu$ in (39) cannot exceed 1. Indeed, if $\mu > 1$, then it follows from (25) that

$$R(s) = s^\mu \ell^\mu(s) = o\{s\},$$

because of which and (39),

$$\Omega_{\mathbf{x}, \mathbf{u}}(s) = o\{s\}.$$

Then

$$\left. \frac{d\Omega_{\mathbf{x}, \mathbf{u}}(s)}{ds} \right|_{s = 0+} \equiv 0,$$

which contradicts the nonconstancy of $\omega(\mathbf{x})$. We have, as a result,

THEOREM 6.7. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-balanced, then the psychometric order $\mu$ is less than or equal to 1.*

No restrictions seem to be imposed on the possible structure of $R(s)$ if $\mu < 1$ (see Fig. 10, where $R(s)$ is chosen to be $s$). One can show, however, that $R(s)$ is subject to certain constraints if $\mu = 1$, that is, if for any line element $(\mathbf{x}, \mathbf{u})$,

$$\Omega_{\mathbf{x}, \mathbf{u}}(s) = o\{s\ell(s)\}. \tag{40}$$

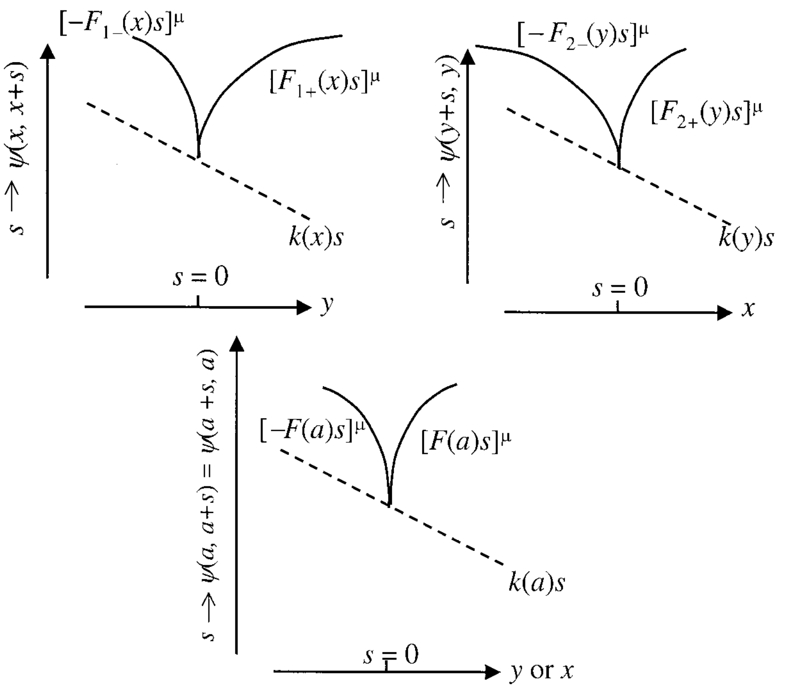


**FIG. 10.** Case IIb, with $\mu < 1$, $R(s) = s$. The format is the same as in Fig. 7. $k(a)$ here is unrelated to $F_1^+(a), F_2^-(a), F_2^+(a), F_1^-(a)$. In the order-balanced case, $F^+(x) = F^-(x)$.

We know (see, e.g., Bruckner, 1978, Chap. 4; or Hewitt & Stromberg, 1965, Chap. 5) that the Dini derivatives

$$\limsup_{s \to 0+} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s)}{s}, \qquad \liminf_{s \to 0+} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s)}{s}$$

exist for every $(\mathbf{x}, \mathbf{u})$ and that at least for some $(\mathbf{x}, \mathbf{u})$ they do not vanish simultaneously (the latter happens if and only if the function is constant). Choosing such an $(\mathbf{x}, \mathbf{u})$ and denoting one of the nonzero Dini derivatives by $D$ (a finite quantity, $\infty$, or $-\infty$), one can find a sequence $\{s_i\}|_{i=1}^{\infty} \to 0$ for which

$$\lim_{i \to \infty} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s_i)}{s_i} = D \neq 0.$$

At the same time, (40) implies that

$$\lim_{i \to \infty} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s_i)/s_i}{\ell(s_i)} = \lim_{i \to \infty} \frac{\Omega_{\mathbf{x},\mathbf{u}}(s_i)}{s_i \ell(s_i)} = 0,$$

which is only possible if $\ell(s_i) \to \infty$. In turn, this means that

$$\limsup_{s \to 0+} \ell(s) = \infty. \tag{41}$$

I summarize this result as

THEOREM 6.8. *Under nonconstant self-similarity, if the metric functions $F_1$ and $F_2$ are cross-balanced and $\mu = 1$, then $R(s) = s\ell(s)$ with* $\limsup \ell(s) = \infty$ *(as $s \to 0+$).*

As $\ell$ cannot attain negative values, its limit superior in general can only be 0, a finite positive number, or infinity, the respective examples being $\ell(s) = 1/\log(1/s)$, $\ell(s) = 1$, and $\ell(s) = \log(1/s)$. The result just obtained tells us that

$$\mu = 1, \qquad R(s) = s \log \frac{1}{s}$$

is a possible combination in the case being considered, whereas

$$\mu = 1, \qquad R(s) = s$$

is not. Note that the latter is the only possible solution for Case IIa.

## 7. CONCLUSION

To summarize our main results, the recognition of the simple fact that stimuli presented pairwise for same–different judgments belong to distinct observation areas (essentially, different stimulus spaces) forces one to modify the underlying assumptions of MDFS.

