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About Mathematical Psychology

There are three fuzzy and interrelated understandings of what mathematical psychology is: part of mathematics, part of psychology, and analytic methodology. We call them "fuzzy" because we do not offer a rigorous way of defining them. As a rule, a work in mathematical psychology, including the chapters of this handbook, can always be argued to conform to more than one if not all three of these understandings (hence our calling them "interrelated"). It seems safer therefore to think of them as three constituents of mathematical psychology that may be differently expressed in any given line of work.

1. Part of mathematics

Mathematical psychology can be understood as a collection of mathematical developments inspired and motivated by problems in psychology (or at least those traditionally related to psychology). A good example for this is the algebraic theory of semiorders proposed by R. Duncan Luce (1956). In algebra and unidimensional topology there are many structures that can be called orders. The simplest one is the total, or linear order $(S, \preceq)$ characterized by the following properties: for any $a, b, c \in S$,

(O1) $a \preceq b$ or $b \preceq a$;

(O2) if $a \preceq b$ and $b \preceq c$ then $a \preceq c$;

(O3) if $a \preceq b$ and $b \preceq a$ then $a = b$.

The ordering relation here has the intuitive meaning of "not greater than." One can, of course, think of many other kinds of order. For instance, if we replace the property (O1) with

(O4) $a \preceq a$,

we obtain a weaker (less restrictive) structure, called a partial order. If we add to the properties (O1-O3) the requirement that every nonempty subset X of

S possesses an element a_X such that $a_X \preceq a$ for any $a \in X$, then we obtain a stronger (more restrictive) structure called a well-order. Clearly, one needs motivation for introducing and studying various types of order, and for one of them, semiorders, it comes from psychology.[1]

Luce (1956) introduces the issue by the following example:

> Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $(1 + \frac{i}{100})x$ grams of sugar, $i = 0, 1, \ldots, 400$, where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup $i + 1$, for any i, but by choice he is not indifferent between $i = 0$ and $i = 400$ (p. 179).

This example involves several idealizations, e.g., Luce ignores here the probabilistic nature of a person's judgments of sweetness/bitterness, treating the issue as if a given pair of cups of coffee was always judged in one and the same way. However, this idealized example leads to the idea that there may be an interesting order such that $a \prec b$ only if a and b are "sufficiently far apart"; otherwise a and b are "too similar" to be ordered ($a \sim b$). Luce formalizes this idea by the following four properties of the structure $(S, \prec, \sim)$: for any $a, b, c, d \in S$,

(SO1) exactly one of three possibilities obtains: either $a \prec b$, or $b \prec c$ or else $a \sim b$;

(SO2) $a \sim a$;

(SO3) if $a \prec b$, $b \sim c$ and $c \prec d$ then $a \prec d$;

(SO4) if $a \prec b$, $b \prec c$ and $b \sim d$ then either $a \sim d$ or $c \sim d$ does not hold.

There are more compact ways of characterizing semiorders, but Luce's seems most intuitive.

Are there familiar mathematical entities that satisfy the requirements (SO1-SO4)? Consider a set of reals and suppose that A is a set of half-open intervals $[x, y)$ with the following property: if $[x_1, y_1)$ and $[x_2, y_2)$ belong to A, then $x_1 \leq x_2$ holds if and only if $y_1 \leq y_2$. Let's call the intervals in A monotonically ordered. Define $[x, y) \prec [v, w)$ to mean $y \leq v$. Define $[x, y) \sim [v, w)$ to mean that the two intervals overlap. It is easy to check then that the system of monotonically ordered intervals with the $\prec$ and $\sim$ relations just defined forms a semiorder.

As it turns out, under certain constraints imposed on S, the reverse of this statement is also true. To simplify the mathematics, let us assume that S can be one-to-one mapped onto an interval (finite or infinite) of real numbers. Thus, in Luce's example with cups of coffee we can assume that each cup is uniquely characterized by the weight of sugar in it. Then all possible cups of coffee form a set S that is bijectively mapped onto an interval of reals between 1 and 5 cubes

[1] The real history, as often happens, is more complicated, and psychology was the main but not the only source of motivation here (see Fisher & Monjardet, 1992).

of sugar (ignoring discreteness due to the molecular structure of sugar). Under this assumption, it follows from a theorem proved by Peter Fishburn (1973) that the semiorder $(S, \prec, \sim)$ has a monotonically ordered representation. The latter means that there is a monotonically ordered set A of real intervals and a function $f : S \rightarrow A$ such that, for all $a, b \in S$,

$$a \prec b \text{ if and only if } f(a) = [x_1, y_1), f(b) = [x_2, y_2), \text{ and } y_1 \leq x_2; \tag{1}$$

$$a \sim b \text{ if and only if } f(a) = [x_1, y_1), f(b) = [x_2, y_2), \text{ and } [x_1, y_1) \cap [x_2, y_2) \neq \emptyset. \tag{2}$$

