One of the big issues in perception (and cognition in general) is identifying how the brain represents information.

In today’s lecture I want to do two things:
1) Discuss a bit about ways of representing information
2) Discuss a mathematical property that is a commonly used tool in perception

Fourier analysis
Don’t freak out!

Joseph Fourier
- French mathematician (1768-1830)
- Involved in the French revolution
- An administrator for Napoleon Bonaparte
- Discovered a mathematical property that even other mathematicians initially found difficult to accept

Sines and cosines
- Take a circle with its center at position (0,0)

Sines and cosines
- Take a circle
- Draw a line from the middle at an angle
- Draw the triangle where the line crosses the circle

Fourier Analysis
PSY 310
Greg Francis
Lecture 08
It’s all waves!

Representation of information
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- In today’s lecture I want to do two things
  1) Discuss a bit about ways of representing information
  2) Discuss a mathematical property that is a commonly used tool in perception
- Fourier analysis
- Don’t freak out!

Sines and cosines
- Take a circle
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PSY 310: Sensory and Perceptual Processes
Sines and cosines

- The cosine function \( \cos(\theta) \) is the proportion of the radius of the circle in the x-direction.

- The sine function \( \sin(\theta) \) is the proportion of the radius of the circle in the y-direction.

So

\[
\cos(0) = 1 \\
\sin(0) = 0
\]

So

\[
\cos\left(\frac{\pi}{2}\right) = 0 \\
\sin\left(\frac{\pi}{2}\right) = 1
\]

So

\[
\cos(\pi) = -1 \\
\sin(\pi) = 0
\]
Cos function

- If you plot the cosine function as theta varies, you get this other nice wave.

\[ \cos(\theta) \]

Properties

- There are some interesting properties of these functions.

\[ \cos(\theta) = \cos(-\theta) \]

Sine function

- You can speed up the wave by multiplying theta.

\[ \sin(2\theta) \]

Sine function

- You can also change the height of the wave.

Amplitude

\[ 0.3 \sin(\theta) \]
**Frequency**

- Suppose you are only interested in theta values between -π and π
  - You can easily generalize to other ranges, but the equations look worse
  - In equations of the form
    \[ \sin(nθ) \cos(nθ) \]
  - We say the frequency of the wave is \( n \)
  - This is how many times the wave cycles (comes back to its starting value)

**Orthonormality**

- This property is harder to prove, but easy to show examples
  - If you multiple any sine and cosine functions and take the integral, you get either zero or \( \pi \)
    \[
    \int_{-\pi}^{\pi} \sin(nθ)\cos(mθ)dθ = \begin{cases} 
    \pi & \text{if } n = m \\
    0 & \text{if } n \neq m
    \end{cases}
    \]
    \[
    \int_{-\pi}^{\pi} \cos(nθ)\cos(mθ)dθ = \begin{cases} 
    \pi & \text{if } n = m \\
    0 & \text{if } n \neq m
    \end{cases}
    \]
    \[
    \int_{-\pi}^{\pi} \sin(nθ)\sin(mθ)dθ = \begin{cases} 
    \pi & \text{if } n = m \\
    0 & \text{if } n \neq m
    \end{cases}
    \]

**Orthonormality**

- Note the positive parts are mirror image of the negative parts
  - \( \sin(2x)\cos(x) \)
  - \( \cos(2x)\cos(6x) \)

**Orthonormality**

- Note the positive parts are mirror image of the negative parts
  - \( \sin(2x)\sin(4x) \)

**Orthonormality**

- It’s less obvious here, but each positive part covers two negative parts
  - \( \cos(2x)\cos(6x) \)

**Orthonormality**

- If the functions are the same, the integral equals \( \pi \)
  - \( \sin(2x)\sin(2x) \)
Orthonormality

- We can sort of ask how much of one function is made by another function.
- E.g., sin(2x) has no part of it that is made of sin(3x), or cos(7x).
- What about other functions? How much of the function is made up of sine or cosine functions of different frequencies?
- Linear: f(x) = mx + b
- Parabola: f(x) = ax^2 + b
- Whatever

Fourier

- Proved that you can write (almost) any function as a series of properly weighted (amplified) sine and cosine functions of different frequencies.

\[ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \]

- The trick is to find the proper values of a_0, a_n, and b_n.

Finding coefficients

- The first term is fairly easy to find, it's just the average.

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \]

- To find the other a_n terms, just multiply by a cosine function of the n-th frequency.

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \]

- To find the b_n terms, just multiply by a sine function of the n-th frequency.

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \]

Example

- Suppose our function is the absolute value function.

\[ f(x) = \frac{|x|}{\pi} \]

- Divided by \( \pi \) to keep everything between -1 and 1.

Example

- Suppose our function is the absolute value function.

\[ a_0 = \frac{1}{2} \]

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Example

- Suppose our function is the absolute value function.

\[ a_0 = \frac{1}{2} \]

- Divided by \( \pi \) to keep everything between -1 and 1.

\[ f(x) = \frac{|x|}{\pi} \]

\[ a_n = \frac{4}{\pi^2} \]

- Divided by \( \pi \) to keep everything between -1 and 1.

\[ f(x) = \frac{|x|}{\pi} \]
Example
- Suppose our function is the absolute value function
  - Divided by \( \pi \) to keep everything between -1 and 1.

\[
 f(x) = \frac{|x|}{\pi}
\]

\[
 a_0 = \frac{1}{2} \quad a_1 = \frac{-4}{\pi^2} \quad a_2 = 0 \quad a_n = a_{-n} \cos(n)
\]

Why does this matter for perception?
- (1) We are curious as to how the brain represents information
- (2) We can do essentially the same thing in 2-dimensions
- (3) It is useful to analyze images and receptive fields in terms of the spatial frequency

Representation
- A function can be described either in space (x)

\[
 f(x) = \frac{|x|}{\pi}
\]

Representation
- A function can be described either in space (x)
  - Or in terms of the Fourier coefficients \((a_0, a_1, \ldots, b_1, b_2, \ldots)\)

\[
 f(x) = \frac{|x|}{\pi} \quad a_0 = \frac{1}{2} \quad a_n = \frac{-4}{(n\pi)^2} \quad b_n = 0
\]

Representation
- Perhaps the brain just converts the spatial image into something like Fourier coefficients
  - No loss of information!
2-dimensional images
- It’s a bit more complicated, but you can do essentially the same kind of analysis with 2-D images

2-dimensional images
- You can isolate the low frequency information in the image

2-dimensional images
- You can isolate the high frequency information in the image

2-dimensional images
- You can isolate any range of frequencies you want

Conclusions
- Fourier analysis
- You can represent any function (image) as a combination of sine and cosine functions
- In principle, the brain might convert visual images into some completely different representation of information

Next time
- The visual system's response to spatial frequency
- Receptive fields
- Images