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Multidimensional Fechnerian Scaling: Perceptual Separability[1]

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A new definition of the perceptual separability of stimulus dimensions is given in terms of discrimination probabilities. Omitting technical details, stimulus dimensions are considered separable if the following two conditions are met: (a) the probability of discriminating two sufficiently close stimuli is computable from the probabilities with which one discriminates the projections of these stimuli on the coordinate axes; (b) the psychometric differential for discriminating two sufficiently close stimuli that differ in one coordinate only does not depend on the value of their matched coordinates (the psychometric differential is the difference between the probability of discriminating a comparison stimulus from a reference stimulus and the probability with which the reference is discriminated from itself). Thus defined perceptual separability is analyzed within the framework of the regular variation version of multidimensional Fechnerian scaling. The result of this analysis is that the Fechnerian metric of a stimulus space with perceptually separable dimensions has the structure of a Minkowski power metric with respect to these dimensions. The exponent of this metric equals the psychometric order of the stimulus space, or 1, whichever is greater.

1. INTRODUCTION

In this paper I apply the general theory of multidimensional Fechnerian scaling (MDFS) to one of the classical problems of cognitive psychology, that of distinguishing the situations when two or more physical dimensions of a stimulus are processed separately from the ones in which they are processed integrally.

To understand the intuition underlying the problem, consider a two-dimensional continuous stimulus space, say, the space of rectangular visual stimuli continuously varying in the lengths of their sides, *a* and *b*, all other stimulus parameters being held fixed. Each stimulus here can be described by two coordinates, (x^1, x^2), taking

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on their values within a region of Re^2 (see Comment 1 in the Appendix). Although the number of stimulus dimensions, in this case 2, is a topological invariant (i.e., it is constant under all one-to-one transformations of the stimulus space continuous together with their inverses), the choice of the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ can be made in an infinity of ways. One can put $x^1 = a$, $x^2 = b$ (lengths of the two sides), or $x^1 = \frac{a}{b}$, $x^2 = ab$ (aspect ratio and area), or one can even choose dimensions for which one has no conventional geometric terms, say, $x^1 = \exp(a+b)$, $x^2 = \log(ab)$. If one imposes a certain subjective (computed from perceptual judgments) metric upon this stimulus space, the metric must be invariant with respect to the choice of stimulus dimensions. Put differently, a distance between two stimuli computed from perceptual judgments should not depend on one's choice of the physical parameters by which these stimuli are identified. In MDFS this invariance is achieved automatically, by the procedure of computing Fechnerian distances, provided that the admissible transformations of stimulus dimensions,

$$\hat{x}^1 = \hat{x}^1(x^1, x^2), \hat{x}^2 = \hat{x}^2(x^1, x^2),$$

are diffeomorphic (i.e., one-to-one and smooth together with their inverses).

One's specific choice of the dimensions describing rectangular visual stimuli, however, may interest one from another point of view. One might hypothesize that with some choice of $\langle x^1, x^2 \rangle$, say, $x^1 = \frac{a}{b}$ (aspect ratio), $x^2 = ab$ (area), the two dimensions are processed "separately," so that all perceptual distinctions between two rectangles can be, in some sense, "computed" from the perceptual distinctions between their aspect ratios (irrespective of area) and their areas (irrespective of aspect ratio), whereas, one might hypothesize, such a reduction to individual dimensions cannot be achieved with other choices, say, $x^1 = a$, $x^2 = b$, in which case the dimensions have to be viewed as processed "integrally."

Ashby and Townsend (1986) analyze several theoretical concepts (separability, orthogonality, independence, performance parity) proposed in the literature in an attempt to capture this intuitive distinction. They propose to interpret these concepts within the framework of the General Recognition Theory (Ashby & Perrin, 1988), as different aspects of the mapping of stimuli into hypothetical random variables taking on their values in some perceptual space. If one can define in this space two coordinate axes, $\langle p^1, p^2 \rangle$ (or two subspaces spanning two distinct sets of axes), such that the p^1-component and p^2-component of the random variable representing a stimulus (x^1, x^2) depend only on x^1 and x^2, respectively, then one can say that the dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are perceptually separable (see Comment 2 in the Appendix). With this definition, the stochastic relationship between the p^1- and p^2-components of the random variable representing (x^1, x^2) may depend on the (x^1, x^2) in an arbitrary fashion, provided the selective correspondence

$$x^1 \to p^1, x^2 \to p^2$$

is satisfied on the level of marginal distributions. Thomas (1996) adapts this approach to the situation in which pairs of stimuli are judged on the same-different scale, which is especially relevant to the Fechnerian analysis to be presented.

A different attempt to rigorously define perceptual separability (or separability of perceptual dimensions, which is a better term in this case) is made by Shepard (1987), within the framework of multidimensional scaling. Shepard assumes that stimuli are represented in a perceptual space by points separated by distances negative-exponentially related to some stimulus generalization measure that, for our purposes, can be thought of as the probability of confusing one stimulus with another. It is traditionally postulated in multidimensional scaling, or derived from equivalent premises (Beals, Krantz, & Tversky, 1968; Tversky & Krantz, 1970), that one can define in this perceptual space coordinate axes $p^1, ..., p^k$, with respect to which the interstimulus distances D in the space form a Minkowski power metric:

$$D^r[(p^1, ..., p^k), (q^1, ..., q^k)] = \sum_{i=1}^{k} |p^i - q^i|^r, \qquad r \geqslant 1.$$

Based on multidimensional scaling of several stimulus spaces, Shepard (1987) suggests that the exponent r of this power metric equals 1 ("city-block" metric) if the stimuli are represented by separable dimensions, and it equals 2 (Euclidean metric) if they are not. Although the relationship between subjective distances and stimulus confusion probabilities is central to Shepard's theory, he does not define the perceptual separability in terms of these confusion probabilities, relying instead on operational criteria external to his theory (such as those described in Garner, 1974).

In this work I present a new approach to the issue of perceptual separability of stimulus dimensions, based on the theory of MDFS (Dzhafarov, 2002, in press; Dzhafarov & Colonius, 1999, 2001). In MDFS, subjective (Fechnerian) distances among stimuli are computed from the probabilities with which stimuli are discriminated from their close neighbors in a continuous stimulus space. Accordingly, the concepts explicating the intuitive idea of perceptual separability are formulated in this work solely in terms of discrimination probabilities. Specifically, I propose to treat dimensions $\langle x^1 \rangle$ and $\langle x^2 \rangle$ as perceptually separable if the following two conditions are met:

(a) the probability with which a stimulus $\mathbf{x} = (x^1, x^2)$ is discriminated from nearby stimuli $\mathbf{y} = (y^1, y^2)$ can be computed from the probabilities with which $\mathbf{x}$ is discriminated from $\mathbf{y}_1 = (y^1, x^2)$ (differing from $\mathbf{x}$ along the first dimension only) and from $\mathbf{y}_2 = (x^1, y^2)$ (differing from $\mathbf{x}$ along the second dimension only);

(b) the difference between the probabilities with which $\mathbf{x} = (x^1, x^2)$ is discriminated from nearby $\mathbf{y}_1 = (y^1, x^2)$ and with which $\mathbf{x}$ is discriminated from itself does not depend on x^2; and analogously for $\mathbf{x} = (x^1, x^2)$ and nearby $\mathbf{y}_2 = (x^1, y^2)$.