Although as a source of inspiration for abstract mathematics psychology cannot compete with physics, it has motivated several lines of mathematical development. Thus, a highly sophisticated study of m-point-homogeneous and n-point-unique monotonic homeomorphisms (mappings that are continuous together with their inverses) of conventionally ordered real numbers launched by Louis Narens (1985) and Theodore M. Alper (1987) was motivated by a well-known classification of measurements by Stanley Smith Stevens (1946). In turn, this classification was inspired by the variety of measurement procedures used in psychology, some of them clearly different from those used in physics. Psychology has inspired and continues to inspire abstract foundational studies in the representational theory of measurement (essentially an area of abstract algebra with elements of topology), probability theory, geometries based on nonmetric dissimilarity measures, topological and pre-topological structures, etc. Finally and prominently, the modern developments in the area of functional equations, beginning with the highly influential work by János Aczél (1966), have been heavily influenced by problems in or closely related to psychology.

2. Part of psychology

According to this understanding, mathematical psychology is simply psychological theorizing and model-building in which mathematics plays a central role (but does not necessarily evolve into new mathematical developments). A classical example of work that falls within this category is Gustav Theodeor Fechner's derivation of his celebrated logarithmic law in the *Elemente der Psychophysik* (1861, Ch. 17).[2] From this book (and this law) many date the beginnings of scientific psychology. The problem Fechner faced was how to relate "magnitude of physical stimulus" to "magnitude of psychological sensation," and he came up with a principle: *equal ratios of stimulus magnitudes correspond to equal differences of sensation magnitudes.* This means that for any stimulus values x_1, x_2 (real numbers at or above some positive threshold value x_0) we have

$$\psi(x_2) - \psi(x_1) = F\left(\frac{x_2}{x_1}\right), \tag{3}$$

[2] The account that follows is not a reconstruction but Fechner's factual derivation (pp. 34-36 of vol. 2 of the *Elemente*). It has been largely overlooked in favor of the less general and less clearly presented derivation of the logarithmic law in Chapter 16 (see Dzhafarov & Colonius, 2012).

where ψ is the hypothetical psychophysical function (mapping stimulus magnitudes into positive reals representing sensation magnitudes), and F is some unknown function.

Once the equation was written, Fechner investigated it as a purely mathematical object. First, he observed its consequence: for any three suprathreshold stimuli x_1, x_2, x_3,

$$F\left(\frac{x_3}{x_1}\right) = F\left(\frac{x_3}{x_2}\right) + F\left(\frac{x_2}{x_1}\right). \tag{4}$$

Second, he observed that $u = {}^{x_2}/_{x_1}$ and $v = {}^{x_3}/_{x_2}$ can be any positive reals, and ${}^{x_3}/_{x_1}$ is the product of the two. We have therefore, for any $u > 0$ and $v > 0$,

$$F(uv) = F(u) + F(v). \tag{5}$$

This is an example of a simple functional equation: the function is unknown, but it is constrained by an identity that holds over a certain domain (positive reals).

Functional equations were introduced in pure mathematics only 40 years before Fechner's publication, by Augustin-Louis Cauchy, in his famous *Cours d'analyse* (1821). Cauchy showed there that the only continuous solution for (3) is the logarithmic function

$$F(x) \equiv k \log x,\ x > 0, \tag{6}$$

where k is a constant. The functional equations of this kind were later called the Cauchy functional equations. We know now that one need not even assume that F is continuous. Thus, it is clear from (1) that F must be positive for on at least some interval of values for ${}^{x_2}/_{x_1}$: if x_2 is much larger than x_1, it is empirically plausible to assume that $\psi(x_2) > \psi(x_1)$. This alone is sufficient to derive (4) as the only possible solution for (3), and to conclude that k is a positive constant.

The rest of the work for Fechner was also purely mathematical, but more elementary. Putting in (1) $x_2 = x$ (an arbitrary value) and $x_1 = x_0$ (the threshold value), one obtains

$$\psi(x) - \psi(x_0) = \psi(x) = k \log\left(\frac{x}{x_0}\right), \tag{7}$$

which is the logarithmic law of psychophysics. Fechner thus used sophisticated by standards of his time mathematical work by Cauchy to derive the first justifiable quantitative relation in the history of psychology. The value of Fechner's reasoning is entirely in psychology, bringing nothing new to mathematics, but the reasoning itself is entirely mathematical.