The most important of these modifications is the hypothesis that the discrimination probability functions $\psi(\mathbf{x}, \mathbf{y})$ possess regular minima, which means that (following, if necessary, an appropriate transformations of the physical measurements for the stimuli belonging to one of the observation areas)

$$\psi(\mathbf{x}, \mathbf{x}) < \begin{cases} \psi(\mathbf{x}, \mathbf{y}) \\ \psi(\mathbf{y}, \mathbf{x}) \end{cases},$$

for any $\mathbf{y} \neq \mathbf{x}$. Intuitively, it seems highly implausible that this fundamental constraint may be violated, and the available empirical data corroborate it.

The remaining emendations in the underlying assumptions of MDFS are relatively straightforward, their thrust being in that one has to deal with two Fechnerian metrics, $G_1(\mathbf{a}, \mathbf{b})$ and $G_2(\mathbf{a}, \mathbf{b})$ (one for each of the observation areas, $\mathfrak{I}_1$ and $\mathfrak{I}_2$), induced by two Fechner–Finsler metric functions,

$$F_1(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\sqrt[\mu]{\psi(\mathbf{x}, \mathbf{x} + \mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x})}}{R(s)}$$

$$F_2(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\sqrt[\mu]{\psi(\mathbf{x} + \mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x})}}{R(s)},$$

where $\mu > 0$ is the psychometric order of the stimulus space, and $R(s)$ is a unit-regularly varying function. $F_1$, $F_2$, and $\mu$ are interrelated by

$$F_1(\mathbf{x}, \mathbf{u})^\mu + F_1(\mathbf{x}, -\mathbf{u})^\mu \equiv F_2(\mathbf{x}, \mathbf{u})^\mu + F_2(\mathbf{x}, -\mathbf{u})^\mu.$$

As it turns out, the theory of MDFS leads to very different results depending on two circumstances: (a) whether the minimum level function $\psi(\mathbf{x}, \mathbf{x})$ is constant; and

(b) whether the metric functions $F_1$ and $F_2$ are cross-balanced. The cross-balance means

$$F_1(\mathbf{x}, \mathbf{u}) \equiv F_2(\mathbf{x}, -\mathbf{u}),$$

and it is implied by but does not imply the constancy of $\psi(\mathbf{x}, \mathbf{x})$ (constant self-similarity property). This leaves us with three cases to consider.

*Case I: Constant self-similarity holds* (hence the metric function are cross-balanced). In this case $\mu$ can be any positive number and $R(s)$ any unit-regularly varying function. Empirical data definitely reject the possibility that this case may hold universally.

*Case IIa: Constant self-similarity does not hold and the metric functions are cross-unbalanced.* This case leads to the linear version of MDFS ($\mu = 1$, $R(s) \equiv s$) as the only possibility:

$$F_1(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\psi(\mathbf{x}, \mathbf{x} + \mathbf{u}s) - \psi(\mathbf{x}, \mathbf{x})}{s}$$

$$F_2(\mathbf{x}, \mathbf{u}) = \lim_{s \to 0+} \frac{\psi(\mathbf{x} + \mathbf{u}s, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x})}{s}.$$

Among the most remarkable consequences of these equations is that the slope of the minimum level function in any direction $\mathbf{u}$,

$$\lim_{s \to 0+} \frac{\psi(\mathbf{x} + \mathbf{u}\mathbf{s}, \mathbf{x} + \mathbf{u}\mathbf{s}) - \psi(\mathbf{x}, \mathbf{x})}{\mathbf{s}},$$

equals

$$F_1(\mathbf{x}, \mathbf{u}) - F_2(\mathbf{x}, -\mathbf{u}) = F_2(\mathbf{x}, \mathbf{u}) - F_1(\mathbf{x}, -\mathbf{u})$$

and that

$$G_1(\mathbf{a}, \mathbf{b}) - G_2(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) - G_1(\mathbf{b}, \mathbf{a}) = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}).$$

*Case IIb: Constant self-similarity does not hold but the metric functions are cross-balanced.* In this case either $\mu < 1$, with $R(s)$ being an arbitrary unit-regularly varying function, or $\mu = 1$, with some restrictions on the possible structure of $R(s)$. These restrictions, in particular, rule out the possibility $\mu = 1$, $R(s) \equiv s$ arrived at in the previous case.

It is very surprising that Cases I and IIb cannot be viewed as special or limit forms of Case IIa and that Case I cannot be viewed as a special or limit form of Case IIb. Indeed, suppose that neither the constant self-similarity nor the cross-balance of the metric functions are inherent properties of discrimination; they may happen to hold but do not hold generally. Then one can begin with Case IIa and hope to achieve Case IIb or Case I by, respectively, gradually diminishing the cross-unbalance of the metric functions or gradually flattening the minimum level function. This would not work, however. Within Case IIa the cross-unbalance, $F_1(\mathbf{x}, \mathbf{u}) - F_2(\mathbf{x}, -\mathbf{u})$, and the slope of the minimum level function can only be reduced to zero simultaneously, so one cannot get to Case IIb at all. What one eventually reaches when they both reach zero (at all **x**, **u**) is a special form of Case I, with $\mu = 1$, $R(s) \equiv s$, all other combinations of $\mu$ and $R(s)$ being lost. Analogously one shows that Case IIb cannot be gradually transformed into Case I without losing combinations of $\mu$ and $R(s)$ that otherwise are perfectly compatible with Case I. With some caution, one could say that Cases I and IIb are singularities with respect to Case IIa, and Case I is a singularity with respect to Case IIb.

While Case I is incompatible with the empirical data mentioned in Section 3, it remains to be seen whether one of the Cases IIa and IIb can be ruled out by empirical evidence in favor of the other.

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