If the probabilities with which each stimulus is discriminated from nearby stimuli are known, then MDFS allows one to uniquely compute the Fechnerian distances among all stimuli comprising the stimulus space. The main result obtained in this work (when specialized to two-dimensional spaces) is as follows: given that $\langle x^1 \rangle$ and $\langle x^2 \rangle$ are separable, the Fechnerian distance between any two (not necessarily close) stimuli $\mathbf{a} = (a^1, a^2)$ and $\mathbf{b} = (b^1, b^2)$ is related to the corresponding coordinatewise

Fechnerian distances, between $\mathbf{a}$ and $\mathbf{b}_1 = (b^1, a^2)$, and between $\mathbf{a}$ and $\mathbf{b}_2 = (a^1, b^2)$, is a Minkowski power metric,

$$G(\mathbf{a}, \mathbf{b})^r = G(\mathbf{a}, \mathbf{b}_1)^r + G(\mathbf{a}, \mathbf{b}_2)^r, \qquad r \geqslant 1,$$

where G denotes the Fechnerian metric. Put more concisely, the Fechnerian metric in a stimulus space with perceptually separable dimensions is a Minkowski power metric with respect to these dimensions.

This result may appear similar to Shepard's (1987) theory. The resemblance, however, is superficial. First, in MDFS the metric is imposed directly on the stimulus space, rather than on a hypothetical perceptual space (that may even have a different dimensionality). Second, it is the power function form per se of the Fechnerian metric that is indicative of perceptual separability, rather than a specific value of the exponent r. I show in this paper that the value of r is determined by the value of the fundamental characteristic of MDFS, the psychometric order of stimulus space, μ. Specifically, $r = \mu$ if $\mu \geqslant 1$, and $r = 1$ otherwise. Roughly, the psychometric order μ determines the degree of flatness/cuspidality of discrimination probability functions at their minima, and this characteristic has nothing to do with perceptual separability.

The theory presented below is formulated for the mutual perceptual separability of all n dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ of an n-dimensional stimulus space. If one is interested in the mutual perceptual separability of some specified $k < n$ dimensions in an n-dimensional stimulus space, the premises and the results of the theory should be understood as applying to this k-dimensional stimulus subspace at *some* fixed values of the remaining $n-k$ coordinates (or, as a special case, at *all* possible values of these coordinates). With obvious notational changes the theory also applies to the perceptual separability of several stimulus subspaces spanning several dimensions each,

$$\underbrace{\langle x^1 \rangle, \ldots, \langle x^{n_1} \rangle}_{\langle \mathbf{x}^1 \rangle}, \underbrace{\langle x^{n_1+1} \rangle, \ldots, \langle x^{n_2} \rangle}_{\langle \mathbf{x}^2 \rangle} \ldots, \underbrace{\langle x^{n_{k-1}+1} \rangle, \ldots, \langle x^{n_k} \rangle}_{\langle \mathbf{x}^k \rangle}.$$

With this generalization, however, one loses the direct interpretability of the results in terms of the horizontal and vertical cross-sections of psychometric functions (as described below), which is the main reason why the discussion in this paper is confined to individual stimulus dimensions.

Some familiarity with the general theory of MDFS is desirable (especially, Dzhafarov, 2002; Dzhafarov & Colonius, 2001), but an effort has been made to keep the presentation as self-contained as possible.

Throughout the paper I use the following notational conventions. Given an n-dimensional space of vectors (stimuli or directions of stimulus change), the *unit coordinate vectors* $\mathbf{1}_i$ $(i = 1, \ldots, n)$ are defined as

$$\begin{bmatrix} \mathbf{1}_1 = (1, 0, \ldots, 0) \\ \mathbf{1}_2 = (0, 1, \ldots, 0) \\ \cdots \\ \mathbf{1}_n = (0, 0, \ldots, 1) \end{bmatrix}.$$

For any vector $\mathbf{v} = (v^1, \ldots, v^n)$ in this space, the *coordinate projections* $\mathbf{v}_i$ $(i = 1, \ldots, n)$ of $\mathbf{v}$ are defined as

$$\begin{bmatrix} \mathbf{v}_1 = (v^1, 0, \ldots, 0) = v^1\mathbf{1}_1 \\ \mathbf{v}_2 = (0, v^2, \ldots, 0) = v^2\mathbf{1}_2 \\ \cdots \\ \mathbf{v}_n = (0, 0, \ldots, v^n) = v^n\mathbf{1}_n \end{bmatrix}.$$

Clearly,

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} v^i\mathbf{1}_i.$$

2. PERCEPTUAL SEPARABILITY AND DISCRIMINATION PROBABILITIES

Consider an n-dimensional stimulus space $\mathfrak{M}$, an open connected region of Re^n, and let $\langle x^1, \ldots, x^n \rangle$ be a coordinate system imposed on this space. The stimulus space is assumed to be endowed with psychometric (discrimination probability) functions

$$\psi_{\mathbf{x}}(\mathbf{y}) = \Pr[\text{stimulus } \mathbf{y} \text{ is distinguished from stimulus } \mathbf{x}],$$

where $\mathbf{x} = (x^1, \ldots, x^n) \in \mathfrak{M}$, $\mathbf{y} = (y^1, \ldots, , y^n) \in \mathfrak{M}$. The stimulus space together with the psychometric functions defined on it is referred to as the *discrimination system* $\langle \mathfrak{M}, \psi \rangle$.

Given a stimulus $\mathbf{x} = (x^1, \ldots, x^n)$ and a direction-of-change vector $\mathbf{u} = (u^1, \ldots, u^n) \neq \mathbf{0}$, the quantity

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) = \psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) - \psi_{\mathbf{x}}(\mathbf{x}), \qquad s \geqslant 0 \tag{1}$$

is referred to as the *psychometric differential* (at $\mathbf{x}, \mathbf{u}$), and it plays a central role in Fechnerian computations. The underlying assumptions of MDFS ensure (if necessary, following a certain procedure of "renaming" reference stimuli, described in Dzhafarov & Colonius, 1999, 2001) that the psychometric differentials are positive (for $s > 0$) and they continuously decrease to zero with $s \to 0+$ (see Comment 3 in the Appendix).

Definition 2.1. (Refer to Fig. 1.) The dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ are *weakly perceptually separable* in the discrimination system $\langle \mathfrak{M}, \psi \rangle$ if for any stimulus $\mathbf{x}$ one can find an open neighborhood $\mathfrak{N}_{\mathbf{x}} \subseteq \mathfrak{M}$ of $\mathbf{x}$, such that whenever $\mathbf{x} + \mathbf{u}s \in \mathfrak{N}_{\mathbf{x}}$,

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) = H_{\mathbf{x}}[\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_1 s), \ldots, \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_n s)], \tag{2}$$

where $H_{\mathbf{x}}(a_1, \ldots, a_n)$ is some function continuously differentiable in a vicinity of $a_1 = \cdots = a_n = 0$.

Recall that $\mathbf{u}_i$ $(i = 1, \ldots, n)$ are coordinate projections of $\mathbf{u}$, and observe that the neighborhood $\mathfrak{N}_{\mathbf{x}}$ can always be chosen so that all $\mathbf{x} + \mathbf{u}_i s$ are stimuli within $\mathfrak{N}_{\mathbf{x}}$.