There are many other problems and areas in psychology whose analysis falls within the considered category because it essentially consists of purely mathematical reasoning. Thus, analysis of response times that involves distribution or quantile functions is one such area, and so are some areas of psychophyscis (especially, theory of detection and discrimination), certain paradigms of decision making, memory and learning, etc.

3. Analytic methodology

A third way one can think of mathematical psychology is as an applied, or service field, a set methodological principles and techniques of experimental design, data analysis, and model assessment developed for use by psychologists. The spectrum of examples here extends from purely statistical research to methodology based on substantive theoretical constructs falling within the scope of the first of our three understandings of mathematical psychology.

A simple but representative example of the latter category is H. Richard Blackwell's (1953) correction-for-guessing formula and recommended experimental design. Blackwell considered a simple detection experiment: an observer is shown a stimulus that may have a certain property and asked whether she is aware of this property being present (Yes or No). Thus, the property may be an increment of intensity ΔB in the center of a larger field of some fixed intensity B. Depending on the value of ΔB, the observer responds Yes with some probability p. Blackwell found that this probability $p\left(\Delta B\right)$ was not zero even at $\Delta B = 0$. It was obvious to Blackwell (but not to the detection theorists later on) that this indicated that the observer was "guessing" that the signal was there, with probability $p_g = p\left(0\right)$. It is clear, however, that the observer cannot distinguish the situation in which $\Delta B = 0$ (and therefore, according to Blackwell, she could not perceive an intensity increment) from one in which $\Delta B > 0$ but she failed to detect it. Assuming that ΔB is detected with probability $p_d\left(\Delta B\right)$, we have the following tree of possibilities:

$$\begin{tikzcd}
& \Delta B \arrow[dl, Rightarrow, "p_{d(\Delta B)}"'] \arrow[dr, Rightarrow, "1-p_{d(\Delta B)}"] & \\
\text{detected} \arrow[d, Rightarrow, "1"'] & & \text{not detected} \arrow[d, Rightarrow, "1-p_g"] \arrow[dll, Rightarrow, "p_g"'] \\
\text{Yes} & & \text{No}
\end{tikzcd}$$

We can now express the probability $p\left(\Delta B\right)$ of the observer responding Yes to ΔB through the probability $p_d\left(\Delta B\right)$ that she detects ΔB and the probability p_g that she says Yes even though she has not detected ΔB:

$$p\left(\Delta B\right) = p_d\left(\Delta B\right) + \left(1 - p_d\left(\Delta B\right)\right) p_g. \tag{8}$$

The value of $p_d\left(\Delta B\right)$ decreases with decreasing ΔB, reaching zero at $\Delta B = 0$. At this value therefore the formula turns into

$$p\left(0\right) = p_g, \tag{9}$$

as it should. That is, p_g is directly observable (more precisely, can be estimated from data): it is the probability with which the observer says Yes to "catch" or "empty" stimuli, those with $\Delta B = 0$. Blackwell therefore should insist that catch trials be an integral part of experimental design in any Yes/No detection experiment. Once $p_g = p(0)$ is known (estimated), one can "correct" the observed (estimated) probability $p(\Delta B)$ for any nonzero ΔB into the true probability of detection:

$$p_d(\Delta B) = \frac{p(\Delta B) - p(0)}{1 - p(0)}. \tag{10}$$

We end up therefore with a strong recommendation on experimental design (which is universally followed by all experimenters) and a formula for finding true detection probabilities (which is by now all but abandoned). Blackwell's work therefore is an example of a methodological development to be used in experimental design and data analysis. At the same time, however, it is also a substantive model of sensory detection, and as such falls within the category of work in psychology in which mathematics plays a central role. The mathematics here is technically simple but ingeniously applied.

The list of methodological developments based on substantive psychological ideas is long. Other classical examples it includes are Louis Leon Thurstone's (1927) analysis of pairwise comparisons and Georg Rasch's analysis of item difficulty and responder aptitude (1960).

On the other pole of the spectrum we find methodological developments that have purely data-analytic character, and their relation to psychology is determined by historical tradition rather than internal logic of these developments. For instance, nowadays we see a rapid growth of sophisticated Bayesian data-analytic and model-comparison procedures, as well as those based on resampling and permutation techniques. Some psychologists prefer to consider all these applied-statistical developments part of psychometrics rather than mathematical psychology. The relationship between the two disciplines is complex, but they are traditionally separated, with different societies and journals.

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