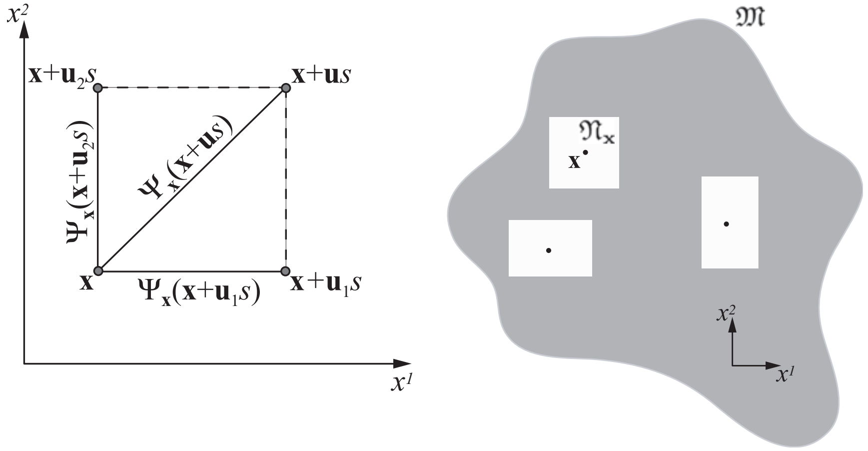

FIG. 1. A diagram for Definition 2.1 (weak separability, $n=2$).

Observe also that the functions $H_{\mathbf{x}}(a_1, ..., a_n)$ are allowed to be different for different $\mathbf{x}$. One consequence of this is that Definition 2.1 can be equivalently reformulated in terms of psychometric functions per se, rather than psychometric differentials. Since $\psi_{\mathbf{x}}(\mathbf{x})$ is a constant for a given $\mathfrak{N}_{\mathbf{x}}$, (2) holds if and only if

$$\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = h_{\mathbf{x}}[\psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s), ..., \psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_n s)], \tag{3}$$

where, obviously, $h_{\mathbf{x}}(a_1, ..., a_n) = H_{\mathbf{x}}[(a_1 - \psi_{\mathbf{x}}(\mathbf{x})), ..., (a_n - \psi_{\mathbf{x}}(\mathbf{x}))] + \psi_{\mathbf{x}}(\mathbf{x})$. The use of (2) in the formal definition, however, is more convenient, as this simplifies computations and makes Definition 2.1 more directly linkable to Definition 2.2 below.

We need the following preliminary result. (Symbol $\sim$ in the proof indicates that the two expressions it connects are asymptotically equal; i.e., their ratio tends to 1.)

LEMMA 2.1. *The function* $H_{\mathbf{x}}(a_1, ..., a_n)$ *in Definition* 2.1 *has the following properties:*

(i) $H_{\mathbf{x}}(\mathbf{0}) = 0$;

(ii) $H_{\mathbf{x}}[\mathbf{1}_i \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s)] = \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s), \quad i = 1, ..., n$;

(iii) $\frac{\partial}{\partial a_i} H_{\mathbf{x}}(a_1, ..., a_n) |_{a_1 = \cdots = a_n = 0} = 1, \quad i = 1, ..., n$.

Proof. Property (i) is obtained from (2) by putting $s = 0$ and observing that $\Psi_{\mathbf{x}}(\mathbf{x}) \equiv 0$; (ii) is obtained by putting in (2) $\mathbf{u} = \mathbf{u}_i$. From (i) and (ii) it follows that

$$\begin{aligned}\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s) &= H_x[\mathbf{1}_i \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s)] - H_x(\mathbf{0}) \\ &= H_x[0, ..., \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s), ..., 0] - H_x(0, ..., 0) \\ &\sim \frac{\partial}{\partial a_i} H_{\mathbf{x}}(a_1, ..., a_n)|_{a_1 = \cdots = a_n = 0} \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s), \qquad s \to 0+\end{aligned}$$

from which it follows that the partial derivative is 1. ∎

As a simple example of $H_{\mathbf{x}}$, consider the discrimination system in which

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = 1 - \prod_{i=1}^{n} [1 - \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s)]. \tag{4}$$

The function $H_{\mathbf{x}}$ here is the same for all values of $\mathbf{x}$:

$$H_{\mathbf{x}}(a_1, ..., a_n) = H(a_1, ..., a_n) = 1 - \prod_{i=1}^{n} (1 - a_i).$$

If, in addition, $\psi_{\mathbf{x}}(\mathbf{x}) \equiv 0$, one can substitute ψ for Ψ, and the equation can be interpreted as saying that the discriminations along individual dimensions are stochastically independent and that two stimuli are discriminated whenever they are discriminated along one of these dimensions (the simplest "probability summation" model).

A wealth of special cases for $H_{\mathbf{x}}$ can be obtained by choosing an arbitrary strictly monotone and differentiable function $T_{\mathbf{x}}(a)$, $0 \leqslant a \leqslant 1$, vanishing at $a = 0$, and by putting

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = T_{\mathbf{x}}^{-1}[T_{\mathbf{x}}(\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s)) + \cdots + T_{\mathbf{x}}(\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_n s))]. \tag{5}$$

In particular, this equation reduces to (4) if

$$T_{\mathbf{x}}(a) = T(a) = \log(1-a).$$

The following lemma is fundamental for the Fechnerian analysis of perceptual separability. (The term in the small means at the limit, asymptotically.)

LEMMA 2.2 (Additivity in the small). *If $\langle x^1 \rangle, ..., \langle x^n \rangle$ are weakly perceptually separable in $\langle \mathfrak{M}, \psi \rangle$, then*

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) \sim \sum_{i=1}^{n} \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s), \quad s \to 0+. \tag{6}$$

Proof. From (2) and Lemma 2.1 (i and ii), using the continuous differentiability of H_x,

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) = H_x[\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_1 s), ..., \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_n s)] - H_x(0, ..., 0)$$
$$\sim \sum_{i=1}^{n} \frac{\partial}{\partial a_i} H_{\mathbf{x}}(a_1, ..., a_n) \,|_{a_1 = \cdots = a_n = 0} \Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s), \qquad s \to 0+,$$

and the statement of the lemma follows from (iii) of Lemma 2.1. ∎

Since

$$\mathbf{u} = \sum_{i=1}^{n} \mathbf{u}_i,$$

one recognizes in (6) an asymptotic version of the conventional factorial additivity (of the main effects of changes along the individual dimensions upon Ψ).

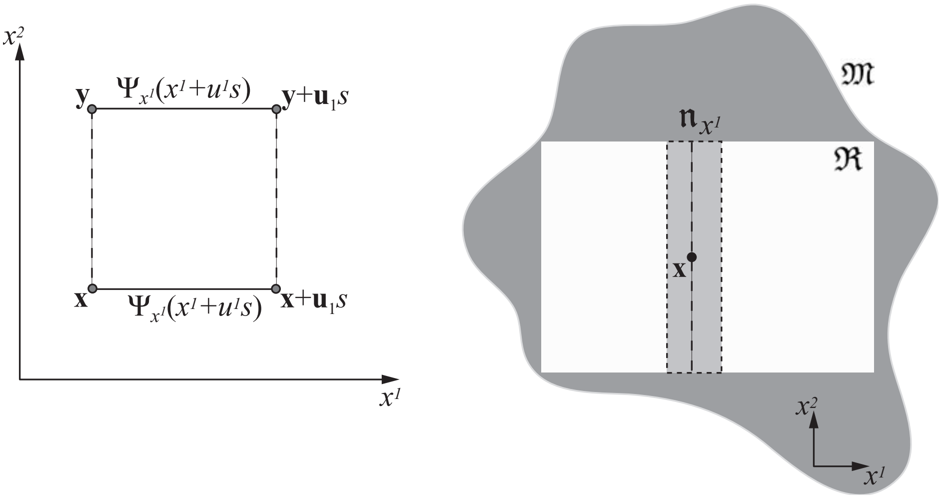

FIG. 2. A diagram for Definition 2.2 (detachability, shown for the horizontal axis only, $n = 2$).

Plainly, if $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ are weakly perceptually separable in the discrimination system $\langle \mathfrak{M}, \psi \rangle$, then for any open connected subregion $\mathfrak{R} \subseteq \mathfrak{M}$, $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ are weakly perceptually separable in the discrimination system $\langle \mathfrak{R}, \psi \rangle$. Our next definition requires that we confine the consideration to some open (hyper-) rectangular area

$$\mathfrak{R} = \prod_{i=1}^{n} (x^i_{\inf}, x^i_{\sup}) \subseteq \mathfrak{M}, \tag{7}$$

where some or all of the endpoints $x^i_{\inf}$, $x^i_{\sup}$ may stand for $-\infty$ and ∞, respectively.

Definition 2.2 (Refer to Fig. 2.). The dimension $\langle x^i \rangle$, $i \in \{1, \ldots, n\}$, is *detachable* from the discrimination system $\langle \mathfrak{R}, \psi \rangle$, if for any value of x^i one can find an open vicinity $\mathfrak{n}_{x^i} \subseteq (x^i_{\inf}, x^i_{\sup})$, such that whenever $x^i + u^i s \in \mathfrak{n}_{x^i}$, $\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_i s)$ does not depend on any components of $\mathbf{x}$ except for $\mathbf{x}^i$. In other words, whenever $\mathbf{x} \in \mathfrak{R}$ and $x^i + u^i s \in \mathfrak{n}_{x^i}$,

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_i s) = \Psi_{x^i}(x^i + u^i s). \tag{8}$$

If this equation holds for all dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$, we say that these dimensions are *mutually detachable in* $\langle \mathfrak{R}, \psi \rangle$. (See Comment 4 in the Appendix.)

The reason for confining the definition of detachability to a rectangular area $\mathfrak{R}$ is straightforward. Discrimination probabilities are not defined outside the stimulus space $\mathfrak{M}$. The requirement that $\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_i s)$ be independent of all components of $\mathbf{x}$ other than x^i implies that if $\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_i s)$ is defined for

$$\mathbf{x} + \mathbf{u}_i s = (x^1, \ldots, x^{i-1}, x^i + u^i s, x^{i+1}, \ldots, x^n), \qquad 0 \leqslant s < s_{\sup},$$

then it should also be defined for any

$$\mathbf{x}_a + \mathbf{u}_i s = (a^1, \ldots, a^{i-1}, x^i + u^i s, a^{i+1}, \ldots, a^n), \qquad 0 \leqslant s < s_{\sup}.$$

This is equivalent to saying that if the straight line segment $\mathbf{x}+\mathbf{u}_i s$ $(0 \leqslant s < s_{\text{sup}})$ lies within $\mathfrak{M}$, then any segment $\mathbf{x}_a + \mathbf{u}_i s$ $(0 \leqslant s < s_{\text{sup}})$ also lies within $\mathfrak{M}$. This is only possible, however, if $\mathfrak{M}$ is a rectangular region (finite or infinite), and if it is not, then the consideration should be confined to some rectangular subregion of $\mathfrak{M}$.

With this restriction to a rectangular area, the definition of perceptual separability proposed in this paper is simply the conjunction of Definitions 2.1 and 2.2.

Definition 2.3. The dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ are *perceptually separable* in the discrimination system $\langle \mathfrak{R}, \psi \rangle$ if they are *weakly perceptually separable* and *mutually detachable* in $\langle \mathfrak{R}, \psi \rangle$.

The following two immediate consequences of this definition are significant for the subsequent development.

Lemma 2.3. *If $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ are perceptually separable in $\langle \mathfrak{R}, \psi \rangle$, then any diffeomorphic transformations ("recalibrations") of these axes,*

$$\langle \hat{x}^1 = \hat{x}^1(x^1) \rangle, \ldots, \langle \hat{x}^n = \hat{x}^n(x^n) \rangle,$$

are perceptually separable in the corresponding $\langle \hat{\mathfrak{R}}, \hat{\psi} \rangle$

Proof. On observing that with $\hat{x}^i = \hat{x}^i(x^i)$ $(i = 1, \ldots, n)$ the coordinate projections of direction vectors transform as

$$\hat{\mathbf{u}}_i = \frac{\partial \hat{x}^i}{\partial x^i} \mathbf{u}_i, \qquad i = 1, \ldots, n$$

(which follows from the contravariant transformation formula for direction vectors, see Dzhafarov & Colonius, 2001), and that the rectangular area $\mathfrak{R}$ remains rectangular in new coordinates, it is obvious that Definitions 2.1 and 2.2 are satisfied for $\hat{\Psi}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}} + \hat{\mathbf{u}}_i s) = \Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}_i s)$. ∎

Lemma 2.4 (Detachable additivity in the small). *If $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ are perceptually separable in $\langle \mathfrak{R}, \psi \rangle$, then, within $\mathfrak{R}$,*

$$\Psi_{\mathbf{x}}(\mathbf{x} + \mathbf{u}s) \sim \sum_{i=1}^{n} \Psi_{x^i}(x^i + u^i s), \qquad s \to 0+. \tag{9}$$

Proof. Follows from Definition 2.2 and Lemma 2.2. ∎

In the next section I show that these two properties imply that the Fechnerian metric in a stimulus space with perceptually separable dimensions is a Minkowski power metric with respect to these dimensions.

3. FECHNERIAN ANALYSIS OF PERCEPTUAL SEPARABILITY

The general theory of MDFS is based on four assumptions about the shapes of the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ (Dzhafarov, 2002; Dzhafarov & Colonius, 2001). Rather than recapitulating them here, I briefly and rather informally mention the aspects and consequences of these assumptions that are directly relevant for the present analysis.

First of all, the assumptions of MDFS ensure that the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ look more or less as shown in Fig. 3 (ignore for now the values of μ): for any $\mathbf{x}$, $\psi_{\mathbf{x}}(\mathbf{y})$ is continuous, attains its global minimum at some point, and increases as one moves a small distance away from this point in any direction. Note that $\psi_{\mathbf{x}}(\mathbf{y})$ is generally allowed to be different from $\psi_{\mathbf{y}}(\mathbf{x})$ (lack of symmetry), and $\psi_{\mathbf{x}}(\mathbf{x})$ is allowed to vary with $\mathbf{x}$ (nonconstant self-similarity). As mentioned in the previous section (and Comment 3 in the Appendix), by means of a certain construction one can always make the minimum of $\psi_{\mathbf{x}}(\mathbf{y})$ to be attained at $\mathbf{y}=\mathbf{x}$, making thereby all psychometric differentials $\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s)$ positive at $s>0$ and continuously decreasing to zero with $s\rightarrow 0+$.

The assumptions underlying MDFS also ensure that all psychometric differentials can be asymptotically decomposed as

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) \sim [F(\mathbf{x},\mathbf{u})\, R(s)]^{\mu}, \qquad s\rightarrow 0+ \tag{10}$$

with the following meaning of the right-hand terms. The constant $\mu>0$, referred to as the *psychometric order* of the stimulus space, is one and the same for all reference stimuli $\mathbf{x}$ and directions of transition $\mathbf{u}$, and it is determined by psychometric

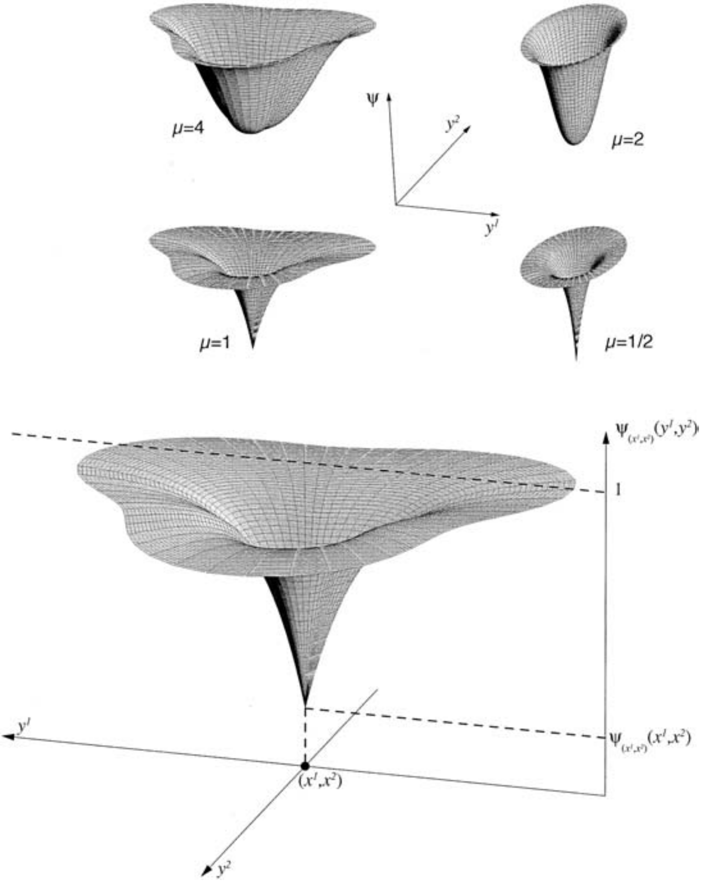

FIG. 3. Possible appearances of psychometric functions ($n=2$).

differentials uniquely. $R(s)$ is some function *regularly varying* at the origin (i.e., as $s \to 0+$) with the unit exponent (Dzhafarov, 2002). The structure of this function is irrelevant for the present discussion, except that the function is positive at $s > 0$ and converging to zero with $s \to 0+$ (see Comment 5 in the Appendix). The critical fact is that this function, too, is one and the same for all psychometric differentials, and it is determined by them asymptotically uniquely. The latter means that $R(s)$ in (10) can only be replaced by $R^*(s) \sim R(s)$ (as $s \to 0+$). Finally, $F(\mathbf{x}, \mathbf{u})$ in (10) is the (Fechner–Finsler) *metric function*, also determined uniquely (see Comment 6 in the Appendix). $F(\mathbf{x}, \mathbf{u})$ is positive (for $\mathbf{u} \neq \mathbf{0}$), continuous, and Euler homogeneous, the latter meaning that, for any k,

$$F(\mathbf{x}, k\mathbf{u}) = |k|\, F(\mathbf{x}, \mathbf{u}). \tag{11}$$

The metric function is all one needs to compute Fechnerian distances. Briefly, the logic of this computation is as follows. Connecting any two points (stimuli) $\mathbf{a}$ and $\mathbf{b}$ by a piecewise smooth path $\mathbf{x}(t)$: $[a, b] \to \mathfrak{M}$, $\mathbf{x}(a) = \mathbf{a}$, $\mathbf{x}(b) = \mathbf{b}$, the *psychometric length* of this path is defined as

$$L\,[\mathbf{x}(t)] = \int_a^b F\,[\mathbf{x}(t), \dot{\mathbf{x}}(t)]\, dt. \tag{12}$$

The Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ is defined as the infimum of $L[\mathbf{x}(t)]$ across all piecewise smooth paths connecting $\mathbf{a}$ and $\mathbf{b}$. Thus defined $G(\mathbf{a}, \mathbf{b})$ is a continuous distance function, invariant with respect to all possible diffeomorphic transformations of coordinates (Dzhafarov & Colonius, 1999, 2001).

The metric function $F(\mathbf{x}, \mathbf{u})$ can be given a simple geometric interpretation in terms of the shapes of psychometric functions. This is achieved through the important concept of a *Fechnerian indicatrix* (Dzhafarov & Colonius, 2001). For a given stimulus $\mathbf{x}$, the Fechnerian indicatrix centered at $\mathbf{x}$ is the contour formed by the direction vectors $\mathbf{u}$ satisfying the equality $F(\mathbf{x}, \mathbf{u}) = 1$. The set of the indicatrices centered at all possible stimuli and the metric function determine each other uniquely. It turns out (Dzhafarov & Colonius, 2001) that the Fechnerian indicatrices are asymptotically similar to the contours formed by horizontally cross-sectioning the psychometric functions $\psi_{\mathbf{x}}(\mathbf{y})$ at a small elevation h from their minima; the smaller h, the better the geometric similarity (see Fig. 4).

Figure 4 and the top panel of Fig. 3 illustrate the geometric meaning of the psychometric order μ. As shown in Dzhafarov and Colonius (2001), if one cross-sections different psychometric functions by vertical planes passing through their minima in various directions, then the cross-sections confined between the minima and some small elevation h are asymptotic replicas of each other, except for the possible difference in scaling coefficient along the horizontal direction. At the very minima of the psychometric functions these cross-sections have a certain degree of flatness/cuspidality, and this degree is determined by the value of μ, from very flat (if μ is large) to pencil-sharp ($\mu = 1$) to needle-sharp (if μ is close to zero). The fact that μ is one and the same for all psychometric differentials means that a specific degree of flatness/cuspidality is shared by all psychometric functions.

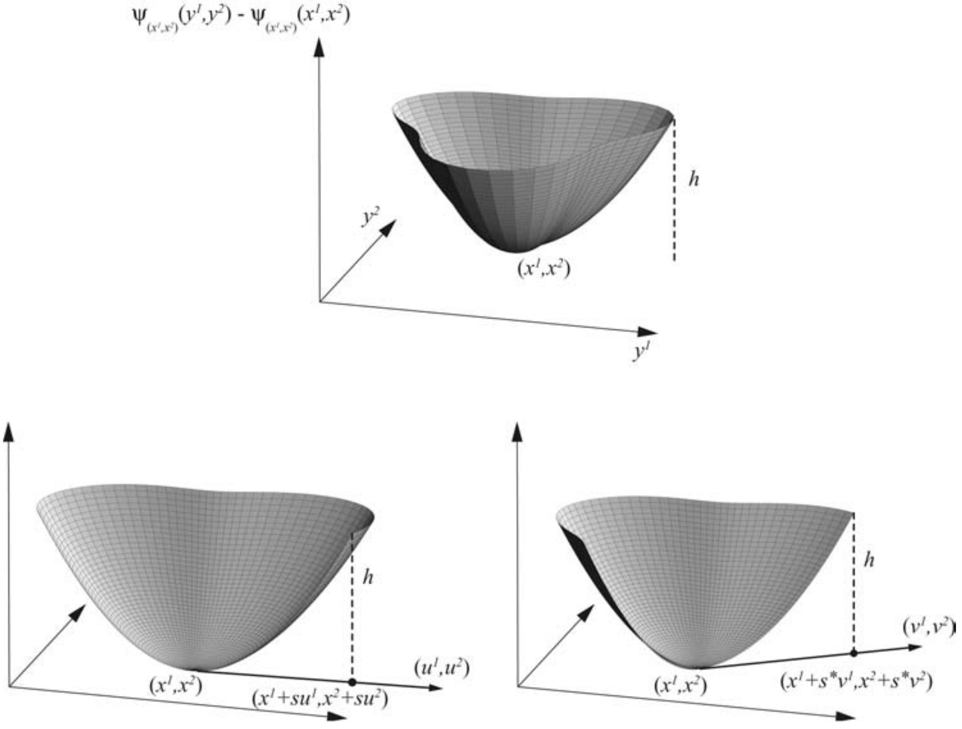

FIG. 4. A horizontal and two vertical cross-sections of a psychometric function at its minimum ($n=2$).

We are prepared now to derive the main result of this work. Let the dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ be perceptually separable in the discrimination system $\langle \mathfrak{R}, \psi \rangle$. On applying (10) to the coordinate projections $\mathbf{u}_i$ of $\mathbf{u}$, one gets, for $i = 1, \ldots, n$,

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s) \sim [F(\mathbf{x}, \mathbf{u}_i)\, R(s)]^{\mu}, \qquad s \to 0+. \tag{13}$$

If, for some i, $\mathbf{u}_i = \mathbf{0}$, (13) still holds, for by continuity the value of $F(\mathbf{x}, \mathbf{u})$ has to be set equal to zero at $\mathbf{u} = \mathbf{0}$.

It follows from Definition 2.2 that $F(\mathbf{x}, \mathbf{u}_i)$ cannot depend on any components of $\mathbf{x}$ other than x^i. One has therefore

$$F(\mathbf{x}, \mathbf{u}_i) = F(\mathbf{x}, \mathbf{1}_i u^i) = |u^i|\, F(\mathbf{x}, \mathbf{1}_i) = F_i(x^i)\, |u^i|, \qquad i = 1, \ldots, n,$$

where we make use of the Euler homogeneity, (11). Equation (13) then can be rewritten as

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}_i s) = \Psi_{x^i}(x^i + u^i s) \sim [F_i(x^i)\, |u^i|\, R(s)]^{\mu} \qquad s \to 0+,$$

for $i = 1, \ldots, n$. Applying this to the right-hand side of (9) in Lemma 2.4, and using (10) for its left-hand side, one gets

$$[F(\mathbf{x}, \mathbf{u})]^{\mu}\, R^{\mu}(s) \sim \sum_{i=1}^{n} [F_i(x^i)\, |u^i|]^{\mu}\, R^{\mu}(s), \qquad s \to 0+,$$

which can only be true if

$$[F(\mathbf{x},\mathbf{u})]^{\mu}=\sum_{i=1}^{n}[F_i(x^i)\,|u^i|]^{\mu}. \tag{14}$$

To see that this structure of the metric function induces the Fechnerian metric with a Minkowski power metric structure, choose an arbitrary point $\mathbf{o}=(o^1,...,o^n)$ and componentwise recalibrate $\langle x^1,...,x^n\rangle$ into

$$\hat{x}^i(x^i)=\int_{o^i}^{x^i}F_i(x)\,dx, \qquad i=1,...,n. \tag{15}$$

According to Lemma 2.3, the axes $\langle\hat{x}^1\rangle,...,\langle\hat{x}^n\rangle$ are perceptually separable. Replacing $\mathbf{x}=(x^1,,...,x^n)$ in (14) with $\hat{\mathbf{x}}=(\hat{x}^1,...,\hat{x}^n)$, in new coordinates, the direction $\mathbf{u}=(u^1,...,u^n)$ attached to $\mathbf{x}$ also acquires new coordinates, $\hat{\mathbf{u}}=(\hat{u}^1,...,\hat{u}^n)$. From (15), these new coordinates are

$$\hat{u}^i=F_i(x^i)\,u^i, \qquad i=1,...,n$$

(see the proof of Lemma 2.3). It follows that $F(\mathbf{x},\mathbf{u})$ in (14), when written in new coordinates as $\hat{F}(\hat{\mathbf{x}},\hat{\mathbf{u}})=F(\mathbf{x},\mathbf{u})$, has the structure

$$\hat{F}(\hat{\mathbf{x}},\hat{\mathbf{u}})=\hat{F}(\hat{\mathbf{u}})=\sqrt[\mu]{|\hat{u}^1|^{\mu}+\cdots+|\hat{u}^n|^{\mu}}\,. \tag{16}$$

The Fechnerian indicatrices corresponding to this metric function

$$\sum_{i=1}^{n}|\hat{u}^i|^{\mu}=1 \tag{17}$$

are Minkowskian; that is, they have the same shape for all stimuli $\mathbf{x}$ at which they are centered (see Comment 7 in the Appendix). This shape is completely determined by the psychometric order μ, as shown in the lower panel of Fig. 5 (filled contours). Recall from the discussion of indicatrices above that these shapes describe the horizontal cross-sections of the psychometric functions (Fig. 5, upper panel) made at a small elevation above their minima. Recall also that μ has another geometric interpretation: it determines the shape (flatness/cuspidality) of the vertical cross-sections of the psychometric functions in the vicinity of their minima (Fig. 5, middle panel). We see, therefore, that *in the case of perceptually separable dimensions the shapes of the horizontal and vertical cross-sections (generally completely independent of each other) are interrelated, being determined by one and the same parameter*, μ.

Figure 5 demonstrates the simple fact that the indicatrices for $\mu\geqslant 1$ are convex in all directions (non-strictly convex when $\mu=1$). A general theory of Fechnerian indicatrices is presented in Dzhafarov and Colonius (2001). Without recapitulating it here, I simply state the fact that if a Minkowskian indicatrix corresponding to $\hat{F}(\hat{\mathbf{u}})$ is convex, then the Fechnerian metric it induces is computed as

$$G(\hat{\mathbf{x}},\hat{\mathbf{y}})=\hat{F}(\hat{\mathbf{x}}-\hat{\mathbf{y}}).$$

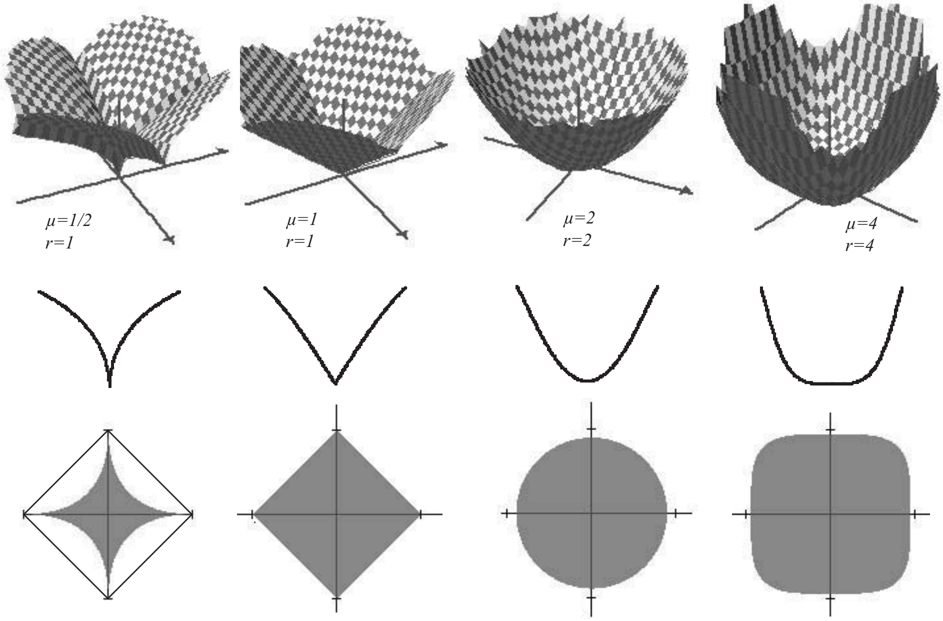

FIG. 5. Psychometric functions in the vicinity of their minima for different values of μ and corresponding r (upper row), their vertical cross-sections (middle row), and horizontal cross-sections (bottom row). The coordinate axes are calibrated in Fechnerian distances along these axes ($n = 1$).

Applying this to (16), with $\mu \geqslant 1$, one gets

$$G(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \hat{F}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) = \sqrt[\mu]{|\hat{x}^1 - \hat{y}^1|^\mu + \cdots + |\hat{x}^n - \hat{y}^n|^\mu}\,. \tag{18}$$

That is, the Fechnerian metric induced by (16) is a Minkowski power metric, with the exponent equal to μ, provided the latter is not less than 1.

One can also see in Fig. 5 that the Fechnerian indicatrix is not convex when $\mu < 1$ (in fact, it is then concave in all directions, except for the coordinate ones). The general theory (Dzhafarov & Colonius, 2001) tells us that the metric induced by a nonconvex indicatrix is the same as the one induced by its *convex closure*, which is the minimal convex contour containing it. In our case it is obvious (see the enclosing contour in Fig. 5 for $\mu = \frac{1}{2}$) that the convex closure of an indicatrix corresponding to any value of $\mu < 1$ is the rhombus described by

$$\sum_{i=1}^{n} |\hat{u}^i| = 1.$$

As a result, when $\mu < 1$, the Fechnerian metric induced by (16) is

$$G(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \hat{F}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) = |\hat{x}^1 - \hat{y}^1| + \cdots + |\hat{x}^n - \hat{y}^n|, \tag{19}$$

the "city-block" metric well known to psychophysicists.

This equation can be combined with (18) in the following statement.

THEOREM 3.1. *If* $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ *imposed on a stimulus space* $\mathfrak{M}$ *are perceptually separable in the discrimination system* $\langle \mathfrak{R}, \psi \rangle$, *then they can be recalibrated (diffeomorphically transformed) into* $\langle \hat{x}^1 \rangle, \ldots, \langle \hat{x}^n \rangle$ *in such a way that*

$$G(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \sqrt[r]{|\hat{x}^1 - \hat{y}^1|^r + \cdots + |\hat{x}^n - \hat{y}^n|^r}, \tag{20}$$

with the exponent $r = \max\{\mu, 1\}$, μ *being the psychometric order of the space.*

In essence, this statement fulfills the goal of the present analysis, except that it is desirable to formulate the main result of this work without mentioning the recalibration procedure (or any specific calibration at all) for the dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$. This can be readily achieved. Recall that the definition of perceptual separability is restricted to some rectangular area $\mathfrak{R} \subseteq \mathfrak{M}$. One consequence of this restriction is that if one chooses a point of origin $\mathbf{o} = (o^1, \ldots, o^n) \in \mathfrak{R}$, then for any $\mathbf{x} = (x^1, \ldots, x^n) \in \mathfrak{R}$ and $\mathbf{y} = (y^1, \ldots, y^n) \in \mathfrak{R}$ their coordinate projections $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^{i=n}$ on the axes drawn through $\mathbf{o}$,

$$\mathbf{x}_i = (o^1, \ldots, o^{i-1}, x^i, o^{i+1}, \ldots, o^n), \; \mathbf{y}_i = (o^1, \ldots, o^{i-1}, y^i, o^{i+1}, \ldots, o^n),$$

are themselves stimuli belonging to $\mathfrak{R}$. When transformed according to (15), these stimuli acquire new coordinates $\hat{\mathbf{x}}_i = \hat{x}^i \mathbf{1}_i$, $\hat{\mathbf{y}}_i = \hat{y}^i \mathbf{1}_i$, and it follows that

$$|\hat{x}^i - \hat{y}^i| = G(\mathbf{x}_i, \mathbf{y}_i), \qquad i = 1, \ldots, n.$$

Using this equality in (20), the development presented in this work can be summarized in the following proposition (refer to Fig. 6).

THEOREM 3.2 (Minkowski power metric structure of Fechnerian metric). *Let the dimensions* $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ *imposed on the stimulus space* $\mathfrak{M}$ *be perceptually separable*

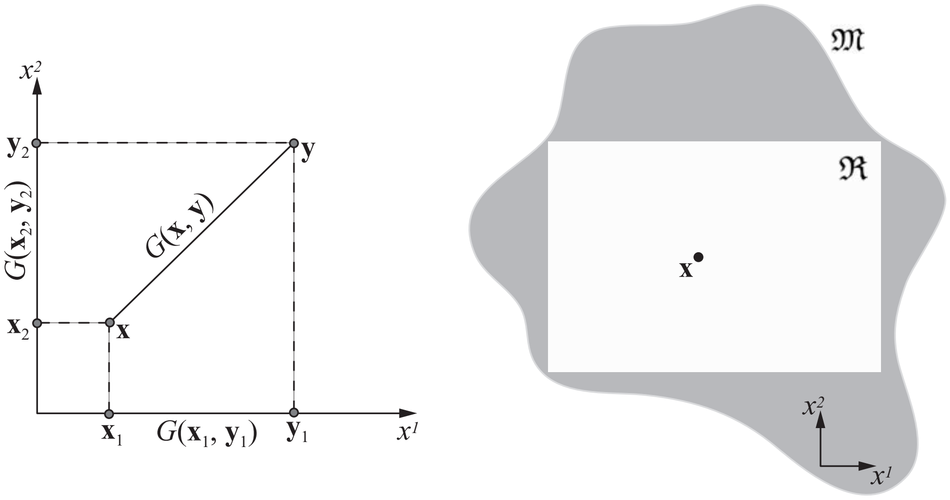

FIG. 6. A diagram for Theorem 3.2 ($n = 2$).

in the discrimination system $\langle \Re, \psi \rangle$. *Then the Fechnerian metric G on this space is a Minkowski power metric with respect to the dimensions* $\langle x^1 \rangle, ..., \langle x^n \rangle$,

$$G(\mathbf{x}, \mathbf{y})^r = G(\mathbf{x}_1, \mathbf{y}_1)^r + \cdots + G(\mathbf{x}_n, \mathbf{y}_n)^r, \tag{21}$$

where $r = \max\{\mu, 1\}$, μ *being the psychometric order of the space.*

Note that the choice of the origin $\mathbf{o} = (o^1, ..., o^n)$ used to define the coordinate projections of $\mathbf{x}$ and $\mathbf{y}$ need not be mentioned, because the Fechnerian distances $G(\mathbf{x}_i, \mathbf{y}_i)$ are invariant with respect to this choice.

4. CONCLUDING COMMENTS

The definition of perceptual separability proposed in this work is mathematically rigorous, based on potentially observable or computable measures of discrimination, and leads to an interesting result within the framework of MDFS: the power-function Minkowskianness of the Fechnerian metric, with the exponent determined by the psychometric order of the stimulus space. The theory predicts definite and non-trivial relationships between the horizontal and vertical cross-sections of the psychometric functions, opening thereby new areas of empirical applicability for the notion of perceptual separability. The question of how these predictions can be experimentally tested, however, is quite complex: the observables of the theory (and MDFS as a whole) are statistical estimates of discrimination probabilities, while its empirical constraints are confined to arbitrarily small areas of the true probabilities. For a general discussion of the empirical status of MDFS and its experimental testability the reader should refer to Dzhafarov (in press) and Dzhafarov and Colonius (1999, 2001).

I conclude this paper with two observations.

4.1. Probability Summation and Probability-Distance Hypothesis

Theorem 3.2 implies an interesting relationship between the notion of perceptual separability, the simple probability summation model described by (4), and the probability-distance hypothesis, according to which the discrimination probability $\psi_{\mathbf{x}}(\mathbf{y})$ is an increasing function of some subjective distance between $\mathbf{x}$ and $\mathbf{y}$. It is shown in Dzhafarov (in press a) that if this probability-distance hypothesis holds, then the subjective distance determining $\psi_{\mathbf{x}}(\mathbf{y})$ (under some constraints, not to be discussed here) coincides with the Fechnerian distance $G(\mathbf{x}, \mathbf{y})$,

$$\psi_{\mathbf{x}}(\mathbf{y}) = f[G(\mathbf{x}, \mathbf{y})]. \tag{22}$$

In view of (21), it is easy to show that the probability-distance hypothesis and the probability summation model with perceptually separable stimulus dimensions $\langle x^1 \rangle, ..., \langle x^n \rangle$ hold together if and only if

$$\psi_{\mathbf{x}}(\mathbf{y}) = 1 - \exp[-kG(\mathbf{x}, \mathbf{y})^r] = 1 - \sum_{i=1}^{n} \exp[-kG(\mathbf{x}_i, \mathbf{y}_i)^r] \tag{23}$$

(see Comment 8 in the Appendix). This can be viewed as a generalized form of Quick's (1974) vector magnitude model for probability summation.

4.2. *Weak Perceptual Separability*

One might argue that the essence of perceptual separability is captured by Definition 2.1 alone, while the detachability requirement can be relaxed or dropped altogether. The Fechnerian metric under the weak separability does not have the Minkowski power metric structure. However, by applying (10) and (13) to Lemma 2.2, one can demonstrate the truth of the following statement.

THEOREM 4.1. (Local Minkowski power metric structure of Fechnerian metric). *Let the dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$ imposed on the stimulus space $\mathfrak{M}$ be weakly perceptually separable in the discrimination system $\langle \mathfrak{M}, \psi \rangle$. Then the (Fechner–Finsler) metric function F on this space has the structure*

$$F(\mathbf{x}, \mathbf{u})^{\mu} = \sum_{i=1}^{n} F(\mathbf{x}, \mathbf{u}_i)^{\mu},$$

where μ is the psychometric order of the space. This, in turn, implies that the Fechnerian metric G is locally (in the small) a Minkowski power metric with respect to the dimensions $\langle x^1 \rangle, \ldots, \langle x^n \rangle$,

$$G(\mathbf{x}, \mathbf{x}+\mathbf{u}s)^{r} \sim \sum_{i=1}^{n} G(\mathbf{x}, \mathbf{x}+\mathbf{u}_i s)^{r}, \qquad s \to 0+,$$

where $r = \max\{\mu, 1\}$.

This result is stronger than it might appear. The shapes of and the relationship between the vertical and horizontal cross-sections of the psychometric functions in the vicinity of their minima remain in the case of weak separability precisely the same as illustrated in Fig. 5, except that the calibration of the axes mentioned in the legend should now be understood as being local and different for different psychometric functions. The notion of weak perceptual separability, therefore, is sufficiently rich in consequences to be of interest by itself.

APPENDIX: TECHNICAL COMMENTS

1. Following the traditional differential-geometric notation adopted in Dzhafarov and Colonius (1999, 2001), I use superscripts rather than subscripts to refer to stimulus coordinates and (later) coordinates of direction vectors. The notation $\langle x^1, \ldots, x^n \rangle$, $\langle x^1 \rangle$, $\langle x^2 \rangle$, etc. refers to frames of reference and axes; whereas $(x^1, \ldots, x^n)$, (x^1), (y^1, y^2), (y^n), etc. denote coordinates of different stimuli with respect to specified frames of reference.

2. It should be noted that the notion of separability is often applied to perceptual dimensions rather than stimulus dimensions. In the General Recognition Theory, due to the one-to-one correspondence between them, this application may

be more linguistic than conceptual. In Shepard's (1987) theory, considered below, the separability is expressly a characterization of perceptual dimensions (as reconstructed by multidimensional scaling). In this paper I strictly adhere to the usage "perceptual separability of stimulus dimensions," by analogy with speaking, say, of the "perceptual discriminability of stimuli" rather than "discriminability of perceptual images."

3. These properties follow from the first assumption of MDFS, according to which $\psi_{\mathbf{x}}(\mathbf{y})$, for a fixed $\mathbf{x}$, reaches its minimum at some point $\mathbf{y}=\mathbf{h}(\mathbf{x})$ diffeomorphically related to $\mathbf{x}$; and that, for any direction $\mathbf{u}$, $\psi_{\mathbf{x}}(\mathbf{y}+\mathbf{u}s)$ strictly decreases as s decreases from a sufficiently small value to $s=0$, at which value it vanishes. The renaming procedure mentioned in the text consists in putting $\psi_{\mathbf{x}}(\mathbf{y}) = \psi^*_{\mathbf{h}(\mathbf{x})}(\mathbf{y})$, which ensures that $\psi^*_{\mathbf{x}}(\mathbf{y})$ reaches its minimum at $\mathbf{y}=\mathbf{x}$. Throughout this paper it is assumed that ψ has already been put in this form.

4. Strictly speaking, the functions in (8) should be denoted by symbols other than Ψ, say, $\Psi^i_{x^i}(x^i+u^i s)$. The use of Ψ, however, does not create confusion if one considers the subscripts at Ψ as part of the function names.

5. The regular variation of $R(s)$ with the unit exponent means that $\frac{R(ks)}{R(s)} \to k$ as $s \to 0+$ (for any $k>0$). For example, $R(s) \equiv s$ is such a function, and in many respects any unit-regularly varying $R(s)$ is indistinguishable from s (Dzhafarov, 2002). The reader who is willing to overlook technical details may, with no serious consequences for understanding this work, assume that $R(s) \equiv s$, and thence (10) has the form

$$\Psi_{\mathbf{x}}(\mathbf{x}+\mathbf{u}s) \sim [F(\mathbf{x},\mathbf{u})\, s]^{\mu}, \qquad s \to 0+.$$

This is the so-called *power function version* of MDFS (Dzhafarov & Colonius, 1999). The more general theory adopted in the present work is called the *regular variation version* of MDFS (Dzhafarov, 2002).

6. Clearly, $F(\mathbf{x},\mathbf{u})$ and $R(s)$ in (10) can be multiplied by reciprocal positive constants without changing the asymptotic equality. The necessity of mentioning this trivial qualification for uniqueness is avoided by putting $F(\mathbf{x}_0,\mathbf{u}_0)=1$ for some arbitrarily chosen $(\mathbf{x}_0,\mathbf{u}_0)$.

7. Indicatrices and the corresponding metric function are called Minkowskian whenever the metric function does not depend on the stimulus, $F(\mathbf{x},\mathbf{u})=F(\mathbf{u})$. The power metric structure arrived at in (16) is just a special case.

8. Putting $g(-x^r)=1-f(x)$, where f is defined in (22) and assumed to be increasing, whence g is increasing too, (4) can be written as

$$g[-G(\mathbf{x},\mathbf{y})^r] = \prod_{i=1}^{n} g[-G(\mathbf{x}_i,\mathbf{y}_i)^r]$$

which is the multiplicative form of the Cauchy functional equation (also known as the Hammel equation). Its only increasing solution is known to be $g(a)=\exp(ka)$, $k>0$.

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